The metric dimension of graph with pendant edges

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Abstract. For an ordered set $W = \{w_1, w_2, \cdots, w_k\}$ of vertices and a vertex v in a connected graph G, the representation of v with respect to W is the ordered k-tuple $r(v|W) = (d(v, w_1), d(v, w_2), \cdots, d(v, w_k))$ where $d(x, y)$ represents the distance between the vertices x and y. The set W is called a resolving set for G if every two vertices of G have distinct representations. A resolving set containing a minimum number of vertices is called a basis for G. The dimension of G, denoted by $\dim(G)$, is the number of vertices in a basis of G. In this paper, we determine the dimensions of some corona graphs $G \odot K_1$, and $G \odot \overline{K}_m$ for any graph G and $m \geq 2$, and a graph with pendant edges more general than corona graphs $G \odot \overline{K}_m$.

1 Introduction

In this paper we consider finite, simple, and connected graphs. The vertex and edge sets of a graph G are denoted by $V(G)$ and $E(G)$, respectively. We refer [6] for the general graph theory notations and terminologies are not described in this paper.

The *distance* $d(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest $u - v$ path in G. For an ordered set $W = \{w_1, w_2, \cdots, w_k\} \subseteq V(G)$ of vertices, we refer to the ordered k-tuple $r(v|W) = (d(v, w_1), d(v, w_2), \cdots, d(v, w_k))$ as the *(metric) representation of v with respect to W.* The set W is called a *resolving set* for G if $r(u|W) = r(v|W)$ implies that $u = v$, for all $u, v \in G$. A resolving set of minimum cardinality for a graph G is called a *minimum resolving set* or a *basis* for G. The *metric dimension* of G, denoted by $\dim(G)$, is the number of vertices in a basis for G.

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The first papers discussing on the notion of a (minimum) resolving set were written by Slater in [10] and [11]. Slater introduced the concept of a resolving set for a connected graph G under the term *location set.* He called the cardinality of a minimum resolving set by the *location number* of G. Independently, Harary and Melter [8] introduced the same concept but used the term *metric dimension*, instead.

Chartrand et. al. [5] determined the bounds of the metric dimensions for any connected graphs and determined the metric dimensions of some well known families of graphs such as trees, paths, and complete graphs. They introduced some following definitions. A *major vertex* v of a graph G is a vertex of degree at least 3 in G. A *terminal vertex of a major vertex* v of G is a pendant vertex u which have a property $d(u,v) < d(u,w)$ for every other major vertex w of G. The *terminal degree* $\text{ter}(v)$ of a major vertex v is the number of terminal vertices of v. A major vertex v of G is an *exterior major vertex* of G if it has positive terminal degree. In the following, we present some known results.

Theorem A [2,7] *Let G be a connected graph of order $n \geq 2$.*

(*i.*) *$dim(G) = 1$ if and only if $G = P_n$*
(*ii.*) *$dim(G) = n-1$ if and only if $G = K_n$*
(*iii.*) *For $n \geq 3$, $dim(C_n) = 2$*
(*iv.*) *For $n \geq 4$, $dim(G) = n-2$ if and only if $G = K_{r,s}, (r,s \geq 1)$, $G = K_r + \overline{K_s}$, $(r \geq 1, s \geq 2)$, or $G = K_r + (K_1 \cup K_s)$, $(r,s \geq 1)$*
(*v.*) *If T is a tree other than a path, then $dim(T) = \sigma(T) - ex(T)$, where $\sigma(T)$ denotes the sum of the terminal degrees of the major vertices of T, and $ex(T)$ denotes the number of the exterior major vertices of T.*

Buczkowski et. al. [1] proved the existence of a graph G with $\dim(G) = k$, for every integer $k \geq 2$. They also in particular determined dimensions of wheels. Chappell et. al. [4] considered relationships between metric dimension with other parameters in a graph. Another researchers in [2, 7] determined the metric dimension of the cartesian products of two graphs and Cayley digraphs.

