The total edge-irregular strengths of the corona product of paths with some graphs

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Abstract. For a simple graph $G = (V(G), E(G))$ with the vertex set $V(G)$ and the edge set $E(G)$, a labelling $\lambda : V(G) \cup E(G) \longrightarrow \{1, 2, ..., k\}$ is called an edge-irregular total k-labelling of G if for any two different edges $e = e_1e_2$ and $f = f_1f_2$ in $E(G)$ we have $wt(e) \neq wt(f)$ where $wt(e) = \lambda(e_1) + \lambda(e) + \lambda(e_2)$. The total edge-irregular strength, denoted by $tes(G)$, is the smallest positive integer k for which G has an edge-irregular total $k-$labelling. In this paper, we determine the total edge-irregular strength of the corona product of paths with some graphs.

Keywords: corona product, cycle, friendship, gear, path, star, total edge-irregular strength, wheel

1 Introduction

All the graphs that we deal with are undirected, simple, and connected. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. The path and the cycle on m vertices are denoted by P_m and C_m, respectively. The star and the wheel on $m + 1$ vertices are denoted by S_m and W_m, respectively. The *gear* G_n is a graph obtained from W_m by subdividing every edge on the cycle. The *friendship* F_m is a graph obtained from W_{2m} by missing every alternate rim edge.

A *labelling* of graph is a map that carries graph elements to the numbers (usually to the positive or non-negative integers). A labelling of graph is called a *vertex labelling*, an *edge labelling*, or a *total labelling*, if the domain

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of the map is the vertex set, the edge set, or the union of vertex and edge sets, respectively.

In [4], Chartrand et al. introduced the notion of edge irregular labelling. For a graph $G = (V(G), E(G))$, the *weight* of a vertex x under an edge labelling $\lambda : E(G) \to \{1, 2, ..., k\}$ is

$$wt(x) = \sum_{xy \in E(G)} \lambda(xy).$$

The edge labelling $\lambda : E(G) \to \{1, 2, ..., k\}$ is called an *edge-irregular k-labelling* of G if every two distinct vertices x and y in $V(G)$ satisfies $wt(x) \neq wt(y)$.

The *irregular strength* $s(G)$ of G is the minimum of the positive integer k for which graph G has an edge-irregular k-labelling.

Some results of the irregular strengths of some graphs can be seen in [2], [3], [5], [6], [7], [8], and [11].

In this paper, we consider a total labelling. For a graph $G = (V(G), E(G))$, the *weight* of an edge $e = e_1e_2$ under a total labelling λ is

$$wt(e) = \lambda(e_1) + \lambda(e) + \lambda(e_2).$$

A total labelling $\lambda : V \cup E \to \{1,\ 2, ..., k\}$ is called an *edge-irregular total k-labelling* of G if every two distinct edges e and f in $E(G)$ satisfies $wt(e) \neq wt(f)$.

The *total edge-irregular strength* $tes(G)$ of G is the minimum of the positive integer k for which G has an edge-irregular total k-labelling.

The notion of the total edge-irregular strength was introduced by Bača et al. [1]. In the same paper, they derive lower and upper bounds of the total edge-irregular strength of any graph G as described in Theorem 1.

Theorem 1. *[1]*
Let $G = (V(G), E(G))$ be a graph with the vertex set $V(G)$ and edge set $E(G)$. Then,

$$\left\lceil \frac{|E(G)| + 2}{3} \right\rceil \leq tes(G) \leq |E(G)|.$$

There are not many graphs of which their total edge-irregular strengths are known. Bača et al. [1] have determined the total edge-irregular strengths for some classes of graphs, namely paths, cycles, stars, wheels, and friendships. In [9] and [10], we determined the total edge-irregular strengths of union graphs of $K_{2,n}$ and lintang graph.

2 Main Results

In this paper, we determine the total edge-irregular strengths of graphs obtained by the corona product of paths with either paths, cycles, stars, gears, friendships, or wheels. The *corona product* of a graph G with a graph H, denoted by $G \odot H$, is a graph obtained by taking one copy of a n-vertex graph G and n copies H_1, H_2, ..., H_n of H, and then joining the i-th vertex of G to every vertex in H_i.

