The computational complexity of λ-backbone colorings of graphs with n-complete backbones

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Abstract. Given an integer $\lambda \geq 2$, a graph $G = (V, E)$ and a spanning subgraph H of G (the backbone of G), a λ-backbone coloring of (G, H) is a proper vertex coloring $V \to \{1, 2, \ldots\}$ of G, in which the colors assigned to adjacent vertices in H differ by at least λ. We study the computational complexity of the problem "Given a graph G with a backbone H, and an integer ℓ, is there a λ-backbone coloring of (G, H) with at most ℓ colors?" Of course, this general problem is NP-complete. In this paper, we consider this problem for collections of pairwise disjoint complete graphs with order n. We show that the complexity jumps from polynomially solvable to NP-complete between $\ell = (n-1)\lambda$ and $\ell = (n-1)\lambda + 1$.

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1 Introduction

In [2] backbone colorings are introduced, motivated and put into a general framework of coloring problems related to frequency assignment. We refer to [2], [3], [4], [6], and [7] for an overview of related research, but we repeat the relevant definitions here. For undefined terminology we refer to [1].

Let $G = (V, E)$ be a graph, where $V = V_G$ is a finite set of vertices and $E = E_G$ is a set of unordered pairs of two different vertices, called edges. A vertex function $f : V \to \{1, 2, 3, \ldots\}$ is called a *proper vertex coloring* of V if $|f(u) - f(v)| \geq 1$ holds for all edges $uv \in E$. A proper vertex coloring $f : V \to \{1, \ldots, k\}$ is called a *k-coloring*, and the *chromatic number* $\chi(G)$ is the smallest integer k for which there exists a k-coloring. A set $V' \subseteq V$

is *independent* if G does not contain edges with both end vertices in V'. By definition, a k-coloring partitions V into k independent sets $V_1, \ldots, V_k$.

Let H be a *spanning subgraph* of G, i.e., $H = (V_G, E_H)$ with $E_H \subseteq E_G$. Given an integer $\lambda \geq 2$, a proper vertex coloring f is a *λ-backbone coloring* of (G, H), if $|f(u) - f(v)| \geq \lambda$ holds for all edges $uv \in E_H$. The *λ-backbone coloring number* $\text{BBC}_\lambda(G, H)$ of (G, H) is the smallest integer ℓ for which there exists a λ-backbone coloring $f : V \to \{1, \ldots, \ell\}$.

We call a spanning subgraph H of a graph G

- an *n-complete backbone* of G if H is a collection of pairwise disjoint complete graphs with order $n \geq 3$;
- a *star backbone* of G if H is a collection of pairwise disjoint stars;
- a *matching backbone* of G if H is a (perfect) matching.
- a *tree backbone* of G if H is a tree;
- a *path backbone* of G if H is a (Hamilton) path;

The following decision problem can be posed.

Problem 4 *Given a graph G, a spanning subgraph H, and an integer ℓ, is $\text{BBC}_\lambda(G, H) \leq \ell$?*

In general this problem is NP-complete. For particular instances of the problem complexity jumps from polynomial to NP-complete have been observed in several cases. If H is a tree backbone of G and $\lambda = 2$ the complexity jump occurs between $\ell = 4$ (easy for all tree backbones) and $\ell = 5$ (difficult even for path backbones) [2]. If H is a star backbone of G and $\lambda \geq 2$ the complexity jump occurs between $\ell = \lambda + 1$ (easy for all star backbones) and $\ell = \lambda + 2$ (difficult even for matching backbones) [3]. In this paper we consider this problem for n-complete backbones.

2 Main results

In this section we show that if H is an n-complete backbone of G for Problem 4, then the complexity jump occurs between $\ell = (n-1)\lambda$ and $\ell = (n-1)\lambda + 1$.

Theorem 5 *Let $\lambda \geq 2$.*

(a) The following problem is polynomially solvable for any $\ell \leq (n-1)\lambda$: Given a graph G and an n-complete backbone $\tilde{K}_n$, decide whether $\text{BBC}_\lambda(G, \tilde{K}_n) \leq \ell$.
(b) The following problem is NP-complete for all $\ell \geq (n-1)\lambda + 1$: Given a graph G and an n-complete backbone $\tilde{K}_n$, decide whether $\text{BBC}_\lambda(G, \tilde{K}_n) \leq \ell$.

Proof. We start with the positive result in statement (a). Let $\ell \leq (n-1)\lambda$ and $G = (V, E)$ be a graph with an n-complete backbone $\tilde{K}_n = (V, E_{\tilde{K}_n})$. Since every vertex in $V_{\tilde{K}_n}$ is in one $V(K_n)$, we need at least n colors with a distance at least λ apart. So there is no a λ-backbone coloring with color set $\{1, 2, \ldots, \ell\}$ for $(G, \tilde{K}_n)$.