Let G and H be two given graphs with G having n vertices, the *corona product* $G \odot H$ is defined as a graph with

$$V(G \odot H) = V(G) \cup \bigcup_{i \in V(G)} V(H_i),$$
$$E(G \odot H) = E(G) \cup \bigcup_{i \in V(G)} (E(H_i) \cup \{iu_i | u_i \in V(H_i)\}),$$

where $H_i \cong H$, for all $i \in V(G)$. If $H \cong \overline{K}_m$, $G \odot H$ is equal to the graph produced by adding n pendant edges to every vertex of G. In particular, if $H \cong K_1$, $G \odot H$ is equal to the graph produced by adding one pendant edge to every vertex of G. Buczkowski et. al. in [1] proved that if G' is a graph obtained by adding a pendant edge to a nontrivial connected graph G, then

$$\dim(G) \leq \dim(G') \leq \dim(G) + 1.$$

Therefore, for $G \odot K_1$ we have:

$$\dim(G) \leq \dim(G \odot K_1).$$

If $G \cong K_1$ and $H \cong C_n$, $G \odot H$ is equal to wheel $W_n = K_1 + C_n$. If $G \cong K_1$ and $H \cong P_n$, $G \odot H$ is equal to fan $F_n = K_1 + P_n$. Buczkowski et. al. and Caceres et. al. in [1, ?], determined the dimensions of wheels and fans, namely: $\dim(W_n) = \lfloor \frac{2n+2}{5} \rfloor$ for $n \notin \{3,6\}$, and $\dim(F_n) = \lfloor \frac{2n+2}{5} \rfloor$ for $n \notin \{1,2,3,6\}$.

Let $G_1 = (V_1,E_1)$ and $G_2 = (V_2,E_2)$ be two graphs. The *Cartesian products* $G_1 \times G_2$ is the graph whose vertex set is $V_1 \times V_2$ and two vertices (x_1,x_2) and (y_1,y_2) being adjacent in $G_1 \times G_2$ if and only if either ($x_1 = y_1$ and $x_2y_2 \in E_2$) or ($x_2 = y_2$ and $x_1y_1 \in E_1$). The graph K_1 or P_1 is a *unit* with respect to the Cartesian product. In other words, for $H = K_1$ or P_1, $H \times G = G$ and $G \times H = G$ for any graph G. Caceres et. al. [3] determined the metric dimension of some cartesian product graphs, namely: $\dim(P_m \times P_n) = 2$, $\dim(P_m \times K_n) = n - 1$, for $n \geq 3$,

$$\dim(P_m \times C_n) = \begin{cases} 2, & \text{if } n \text{ odd;} \\ 3, & \text{if } n \text{ even (and } m \neq 1), \end{cases}$$

$$dim(C_m \times C_n) = \begin{cases} 4, & \text{if } m.n \text{ even;} \\ 3, & \text{otherwise,} \end{cases}$$

$$\text{for } m \geq 4, dim(K_m \times C_n) = \begin{cases} m, & \text{if } m = 4 \text{ and } n \text{ odd;} \\ m-1, & \text{otherwise,} \end{cases}$$

and

$$\text{for } m \leq n, dim(K_m \times K_n) = \begin{cases} n-1, & \text{if } 2m-2 < n; \\ \lfloor \frac{2m+2n+2}{3} \rfloor, & \text{if } 2m-2 \geq n. \end{cases}$$

In this paper, we determine the dimensions of some corona graphs in $G \odot K_1$ and $G \odot \overline{K}_m$, for any graph G and $m \geq 2$. We also consider the dimension of a graph with pendant edges in more general sense than the corona graphs, namely a graph obtained from graph G by adding a (not necessarily the same) number of pendant edges to every vertex of G.

2 Results

In this paper, we first determine the dimension of $(P_n \times P_m) \odot K_1$.