Theorem 2. *For any integer $m, n \geq 2$,*

$$tes(P_m \odot P_n) = \left\lceil \frac{2mn+1}{3} \right\rceil.$$

Proof. Since $|E(P_m \odot P_n)| = 2mn - 1$, by Theorem 1, we obtain that $tes(P_m \odot P_n) \geq \lceil \frac{2mn+1}{3} \rceil$. Next, we will show that $tes(P_m \odot P_n) \leq \lceil \frac{2mn+1}{3} \rceil$. Let the vertex set of $P_m \odot P_n$ be

$$V(P_m \odot P_n) = \{x_i, y_i^j | 1 \leq i \leq m \text{ and } 1 \leq j \leq n\},$$

and the edge set of $P_m \odot P_n$ be

$$\begin{aligned} E(P_m \odot P_n) =& \{x_i y_i^j | 1 \leq i \leq m \text{ and } 1 \leq j \leq n\} \\ & \cup \{y_i^j y_i^{j+1} | 1 \leq i \leq m \text{ and } 1 \leq j \leq n-1\} \\ & \cup \{x_i x_{i+1} | 1 \leq i \leq m-1\}. \end{aligned}$$

Define $r_i = \lceil \frac{2ni+1}{3} \rceil$ for $2 \leq i \leq m$.
We construct an edge-irregular total r_m−labelling λ as follows:

$$\lambda(x_i) = \begin{cases} 1 & \text{for } i = 1 \\ r_i & \text{for } 2 \leq i \leq m, \end{cases}$$

$$\lambda(y_i^j) = \begin{cases} j & \text{for } i = 1 \text{ and } 1 \leq j \leq n \\ r_i + j - n & \text{for } 2 \leq i \leq m \text{ and } 1 \leq j \leq n, \end{cases}$$

$$\lambda(x_i y_i^j) = \begin{cases} j & \text{for } i = 1 \text{ and } 1 \leq j \leq n \\ n(2i-1) + 1 + j - 2r_i & \text{for } 2 \leq i \leq m \text{ and } 1 \leq j \leq n, \end{cases}$$

$$\lambda(y_i^j y_i^{j+1}) = \begin{cases} 1 & \text{for } i = 1 \text{ and } 1 \leq j \leq n-1 \\ 2ni + 1 - 2r_i & \text{for } 2 \leq i \leq m \text{ and } 1 \leq j \leq n-1, \end{cases}$$

$$\lambda(x_i x_{i+1}) = \begin{cases} 2n+1-r_2 & \text{for } i=1 \\ 2ni+2-r_i-r_{i+1} & \text{for } 2 \le i \le m-1. \end{cases}$$

It is easy to see from the definition that the function λ is a map from $V(P_m \odot P_n) \cup E(P_m \odot P_n)$ into $\{1, 2, ..., \lceil \frac{2mn+1}{3} \rceil\}$.
Observe that

$$\begin{aligned} wt(x_i y_i^j) &= \lambda(x_i) + \lambda(x_i y_i^j) + \lambda(y_i^j) \\ &= 2n(i-1) + 2j + 1 \quad \text{for } 1 \le i \le m \text{ and } 1 \le j \le n, \end{aligned}$$

$$\begin{aligned} wt(y_i^j y_i^{j+1}) &= \lambda(y_i^j) + \lambda(y_i^j y_i^{j+1}) + \lambda(y_i^{j+1}) \\ &= 2n(i-1) + 2(j+1) \quad \text{for } 1 \le i \le m \text{ and } 1 \le j \le n-1, \end{aligned}$$

and

$$\begin{aligned} wt(x_i x_{i+1}) &= \lambda(x_i) + \lambda(x_i x_{i+1}) + \lambda(x_{i+1}) \\ &= 2(ni+1) \quad \text{for } 1 \le i \le m-1. \end{aligned}$$

So the weights of edges of $P_m \odot P_n$ under the total labelling λ form consecutive integers from 3 up to $2mn+1$. It means that the there is no two edges that have the same weight.

This implies that $tes(P_m \odot P_n) \le \lceil \frac{2mn+1}{3} \rceil$. □

Theorem 3. *For any integer $m \ge 2$ and $n \ge 3$,*

$$tes(P_m \odot C_n) = \left\lceil \frac{(2n+1)m+1}{3} \right\rceil.$$

Proof. Since $|E(P_m \odot C_n)| = (2n+1)m - 1$, by Theorem 1, we obtain that $tes(P_m \odot C_n) \ge \left\lceil \frac{(2n+1)m+1}{3} \right\rceil$. Next, we will show that $tes(P_m \odot C_n) \le \left\lceil \frac{(2n+1)m+1}{3} \right\rceil$.
Let the vertex set of $P_m \odot C_n$ be

$$V(P_m \odot C_n) = \{x_i, y_i^j | 1 \le i \le m \text{ and } 1 \le j \le n\},$$

and the edge set of $P_m \odot C_n$ be

$$\begin{aligned} E(P_m \odot C_n) = &\{x_i x_{i+1} | 1 \le i \le m-1\} \\ &\cup \{x_i y_i^j | 1 \le i \le m \text{ and } 1 \le j \le n\} \\ &\cup \{y_i^1 y_i^n, y_i^j y_i^{j+1} | 1 \le i \le m \text{ and } 1 \le j \le n-1\}. \end{aligned}$$

