New constructions of A-magic graphs using labeling matrices

Kiki A. Sugeng[1] and Mirka Miller[2]

[1] Department of Mathematics
University of Indonesia
Depok 16424, Indonesia
kikiariyanti@yahoo.com

[2] School of Information Technology and Mathematical Sciences
University of Ballarat, VIC 3353, Australia
m.miller@ballarat.edu.au

Abstract. A simple graph $G(V, E)$ is called A-magic if there is a labeling $f : E \rightarrow A^*$, where A is an Abelian group and $A^* = A - \{0\}$ so that the induced vertex labeling $f^* : V \rightarrow A$, defined as $f^*(v) = \Sigma_{u \in N(v)} f(uv) = k$, for every $v \in V$, k is a constant in A. In this paper we show constructions of new classes of A-magic graphs from known A-magic graphs using labeling matrices.

1 Introduction

Let $G = (V, E)$ be a simple graph. For an arbitrary Abelian group A, a mapping $f : E \rightarrow A^*$, with $A^* = A - \{0\}$, is called a *labeling* of G. This labeling induces a mapping $f^* : V \rightarrow A$, defined as $f^*(v) = \Sigma_{u \in N(v)} f(uv)$, where $N(v) = \{u \in V | uv \in E\}$ is a neighborhood of $v \in V$. Note that $f^*(v)$ is also known as the *weight* of the vertex v. We call a labeling f *A-magic labeling* if for every $v \in V$, $f^*(v) = \Sigma_{u \in N(v)} f(uv) = k$, where k is a constant in A. The number k is called the *magic constant* of G. Using this definition, edge labels do not have to be the same. Magic labeling was introduced by Sedláček [1] in 1963. Sedláček's labeling uses real numbers for edge labels and every edge has a distinct label.

Many researchers have studied magic labeling and its variations. If $f(E) = \{1, 2, 3, \ldots, |E|\} \subset N$, and every edge has a distinct label, then we call f a *vertex magic labeling*, see [2, ?] for details. Another variation of magic labeling is obtained when we allow some edges to have the same label. Let V_4 be the group of the direct sum of $Z_2 \times Z_2$ with binary operation $\oplus$, which can be represented by its Cayley Table, as in Table 1. Lee *et al.* [3] studied the properties of V_4-magic graphs and constructed V_4-magic labelings on several classes of graphs, such as stars, cycles, flowers and bipyramids.

⊕	(0,0)	(0,1)	(1,0)	(1,1)
(0,0)	(0,0)	(0,1)	(1,0)	(1,1)
(0,1)	(0,1)	(0,0)	(1,1)	(1,0)
(1,0)	(1,0)	(1,1)	(0,0)	(0,1)
(1,1)	(1,1)	(1,0)	(0,1)	(0,0)

Table 1. Cayley table of group V_4

Shie and Low [6], using the same principle as in [3], studied Z_3-magic labeling for a complete n-partite graph. Low and Lee [4] explored a group magic labeling of Eulerian graphs. In another paper, Low and Lee [5] proved that some products of several A-magic graphs can be group magic graphs. In that paper they showed that $G_1 \times G_2$, lexicographic product $G_1 \bullet G_2$ and $G_2 \bullet G_1$, tensor product $G_1 \otimes G_2$ (with additional requirements) and normal product $G_1 \times G_2$ (with additional requirements) are A-magic graphs if both G_1 and G_2 are A-magic graphs.

Inspired by a labeling matrix used in [6, ?], we will give new constructions to generate new classes of A-magic graphs using labeling matrices.

2 Basic theory

Let $f : V \to A - \{0\}$ be an A-magic labeling of G, with A an arbitrary Abelian group. Let $\{v_1, \ldots, v_n\}$ be the set of the vertices of G. A *labeling matrix* of f, denoted by $A_f(G) = (a_{ij})$, is a matrix in which each row and each column represents the vertices of G and the entry ij is a label of the edge v_iv_j. Thus

$$a_{ij} = \begin{cases} f(v_iv_j) & \text{if } v_iv_j \text{ is an edge of } G, \\ 0 & \text{if } v_iv_j \text{ is not an edge of } G. \end{cases}$$

Figure 1 shows Z_3-magic labelings for cycle C_4 and complete graph K_4. Graph C_4 in Fig. 1 is a Z_3-magic graph with magic constant 0, and graph K_4 is a Z_3-magic graph with magic constant 1. The labeling matrix of C_4 is

$$A_f = \begin{pmatrix} 0 & 1 & 0 & 2 \\ 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & 1 \\ 2 & 0 & 1 & 0 \end{pmatrix},$$

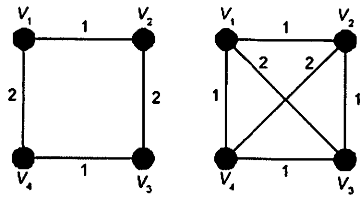

Fig. 1. Z_3-magic labeling of C_4 and K_4.

while the labeling matrix of K_4 is

$$B_f = \begin{pmatrix} 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \end{pmatrix}.$$

3 Construction of new A-magic graphs

Let G_1 and G_2 be A-magic graphs with labeling matrices A_f and B_g, respectively. By creating a bigger labeling matrix using A_f, B_g, diagonal matrices and/or their combination we obtain new matrix that represents new classes of A-magic graphs.