Theorem 1. *For $n \geq 1$ and $1 \leq m \leq 2$, $dim((P_n \times P_m) \odot K_1) = 2$.*

Proof Let $v_{ij} = (v_i, v_j)$ be the vertices of $P_n \times P_m \subseteq (P_n \times P_m) \odot K_1$, where $v_i \in P_n$, $v_j \in P_m$, $1 \le i \le n$, and $1 \le j \le m$. Let u_{ij} be the pendant vertex of v_{ij}. For small numbers n and m, we have $\dim((P_1 \times P_1) \odot K_1) = \dim(P_2) = 1$, $\dim((P_2 \times P_1) \odot K_1) = \dim(P_2) = 1$, and $\dim((P_2 \times P_2) \odot K_1) = 2$.

Case 1. $m = 1$. By using Theorem A (i), we only need to show that $\dim((P_n \times P_1) \odot K_1) \le 2$. Choose a resolving set $B = \{v_{11}, v_{n1}\}$ in $(P_n \times P_1) \odot K_1$. The representation of vertices $v \in (P_n \times P_1) \odot K_1$ by B are

$r(v_{i1}|B) = (i-1, n-i)$, for $2 \le i \le n-1$,

$r(u_{i1}|B) = (d(v_{11}, v_{i1}) + 1, d(v_{n1}, v_{i1}) + 1)$, for $1 \le i \le n$.

All of those representation are distinct. Therefore, $\dim((P_n \times P_1) \odot K_1) = 2$.

Case 2. $m = 2$. Again, by Theorem A (i), we only need to show that $\dim((P_n \times P_2) \odot K_1) \le 2$. Choose a resolving set $B = \{u_{11}, u_{12}\}$ in $(P_n \times P_2) \odot K_1$. The representation of vertices $v \in (P_n \times P_2) \odot K_1$ by B are

$r(v_{i1}|B) = (i, i+1)$ and $r(v_{i2}|B) = (i+1, i)$, for $1 \le i \le n$,

$r(u_{i1}|B) = (d(v_{i1}, u_{11}) + 1, d(v_{i1}, u_{12}) + 1)$

and $r(u_{i2}|B) = (d(v_{i2}, u_{11}) + 1, d(v_{i2}, u_{12}) + 1)$, for $2 \le i \le n$.

All of those representation are distinct. Therefore, $\dim((P_n \times P_2) \odot K_1) = 2$. ■

Open problem 1 *Find the dimension of* $(P_n \times P_m) \odot K_1$*, for* $n \ge 3$ *and* $m \ge 3$.

Theorem 2. *For* $n \ge 3$,

$$dim((K_n \times P_m) \odot K_1) = \begin{cases} n-1, & m = 1, \\ n, & m = 2. \end{cases}$$

Proof Let $v_{ij} = (v_i, v_j)$ be a vertex in $K_n \times P_2$, where $v_i \in K_n$, $v_j \in P_2$, $1 \le i \le n$, and $1 \le j \le 2$. Let u_{ij} be a pendant vertex of v_{ij}.

Case 1. $m = 1$. By a contradiction, we show $\dim((K_n \times P_1) \odot K_1) \ge n-1$. Suppose that B is a basis of $(K_n \times P_1) \odot K_1$ (or $K_n \odot K_1$) with $|B| < n-1$. If $B \subseteq V(K_n)$ then we will have at least two vertices v and w in $V(K_n)$ such that the distance from v and w to all vertices x in B is 1. Otherwise, if there is a vertex y of B with $y \notin V(K_n)$, then we will still have at least two vertices v and w in $V(K_n)$ such that the distance from v and w to the vertices x is 1 and to a vertex y is 2. So, we also have $r(v|B) = r(w|B)$, a contradiction. Now, we show $\dim(K_n \odot K_1) \le n-1$ by choosing a resolving set $B = \{v_1, v_2, \cdots, v_{n-1}\} \subseteq K_n \times P_1$ in $(K_n \times P_1) \odot K_1$. The representation of vertices of graph $(K_n \times P_1) \odot K_1$ by B are

$r(v_{n1}|B) = (1, 1, \cdots, 1)$,

$r(u_{n1}|B) = (2, 2, \cdots, 2)$,

$r(u_{i1}|B) = (\cdots, 2, 1, 2, \cdots)$, for $1 \le i \le n-1$.