Define $r_i = \left\lceil \frac{(2n+1)i+1}{3} \right\rceil$ for $2 \leq i \leq m$.
We construct an edge-irregular total r_m−labelling λ as follows:

$$\lambda(x_i) = \begin{cases} 1 & \text{for } i = 1 \\ r_i + 1 - n & \text{for } 2 \leq i \leq m, \end{cases}$$

$$\lambda(y_i^j) = \begin{cases} j & \text{for } i = 1 \text{ and } 1 \leq j \leq n \\ r_i & \text{for } 2 \leq i \leq m \text{ and } 1 \leq j \leq n, \end{cases}$$

$$\lambda(x_i y_i^j) = \begin{cases} 1 & \text{for } i = 1 \text{ and } 1 \leq j \leq n \\ (2n+1)i - n + j - 2r_i & \text{for } 2 \leq i \leq m \text{ and } 1 \leq j \leq n, \end{cases}$$

$$\lambda(y_i^1 y_i^n) = \begin{cases} 2 & \text{for } i = 1 \\ (2n+1)i + 2 - n - 2r_i & \text{for } 2 \leq i \leq m, \end{cases}$$

$$\lambda(y_i^j y_i^{j+1}) = \begin{cases} n + 2 - j & \text{for } i = 1 \text{ and } 1 \leq j \leq n-1 \\ (2n+1)i + 2 - n + j - 2r_i & \text{for } 2 \leq i \leq m \text{ and } 1 \leq j \leq n-1, \end{cases}$$

$$\lambda(x_i x_{i+1}) = \begin{cases} 3n + 1 - r_2 & \text{for } i = 1 \\ (2n+1)i + 2n - r_i - r_{i+1} & \text{for } 2 \leq i \leq m-1. \end{cases}$$

It is easy to see from the definition that the function λ is a map from $V(P_m \odot C_n) \cup E(P_m \odot C_n)$ into $\left\{1,\ 2,\ 3,\ ...,\ \left\lceil \frac{m(2n+1)+1}{3} \right\rceil \right\}$.
Observe that

$$\begin{aligned} wt(x_i y_i^j) &= \lambda(x_i) + \lambda(x_i y_i^j) + \lambda(y_i^j) \\ &= (2n+1)(i-1) + 2 + j \quad \text{for } 1 \leq i \leq m \text{ and } 1 \leq j \leq n, \end{aligned}$$

$$\begin{aligned} wt(y_i^1 y_i^n) &= \lambda(y_i^n) + \lambda(y_i^n y_i^1) + \lambda(y_i^1) \\ &= (2n+1)i - n + 2 \quad \text{for } 1 \leq i \leq m, \end{aligned}$$

$$\begin{aligned} wt(y_i^j y_i^{j+1}) &= \lambda(y_i^j) + \lambda(y_i^j y_i^{j+1}) + \lambda(y_i^{j+1}) \\ &= (2n+1)i + 2 - n + j \quad \text{for } 1 \leq i \leq m \text{ and } 1 \leq j \leq n-1, \end{aligned}$$

$$\begin{aligned} wt(x_i x_{i+1}) &= \lambda(x_i) + \lambda(x_i x_{i+1}) + \lambda(x_{i+1}) \\ &= (2n+1)i + 2 \quad \text{for } 1 \leq i \leq m-1. \end{aligned}$$

So the weights of edges of $P_m \odot C_n$ under the total labelling λ form consecutive integers from 3 up to $m(2n+1)+1$. It means that the weight of edges are distinct.

This concludes that $tes(P_m \odot C_n) \leq \left\lceil \frac{m(2n+1)+1)}{3} \right\rceil$. □

Theorem 4. *For any integer $m \geq 2$ and $n \geq 3$,*

$$tes(P_m \odot S_n) = \left\lceil \frac{(2m(n+1)+1)}{3} \right\rceil.$$

Proof. Since $|E(P_m \odot S_n)| = 2m(n+1) - 1$, by Theorem 1, we have $tes(P_m \odot S_n) \geq \left\lceil \frac{2m(n+1)+1}{3} \right\rceil$. Next, we will show that $tes(P_m \odot S_n) \leq \left\lceil \frac{2m(n+1)+1}{3} \right\rceil$.