Now let us prove the negative result in statement (b). The reduction is done from the NP-complete classical problem of GRAPH k-COLORABILITY (see Garey & Johnson [5] problem [GT 4] for more information): Given a graph $H = (V_H, E_H)$, does there exist a k-coloring of H? This problem is known to be NP-complete for any integer $k \geq 3$. We distinguish the following cases.

Case 1 $\ell = (n-1)\lambda + t$ for $t = 1, \ldots, \lambda - 1$.
Let $H = (V_H, E_H)$ be an instance of ℓ colorability, and let $v_1, v_2, \ldots, v_m$ denote the vertices in V_H. We create $m(n-1)$ new vertices $u_{i,j}$ ($i = 1, \ldots, m$ and $j = 2, \ldots, n$) and introduce the new edges $v_i u_{i,j}$ ($i = 1, \ldots, m$ and $j = 2, \ldots, n$) and $u_{i,j} u_{i,k}$ ($i = 1, \ldots, m$, $2 \leq j < k \leq n$). The graph that results from this is denoted by G. The new edges form an n-complete backbone $\tilde{K}_n$ of G. We complete the proof by showing that $\chi(H) \leq n \cdot t$ if and only if $\text{BBC}_\lambda(G, \tilde{K}_n) \leq \ell$.

Assume that $\text{BBC}_\lambda(G, \tilde{K}_n) \leq l$ and consider a λ-backbone ℓ-coloring b of $(G, \tilde{K}_n)$. Since for every vertex $v \in V_G$ there are exactly $n-1$ edges in $E_{\tilde{K}_n}$ that are incident with v, colors $p \cdot \lambda + q$ for $p = 0, \ldots, n-2$ and $q = t+1, \ldots, \lambda$ can not be used at all. Then define a $n \cdot t$-coloring c of H by:

- if $b(v) = p \cdot \lambda + k$ for $p = 0, \ldots, n-1$ and $k = 1, \ldots, t$: $c(v) = p \cdot t + k$.

Next, assume that $\chi(H) \leq n \cdot t$, and consider a $n \cdot t$-coloring $f : V_H \to \{1, \ldots, n \cdot t\}$. We define a λ-backbone ℓ-coloring $g : V_G \to \{1, , \ldots, \ell\}$ of $(G, \tilde{K}_n)$ by:

- if $v \in V_H$ and $f(v) = p \cdot t + k$ for $p = 0, \ldots, n-1$ and $k = 1, \ldots, t$: $g(v) = p \cdot \lambda + k$;
- if $p \cdot \lambda + 1 \leq g(v_i) \leq p \cdot \lambda + t$ for $p = 0, \ldots, n-1$:

$$g(u_{i,j}) = \begin{cases} (j-2)\lambda + 1 & \text{for } j \leq p+1 \\ (j-1)\lambda + t & \text{for } j \geq p+2. \end{cases}$$

Case 2 $\ell \geq n\lambda$.
Let $H = (V_H, E_H)$ be an instance of ℓ colorability, and let $v_1, v_2, \ldots, v_m$ denote the vertices in V_H. We create $m(n-1)$ new vertices $u_{i,j}$ ($i = 1, \ldots, m$ and $j = 2, \ldots, n$) and introduce the new edges $v_i u_{i,j}$ ($i = 1, \ldots, m$ and $j = 2, \ldots, n$) and $u_{i,j} u_{i,k}$ ($i = 1, \ldots, m$, $2 \leq j < k \leq n$). The graph

that results from this is denoted by G. The new edges form an n-complete backbone $\tilde{K}_n$ of G. We complete the proof by showing that $\chi(H) \leq \ell$ if and only if $\text{BBC}_\lambda(G, \tilde{K}_n) \leq \ell$.

Indeed, assume that $\text{BBC}_\lambda(G, \tilde{K}_n) \leq \ell$ and consider such a λ-backbone ℓ-coloring. Then the restriction to the vertices in V_H yields a ℓ-coloring of H. Next assume that $\chi(H) \leq \ell$, and consider a ℓ-coloring $f : V_H \to \{1, , \ldots, \ell\}$. We define a λ-backbone ℓ-coloring $g : V_G \to \{1, , \ldots, \ell\}$ of $(G, \tilde{K}_n)$ by:

- $g(v) = f(v)$ for $v \in V_H$;
- if $p \cdot \lambda + 1 \leq g(v_i) \leq (p+1)\lambda$ for $p = 0, \ldots, n-1$ and $i = 1, \ldots, m$:

$$g(u_{i,j}) = \begin{cases} (j-2)\lambda + 1 & \text{for } j \leq p+1 \\ j \cdot \lambda & \text{for } j \geq p+2. \end{cases}$$

This completes the proof.

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