All of those representations are distinct. Therefore, $\dim(K_n \times P_1) \odot K_1) = n-1$.

Case 2. $m = 2$. By a contradiction, we will show that $\dim((K_n \times P_2) \odot K_1) \geq n$. Assume that B is a basis of $(K_n \times P_2) \odot K_1$, with $|B| < n$. If $B \subseteq \{v_{11}, v_{21}, \cdots, v_{n1}\}$ or $B \subseteq \{v_{12}, v_{22}, \cdots, v_{n2}\}$, let B be the previous one, then there exist $k \in \{1, 2, \cdots, n\}$ such that $r(u_{k1}|B) = \{2, 2, \cdots, 2\} = r(v_{k2}|B)$, a contradiction. Otherwise, there exist $k, l \in \{1, 2, \cdots, n\}$ such that $r(u_{kj}|B) = r(u_{lj}|B)$, for $1 \leq j \leq 2$, a contradiction. We will show that $\dim((K_n \times P_2) \odot K_1) \leq n$ by choosing a resolving set $B = \{u_{11}, u_{21}, \cdots, u_{n1}\}$. The representation of the vertices of $(K_n \times P_2) \odot K_1$ by B are

$$r(v_{i1}|B) = \{\cdots, 2, 1, 2, \cdots\},$$

v_{i1} is adjacent with u_{i1} and have distance 2 with the others vertex in B,

$$r(v_{i2}|B) = \{\cdots, 3, 2, 3, \cdots\},$$

$$r(u_{i2}|B) = \{\cdots, 4, 3, 4, \cdots\},$$

It makes all representations of vertices in $(K_n \times P_2) \odot K_1$ are distinct. ■

Open problem 2 *Find the dimension of $(K_n \times P_m) \odot K_1$, for $n \geq 3$ and $m \geq 3$.*

Next, we will use the notion of "distance similar" introduced by Saenpholphat and Zhang in [9] to determine the dimension of corona graph $G \odot \overline{K}_m$, for any graph G and $m \geq 2$. Two vertices u and v of a connected graph G are said to be *distance similar* if $d(u, x) = d(v, x)$ for all $x \in V(G) - \{u, v\}$. Some of their properties can be found in the following observations.

Observation 1 [9] *Two vertices u and v of a connected graph G are distance similar if and only if (1) $uv \notin E(G)$ and $N(u) = N(v)$ or (2) $uv \in E(G)$ and $N[u] = N[v]$.*

Observation 2 [9] *Distance similarity in a connected graph G is an equivalence relation on $V(G)$.*

Observation 3 [9] *If U is a distance similar equivalence class of a connected graph G, then U is either independent in G or in $\overline{G}$.*

Observation 4 [9] *If U is a distance similar equivalence class in a connected graph G with $|U| = p \geq 2$, then every resolving set of G contains at least $p - 1$ vertices from U.*

Corollary 1. *If $G \odot \overline{K}_m$, with $|G| = n$ and $m \geq 2$, $\dim(G \odot \overline{K}_m) = n(m-1)$.*