Let the vertex set of $P_m \odot S_n$ be

$$V(P_m \odot S_n) = \{x_i, y_i^j | 1 \leq i \leq m \text{ and } 1 \leq j \leq n+1\},$$

and the edge set of $P_m \odot S_n$ be

$$\begin{aligned} E(P_m \odot S_n) =& \{y_i^j y_i^{n+1} | 1 \leq i \leq m \text{ and } 1 \leq j \leq n\} \\ & \cup \{x_i y_i^j | 1 \leq i \leq m \text{ and } 1 \leq j \leq n+1\} \\ & \cup \{x_i x_{i+1} | 1 \leq i \leq m-1\}. \end{aligned}$$

From the definition of $V(P_m \odot S_n)$, we conclude that y_i^{n+1} is the center vertex of the $i-$th stars.

Define $r_i = \left\lceil \frac{2i(n+1)+1}{3} \right\rceil$ for $2 \leq i \leq m$.

We construct an edge-irregular total r_m-labelling λ as follows:

$$\lambda(x_i) = \begin{cases} 1 & \text{for } i = 1 \\ r_i & \text{for } 2 \leq i \leq m, \end{cases}$$

$$\lambda(y_i^j) = \begin{cases} j & \text{for } i = 1 \text{ and } 1 \leq j \leq n \\ n & \text{for } i = 1 \text{ and } j = n+1 \\ r_i - n + j & \text{for } 2 \leq i \leq m \text{ and } 1 \leq j \leq n \\ r_i & \text{for } 2 \leq i \leq m \text{ and } j = n+1, \end{cases}$$

$$\lambda(x_i y_i^j) = \begin{cases} 1 & \text{for } i = 1 \text{ and } 1 \leq j \leq n \\ 2 & \text{for } i = 1 \text{ and } j = n+1 \\ 2i(n+1) - n - 2r_i & \text{for } 2 \leq i \leq m \text{ and } 1 \leq j \leq n \\ 2i(n+1) + 1 - n - 2r_i & \text{for } 2 \leq i \leq m \text{ and } j = n+1, \end{cases}$$

$$\lambda(y_i^j y_i^{n+1}) = \begin{cases} 3 & \text{for } i = 1 \text{ and } 1 \leq j \leq n \\ 2i(n+1) + 1 - 2r_i & \text{for } 2 \leq i \leq m \text{ and } 1 \leq j \leq n, \end{cases}$$

$$\lambda(x_i x_{i+1}) = \begin{cases} 2n+3-r_2 & \text{for } i=1 \\ 2i(n+1)+2-r_i-r_{i+1} & \text{for } 2 \le i \le m-1. \end{cases}$$

It is easy to see from the definition that the function λ is a map from $V(P_m \odot S_n) \cup E(P_m \odot S_n)$ into $\left\{1,\ 2,\ \ldots,\ \left\lceil \frac{2m(n+1)+1}{3} \right\rceil\right\}$.
Observe that

$$\begin{aligned} wt(x_i y_i^j) &= \lambda(x_i) + \lambda(x_i y_i^j) + \lambda(y_i^j) \\ &= 2(n+1)i - 2n + j \quad \text{for } 1 \le i \le m \text{ and } 1 \le j \le n+1, \end{aligned}$$

$$\begin{aligned} wt(y_i^j y_i^{n+1}) &= \lambda(y_i^j) + \lambda(y_i^j y_i^{n+1}) + \lambda(y_i^{n+1}) \\ &= 2i(n+1) + 1 - n + j \quad \text{for } 1 \le i \le m \text{ and } 1 \le j \le n, \end{aligned}$$

and

$$\begin{aligned} wt(x_i x_{i+1}) &= \lambda(x_i) + \lambda(x_i x_{i+1}) + \lambda(x_{i+1}) \\ &= 2i(n+1) + 2 \quad \text{for } 1 \le i \le m-1. \end{aligned}$$

So the weights of edges of $P_m \odot S_n$ under the total labelling λ form consecutive integers from 3 up to $2m(n+1)+1$. It means that the weight of edges are distinct.