Theorem 6 *Let G_1 and G_2 be A-magic graphs with labeling matrices A_f and B_g, respectively. If G_1 and G_2 have the same magic constant k then $G_1 \cup G_2$ is an A-magic graph.*

Proof: Let G_1 and G_2 be A-magic graphs with labeling matrices A_f and B_g, respectively, with magic constant k. Define matrix C as

$$C = \begin{pmatrix} A_f & 0 \\ 0 & B_g \end{pmatrix}.$$

Define a new labeling L on $G_1 \cup G_2$ as follows.

$$L(xy) = \begin{cases} f(xy) & \text{if } xy \in E(G_1), \\ g(xy) & \text{if } xy \in E(G_2). \end{cases}$$

Then C represents a labeling matrix of L. Since the sums of all entries in each row/column are the same (every vertex in $G_1 \cup G_2$ has the same weight, equal to the magic constant k), then the induced labeling $L^*(v) = \Sigma_{y \in N(v)} L(xy)$ is a constant. Hence L is an A-magic labeling for $G_1 \cup G_2$.
□

Note that if we have $G_i = G$ for $i = 1, 2, ..., m$ then we can have the following result by doing the process in the proof of the above Theorem repeatedly m times.

Corollary 2 *Let G be an A-magic graph with labeling matrix A_f. Then a matching graph mG is also an A-magic graph, for every positive integer $m > 1$.*

Theorem 7 *Let G_1 and G_2 be A-magic graphs with labeling matrix A_f and B_g, respectively. If G_1 and G_2 have the same number of vertices and have disjoint edges then the composition of G_1 and G_2 is an A-magic graph.*

Proof. Let G_1 and G_2 be A-magic graphs with magic constants k and h , respectively. Since G_1 and G_2 have the same number of vertices but disjoint edges, we can give the same label for vertices in G_1 and G_2. Let $\{v_1, ..., v_n\}$ be the set of vertices of both G_1 and G_2. Define a new graph G such that $\{v_1, ..., v_n\}$ is the set of vertices of G and $E(G) = E(G_1) \cup E(G_2)$. Every vertex v_i now has incidence edges from $E(G_1$ and $E(G_2$. Thus G is a composition of G_1 and G_2. Define a new labeling L on $E(G)$ as

$$L(v_iv_j) = \begin{cases} f(v_iv_j) & \text{if } v_iv_j \in E(G_1), \\ g(v_iv_j) & \text{if } v_iv_j \in E(G_2). \end{cases}$$

Define a matrix $C = (c_{ij})$ as

$$c_{ij} = \begin{cases} f(v_iv_j) & \text{if } v_iv_j \in E(G_1), \\ g(xy) & \text{if } v_iv_j \in E(G_2), \\ 0 & \quad \text{for others.} \end{cases}$$

Since G_1 and G_2 are A-magic graphs, then the weight of a vertex v_i in G_1 is $f^*(v_i) = \Sigma_{v_j \in N(v_i)} f(v_iv_j) = k$ since $v_i \in V(G_1)$ and the weight of a vertex v_i in G_2 is $g^*(v_i) = \Sigma_{v_j \in N(v_i)} g(v_iv_j) = h$ since $v_i \in V(G_2)$. If we consider v_i as a vertex of G then the weight of $v_i \in V(G)$ is $L^*(v_i) = f^*(v_i) + g^*(v_i) = k + h$, for every $v_i \in V(G)$. As a consequence, G is an A-magic graphs. □

Let G_1 and G_2 be A-magic graphs of order n and m, respectively. Let A_f and B_g be labeling matrices of G_1 and G_2 respectively. Let P be a rectangular matrix of order $n \times m$, where all entry of P are elements of A and the sum of all elements of each row and each column of P are the same. Note that P does not have to be a rectangular magic, since some of the elements can be duplicated. Construct a labeling matrix

$$C = \begin{pmatrix} A_f & P \\ P & B_g \end{pmatrix}.$$

Let $G^* = G_1 * G_2$ be a graph that has C as its adjacency matrix.
Then we have

Theorem 8 *Let G_1 and G_2 be A-magic graphs with labeling matrix A_f and B_g, respectively. If G_1 and G_2 have the same magic constant k then the graph $G^* = G_1 * G_2$, as defined above, is an A-magic graph.*

Proof. Let G_1 and G_2 be A-magic graphs with labeling matrices A_f and B_g, respectively, with magic constant k.

Define a new labeling L on $E(G^*)$ following the adjacency matrix C.
Then C represents a labeling matrix of a labeling L on G^*. Suppose that the sum of each row/column of G^* is $a \in A$. Then the sum of all the entries in each row is equal to $k + a$, that is, every vertex in $G_1 \cup G_2$ has the same weight, equal to the magic constant $k + a$. Then L is an A-magic labeling for G^*. □

4 Conclusion

In this paper we show three different constructions to generate new classes of graphs using combinations of labeling matrices of known A-magic graphs. There are still more combinations of labelings that could be obtained using labeling matrices.

References

1. J. Sedláček, *Problem 27*, in Theory of Graphs and Its Applications, *Proc. Symposium Smolenice*, June (1963) 163-167.
2. J. Gallian, A dynamic survey of graph labeling, *The Electronic Journal of Combinatorics* **9** (2005) .
3. S-M Lee, F. Saba, E. Salehi and H. Sun, On the V_4-magic graph, *Congressus Numerantium* **156** (2002) 59-67.
4. R.M. Low and S-M Lee, On group magic eulerian graph, *Journal of Combinatorial Mathematics and Combinatorial Computing* **50** (2004) 141-148.
5. R.M. Low and S-M Lee, On the product of group magic graphs, *Australasian Journal of Combinatorics* **34** (2006) 41-48.
6. W.C. Shie and R.M. Low, Group magicness of complete n-partite graphs, preprint.
7. K.A. Sugeng and M. Miller, Relationships between adjacency matrices and super (a, d)-edge-antimagic total graphs, *J. Combin. Math. Combin. Comput.***55**(2005) 71-82.
8. W.D. Wallis, *Magic Graphs*, Birkhäuser, 2001.