Proof Let d_{ij} be the distance between two vertices v_i and v_j in $G \odot \overline{K}_m$. For every $i \in \{1, 2, \cdots, n\}$, every pair vertices $u, v \in (\overline{K}_m)_i$ holds $d(u, x) = d(v, x)$ for all $x \in V(G \odot \overline{K}_m) - \{u, v\}$. Further, $(\overline{K}_m)_i$ is a distance similar equivalence class of $G \odot \overline{K}_m$. By using Observation 4, we have $\dim(G \odot \overline{K}_m) \geq n(m-1)$. Next, we will show that $\dim(G) \leq n(m-1)$. Let $B = \{B_1, B_2, \cdots, B_n\}$, where B_i is a basis of $K_1 \odot (\overline{K}_m)_i$. Without loss of generality, let $B_i = \{u_{1i}, u_{2i}, \cdots, u_{(m-1)i}\}$, for every $i \in \{1, 2, \cdots, n\}$. The representation of another vertices in $G \odot \overline{K}_m$ are

$$r(u_{mi}|B) = (\cdots, \underbrace{2, 2, \cdots, 2}_{coord.\ u_{mi}\ by\ B_i}, \cdots),$$

$$r(v_i|B) = (\cdots, \underbrace{1, 1, \cdots, 1}_{coord.\ v_i\ by\ B_i}, \cdots).$$

It implies the representation of every vertex v in G by B is unique. Then B is a resolving set. So, $\dim(G) \leq n(m-1)$. ■

For corona product $G \odot H$, if $G \cong K_1$ and $H \cong \overline{K}_m$, $G \odot H$ is equal to star $\text{Star}(m) = K_1 + \overline{K}_m$. If we apply Corollary 1 for $\text{Star}(m)$ then we have $\dim(\text{Star}(m)) = m - 1$. This is the same result if we apply Theorem A (iv) or Theorem A (v) for $\text{Star}(m)$.

Now, we will determine of a graph with pendant edges more general than corona graphs $G \odot \overline{K}_m$. Let G is a connected graph with order n. Let every vertex v_i of G is joining with m_i number of pendant edges, $m_i \geq 2$ and $1 \leq i \leq n$.

Corollary 2. *For $n \geq 2$,*

$$dim(G) = \sum_{i=1}^{n}(m_i - 1).$$

Proof The proof is similar to the one of Corollary 1. ■

Open problem 3 *Find the dimension of $G \odot \overline{K}_m$, with $|G| = n$ and $m \geq 1$.*

References

1. P.S. Buczkowski, G. Chartrand, C. Poisson, and P. Zhang, On k-dimensional graphs and their bases, *Period. Math. Hungar.*, **46:1** (2003), 9-15.
2. J. Caceres, C. Hernando, M. Mora, M.L. Puertas, I.M. Pelayo, C. Seara, and D. R. Wood, On the metric dimension of cartesian products of graphs, *Discrete Math.*, submitted.
3. J. Caceres, C. Hernando, M. Mora, M.L. Puertas, I.M. Pelayo, and C. Seara, On the metric dimension of some families of graphs, *Electronic Notes in Discrete Math.*, **22** (2005), 129-133.

4. G.G. Chappell, J. Gimbel, and C. Hartman, Bounds on the metric and partition dimensions of a graph, preprint.
5. G. Chartrand, L. Eroh, M.A. Johnson, and O.R. Oellermann, Resolvability in graphs and the metric dimension of a graph, *Discrete Appl. Math.*, **105** (2000), 99 - 113.
6. G. Chartrand, and L. Lesniak, *Graphs and Digraphs*, 3rd ed., Chapman and Hall/CRC, 2000.
7. M. Fehr, S. Gosselin, and O.R. Oellermann, The metric dimension of Cayley digraphs, *Discrete Math.*, **306** (2006), 31-41.
8. F. Harary and R.A. Melter, On the metric dimension of a graph, *Ars. Combin.*, **2** (1976), 191-195.
9. V. Saenpholphat and P. Zhang, Some results on connected resolvability in graphs, *Congr. Numer.*, **158** (2002), 5-19.
10. P.J. Slater, Leaves of trees, *Congr. Numer.*, **14** (1975), 549-559.
11. P.J. Slater, Dominating and reference sets in graphs, *J. Math. Phys. Sci.*, **22** (1988), 445-455.