This implies that $tes(P_m \odot S_n) \le \left\lceil \frac{2m(n+1)+1}{3} \right\rceil$. □

Theorem 5. *For any integer $m \ge 2$ and $n \ge 2$,*

$$tes(P_m \odot G_n) = \left\lceil \frac{m(5n+2)+1}{3} \right\rceil.$$

Proof. Since $|E(P_m \odot G_n)| = m(5n+2) - 1$, by Theorem 1, we obtain that $tes(P_m \odot G_n) \ge \lceil \frac{m(5n+2)+1}{3} \rceil$. Next, we will show that $tes(P_m \odot G_n) \le \lceil \frac{m(5n+2)+1}{3} \rceil$.
Denote

$$V(P_m \odot G_n) = \{x_i, y_i^j | 1 \le i \le m \text{ and } 1 \le j \le 2n+1\},$$

as the vertex set of $P_m \odot G_n$ and

$$\begin{aligned} E(P_m \odot G_n) =& \{x_i y_i^j | 1 \le i \le m \text{ and } 1 \le j \le 2n+1\} \\ & \cup \{y_i^j y_i^{j+1}, y_i^1 y_i^{2n} | 1 \le i \le m \text{ and } 1 \le j \le 2n-1\} \\ & \cup \{y_i^{2l-1} y_i^{2n+1} | 1 \le i \le m \text{ and } 1 \le l \le n\} \\ & \cup \{x_i x_{i+1} | 1 \le i \le m-1\}. \end{aligned}$$

as the edge set of $P_m \odot G_n$. From the definition of $V(P_m \odot G_n)$, we conclude that y_i^{2n+1} is the center vertex of the $i-$th gears.
Define $r_i = \left\lceil \frac{i(5n+2)+1}{3} \right\rceil$ for $1 \leq i \leq m$.
We construct an edge-irregular total r_m-labelling λ as follows:

$$\lambda(x_i) = \begin{cases} 1 & \text{for } i = 1 \\ 2n & \text{for } i = 2 \\ r_i & \text{for } 3 \leq i \leq m, \end{cases}$$

$$\lambda(y_i^j) = \begin{cases} j & \text{for } i = 1 \text{ and } 1 \leq j \leq 2n+1 \\ r_i & \text{for } 2 \leq i \leq m \text{ and } 1 \leq j \leq 2n+1, \end{cases}$$

$$\lambda(x_i y_i^j) = \begin{cases} 1 & \text{for } i = 1 \text{ and } 1 \leq j \leq 2n+1 \\ 3n+4+j-r_2 & \text{for } i = 2 \text{ and } 1 \leq j \leq 2n+1 \\ (5n+2)i-5n+j-2r_i & \text{for } 3 \leq i \leq m \text{ and } 1 \leq j \leq 2n+1, \end{cases}$$

$$\lambda(y_i^1 y_i^{2n}) = \begin{cases} 3 & \text{for } i = 1 \\ (5n+2)i-3n+2-2r_i & \text{for } 2 \leq i \leq m, \end{cases}$$

$$\lambda(y_i^j y_i^{j+1}) = \begin{cases} 3+2n-j & \text{for } i = 1 \text{ and } 1 \leq j \leq 2n-1 \\ (5n+2)i-3n+2+j-2r_i & \text{for } 2 \leq i \leq m \\ & \text{and } 1 \leq j \leq 2n-1, \end{cases}$$

$$\lambda(y_i^{2l-1} y_i^{2n+1}) = \begin{cases} 2n+3-l & \text{for } i = 1 \text{ and } 1 \leq l \leq n \\ (5n+2)i-n+1+l-2r_i & \text{for } 2 \leq i \leq m \text{ and } 1 \leq l \leq n, \end{cases}$$

$$\lambda(x_i x_{i+1}) = \begin{cases} 3n+3 & \text{for } i = 1 \\ 8n+6-r_3 & \text{for } i = 2 \\ (5n+2)i+2-r_i-r_{i+1} & \text{for } 3 \leq i \leq m-1. \end{cases}$$

It is easy to see from the definition that the function λ is a map from $V(P_m \odot G_n) \cup E(P_m \odot G_n)$ into $\left\{1,\ 2,\ ...,\ \left\lceil \frac{m(5n+2)+1}{3} \right\rceil \right\}$.
Observe that

$$\begin{aligned} wt(x_i y_i^j) &= \lambda(x_i) + \lambda(x_i y_i^j) + \lambda(y_i^j) \\ &= (5n+2)i-5n+j \text{ for } 1 \leq i \leq m \text{ and } 1 \leq j \leq 2n+1, \end{aligned}$$

$$\begin{aligned} wt(y_i^1 y_i^{2n}) &= \lambda(y_i^1) + \lambda(y_i^1 y_i^{2n}) + \lambda(y_i^{2n}) \\ &= (5n+2)i-3n+2 \text{ for } 1 \leq i \leq m, \end{aligned}$$

$$wt(y_i^j y_i^{j+1}) = \lambda(y_i^j) + \lambda(y_i^j y_i^{j+1}) + \lambda(y_i^{j+1})$$
$$= (5n+2)i - 3n + 2 + j \text{ for } 1 \le i \le m \text{ and } 1 \le j \le 2n-1,$$

$$wt(y_i^{2l-1} y_i^{2n+1}) = \lambda(y_i^{2l-1}) + \lambda(y_i^{2l-1} y_i^{2n+1}) + \lambda(y_i^{2n+1})$$
$$= (5n+2)i - n + 1 + l \text{ for } 1 \le i \le m \text{ and } 1 \le l \le n,$$

and

$$wt(x_i x_{i+1}) = \lambda(x_i) + \lambda(x_i x_{i+1}) + \lambda(x_{i+1})$$
$$= (5n+2)i + 2 \quad \text{for } 1 \le i \le m-1.$$

So the weights of edges of $P_m \odot G_n$ under the total labelling λ form consecutive integers from 3 up to $m(5n+2)+1$. It means that there is no two edges that have the same weight.

It concludes that $tes(P_m \odot G_n) \le \lceil \frac{m(5n+2)+1}{3} \rceil$. □

Theorem 6. *For any integer $m \ge 2$ and $n \ge 2$,*

$$tes(P_m \odot F_n) = \left\lceil \frac{m(5n+2)+1}{3} \right\rceil.$$

Proof. Since $|E(P_m \odot F_n)| = m(5n+2) - 1$, by Theorem 1, we obtain that $tes(P_m \odot F_n) \ge \lceil \frac{m(5n+2)+1}{3} \rceil$. Next, we will show that $tes(P_m \odot F_n) \le \lceil \frac{m(5n+2)+1}{3} \rceil$.

Let the vertex set of $P_m \odot F_n$ be

$$V(P_m \odot F_n) = \{x_i, y_i^j | 1 \le i \le m \text{ and } 1 \le j \le 2n+1\},$$

and the edge set of $P_m \odot F_n$ be

$$E(P_m \odot F_n) = \{x_i y_i^j | 1 \le i \le m \text{ and } 1 \le j \le 2n+1\}$$
$$\cup \{y_i^{2l-1} y_i^{2l}, y_i^{2l-1} y_i^{2n+1}, y_i^{2l} y_i^{2n+1} | 1 \le i \le m \text{ and } 1 \le l \le n\}$$
$$\cup \{x_i x_{i+1} | 1 \le i \le m-1\}.$$

From the definition of $V(P_m \odot F_n)$, we conclude that y_i^{2n+1} is the center vertex of the $i-$th friendships.

Define $r_i = \lceil \frac{i(5n+2)+1}{3} \rceil$ for $1 \le i \le m$.

We construct an edge-irregular total r_m-labelling λ as follows:

$$\lambda(x_i) = \begin{cases} 1 & \text{for } i = 1 \\ 2n & \text{for } i = 2 \\ r_i & \text{for } 3 \le i \le m, \end{cases}$$

$$\lambda(y_i^j) = \begin{cases} j & \text{for } i = 1 \text{ and } 1 \le j \le 2n+1 \\ r_i & \text{for } 2 \le i \le m \text{ and } 1 \le j \le 2n+1, \end{cases}$$

$$\lambda(x_i y_i^j) = \begin{cases} 1 & \text{for } i = 1 \text{ and } 1 \le j \le 2n+1 \\ 3n+4-r_2+j & \text{for } i = 2 \text{ and } 1 \le j \le 2n+1 \\ (5n+2)i-5n+j-2r_i & \text{for } 3 \le i \le m \text{ and } 1 \le j \le 2n+1, \end{cases}$$

$$\lambda(y_i^{2l-1} y_i^{2l}) = \begin{cases} 2n+2-l & \text{for } i = 1 \text{ and } 1 \le l \le n \\ (5n+2)i-3n-1+3l-2r_i & \text{for } 2 \le i \le m \text{ and } 1 \le l \le n, \end{cases}$$

$$\lambda(y_i^{2l-1} y_i^{2n+1}) = \begin{cases} 2+l & \text{for } i = 1 \text{ and } 1 \le l \le n \\ (5n+2)i-3n+3l-2r_i & \text{for } 2 \le i \le m \text{ and } 1 \le l \le n, \end{cases}$$

$$\lambda(y_i^{2l} y_i^{2n+1}) = \begin{cases} 2+l & \text{for } i = 1 \text{ and } 1 \le l \le n \\ (5n+2)i-3n+1+3l-2r_i & \text{for } 2 \le i \le m \text{ and } 1 \le l \le n, \end{cases}$$

$$\lambda(x_i x_{i+1}) = \begin{cases} 3n+3 & \text{for } i = 1 \\ 8n+6-r_3 & \text{for } i = 2 \\ (5n+2)i+2-r_i-r_{i+1} & \text{for } 3 \le i \le m-1. \end{cases}$$

It is easy to see from the definition that the function λ is a map from $V(P_m \odot F_n) \cup E(P_m \odot F_n)$ into $\left\{1,\ 2,\ ...,\ \left\lceil \frac{m(5n+2)+1}{3} \right\rceil \right\}$.
Observe that

$$\begin{aligned} wt(x_i y_i^j) &= \lambda(x_i) + \lambda(x_i y_i^j) + \lambda(y_i^j) \\ &= (5n+2)i - 5n + j \quad \text{for } 1 \le i \le m \text{ and } 1 \le j \le 2n+1, \end{aligned}$$

$$\begin{aligned} wt(y_i^{2l-1} y_i^{2l}) &= \lambda(y_i^{2l-1}) + \lambda(y_i^{2l-1} y_i^{2l}) + \lambda(y_i^{2l}) \\ &= (5n+2)i - 3n - 1 + 3l \quad \text{for } 1 \le i \le m \text{ and } 1 \le l \le n, \end{aligned}$$

For any $i \in \{1,\ 2,\ ...,\ m\}$ and any $l \in \{1,\ 2,\ ...,\ n\}$, we obtain that

$$\begin{aligned} wt(y_i^{2l-1} y_i^{2n+1}) &= \lambda(y_i^{2l-1}) + \lambda(y_i^{2l-1} y_i^{2n+1}) + \lambda(y_i^{2n+1}) \\ &= (5n+2)i - 3n + 3l \quad \text{for } 1 \le i \le m \text{ and } 1 \le l \le n, \end{aligned}$$

$$\begin{aligned} wt(y_i^{2l} y_i^{2n+1}) &= \lambda(y_i^{2l}) + \lambda(y_i^{2l} y_i^{2n+1}) + \lambda(y_i^{2n+1}) \\ &= (5n+2)i - 3n + 1 + 3l \quad \text{for } 1 \le i \le m \text{ and } 1 \le l \le n, \end{aligned}$$

and

$$\begin{aligned} wt(x_i x_{i+1}) &= \lambda(x_i) + \lambda(x_i x_{i+1}) + \lambda(x_{i+1}) \\ &= (5n+2)i + 2 \quad \text{for } 1 \le i \le m-1. \end{aligned}$$

So the weights of edges of $P_m \odot F_n$ under the total labelling λ form consecutive integers from 3 up to $m(5n+2)+1$. It means that the weight of edges are distinct.

This implies that $tes(P_m \odot F_n) \le \lceil \frac{m(5n+2)+1}{3} \rceil$. □

Theorem 7. *For any integer $m \ge 2$ and $n \ge 3$,*

$$tes(P_m \odot W_n) = \left\lceil \frac{(3n+2)m+1}{3} \right\rceil.$$

Proof. Since $|E(P_m \odot W_n)| = (3n+2)m - 1$, by Theorem 1, we obtain $tes(P_m \odot W_n) \ge \lceil \frac{(3n+2)m+1}{3} \rceil$. Next, we will show that $tes(P_m \odot W_n) \le \lceil \frac{(3n+2)m+1}{3} \rceil$.

Let the vertex set of $P_m \odot W_n$ be

$$V(P_m \odot W_n) = \{x_i, y_i^j | 1 \le i \le m \text{ and } 1 \le j \le n+1\},$$

and the edge set of $P_m \odot W_n$ be

$$\begin{aligned} E(P_m \odot W_n) =& \{x_i y_i^j | 1 \le i \le m \text{ and } 1 \le j \le n+1\} \\ & \cup \{y_i^j y_i^{j+1}, y_i^1 y_i^n | 1 \le i \le m \text{ and } 1 \le j \le n-1\} \\ & \cup \{y_i^j y_i^{n+1} | 1 \le i \le m \text{ and } 1 \le j \le n\} \\ & \cup \{x_i x_{i+1} | 1 \le i \le m-1\}. \end{aligned}$$

From the definition of $V(P_m \odot W_n)$, we conclude that y_i^{n+1} is the center vertex of the i-th wheels.

Define $r_i = \left\lceil \frac{(3n+2)i+1}{3} \right\rceil$ for $2 \le i \le m$.

We construct an edge-irregular total $r_m$labelling λ as follows:

$$\lambda(x_i) = \begin{cases} 1 & \text{for } i = 1 \\ r_i - n + 1 & \text{for } 2 \le i \le m, \end{cases}$$

$$\lambda(y_i^j) = \begin{cases} j & \text{for } i = 1 \text{ and } 1 \le j \le n+1 \\ r_i - n + j & \text{for } 2 \le i \le m \text{ and } 1 \le j \le n \\ r_i & \text{for } 2 \le i \le m \text{ and } j = n+1, \end{cases}$$

$$\lambda(x_iy_i^j) = \begin{cases} 1 & \text{for } i=1 \text{ and } 1 \le j \le n+1 \\ (3n+2)i-(n+1)-2r_i & \text{for } 2 \le i \le m \text{ and } 1 \le j \le n \\ (3n+2)i-n-2r_i & \text{for } 2 \le i \le m \text{ and } j=n+1, \end{cases}$$

$$\lambda(y_i^1y_i^n) = \begin{cases} 3 & \text{for } i=1 \\ (3n+2)i+1-n-2r_i & \text{for } 2 \le i \le m, \end{cases}$$

$$\lambda(y_i^jy_i^{j+1}) = \begin{cases} n+3-j & \text{for } i=1 \text{ and } 1 \le j \le n-1 \\ (3n+2)i+1-j-2r_i & \text{for } 2 \le i \le m \text{ and } 1 \le j \le n-1, \end{cases}$$

$$\lambda(y_i^jy_i^{n+1}) = \begin{cases} n+2 & \text{for } i=1 \text{ and } 1 \le j \le n \\ (3n+2)i+1-2r_i & \text{for } 2 \le i \le m \text{ and } 1 \le j \le n, \end{cases}$$

$$\lambda(x_ix_{i+1}) = \begin{cases} 4n+2-r_2 & \text{for } i=1 \\ (3n+2)i+2n-r_i-r_{i+1} & \text{for } 2 \le i \le m-1. \end{cases}$$

It is easy to see that the function λ is a map from $V(P_m \odot W_n) \cup E(P_m \odot W_n)$ into $\left\{1,\ 2,\ \ldots,\ \left\lceil \frac{(3n+2)m+1}{3} \right\rceil \right\}$.
Observe that

$$\begin{aligned} wt(x_iy_i^j) &= \lambda(x_i) + \lambda(x_iy_i^j) + \lambda(y_i^j) \\ &= (3n+2)i - 3n + j \text{ for } 1 \le i \le m \text{ and } 1 \le j \le n+1, \end{aligned}$$

$$\begin{aligned} wt(y_i^1y_i^n) &= \lambda(y_i^1) + \lambda(y_i^1y_i^n) + \lambda(y_i^n) \\ &= (3n+2)i + 2(1-n) \quad \text{for } 1 \le i \le m, \end{aligned}$$

$$\begin{aligned} wt(y_i^jy_i^{j+1}) &= \lambda(y_i^j) + \lambda(y_i^jy_i^{j+1}) + \lambda(y_i^{j+1}) \\ &= (3n+2)i + 2(1-n) + j \quad \text{for } 1 \le i \le m \text{ and } 1 \le j \le n-1, \end{aligned}$$

$$\begin{aligned} wt(y_i^jy_i^{n+1}) &= \lambda(y_i^j) + \lambda(y_i^jy_i^{n+1}) + \lambda(y_i^{n+1}) \\ &= (3n+2)i - n + 1 + j \text{ for } 1 \le i \le m \text{ and } 1 \le j \le n, \end{aligned}$$

and

$$\begin{aligned} wt(x_ix_{i+1}) &= \lambda(x_i) + \lambda(x_ix_{i+1}) + \lambda(x_{i+1}) \\ &= (3n+2)i + 2 \text{ for } 1 \le i \le m. \end{aligned}$$

So the weights of edges of $P_m \odot W_n$ under the total labelling λ form consecutive integers from 3 up to $(3n+2)m+1$. It means that there is no two edges that have the same weight.

It concludes that $tes(P_m \odot W_n) \le \lceil \frac{(3n+2)m+1}{3} \rceil$. □

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