On graphs of maximum degree 3 and defect 4 ⋆

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Abstract. It is well known that apart from the Petersen graph there are no Moore graphs of degree 3. As a cubic graph must have an even number of vertices, there are no graphs of maximum degree 3 and δ vertices less than the Moore bound, where δ is odd. Additionally, it is known that there exist only three graphs of maximum degree 3 and 2 vertices less than the Moore bound. In this paper, we consider graphs of maximum degree 3, diameter $D \geq 2$ and 4 vertices less than the Moore bound, denoted as $(3, D, 4)$-graphs. We obtain all non-isomorphic $(3, D, 4)$-graphs for $D = 2$. Furthermore, for any diameter D, we consider the girth of $(3, D, 4)$-graphs. By a counting argument, it is easy to see that the girth is at least $2D - 2$. The main contribution of this paper is that we prove that the girth of a $(3, D, 4)$-graph is at least $2D - 1$.
Finally, for $D > 4$, we conjecture that the girth of a $(3, D, 4)$-graph is $2D$.

Keywords: Degree/diameter problem, cubic graphs, Moore bound, Moore graphs, defect.

1 Introduction

Derived from the need of designing ever larger interconnection networks with constraints on the specification of the network, many graph-theoretical problems have arisen. One of these problems is the *degree/diameter problem*, which can be enunciated as follows

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Degree/diameter problem: Given natural numbers Δ and D, find the largest possible number of vertices $n_{\Delta,D}$ in a graph of maximum degree Δ and diameter at most D.

This problem is also known as the (Δ, D)*-graph problem*. An upper bound for $n_{\Delta,D}$ is given by the following expression; see [2].

$$\begin{aligned} n_{\Delta,D} &\leq 1 + \Delta + \Delta(\Delta-1) + \cdots + \Delta(\Delta-1)^{D-1} \\ &= 1 + \Delta(1 + (\Delta-1) + \cdots + (\Delta-1)^{D-1}) \\ &= \begin{cases} 1 + \Delta \frac{(\Delta-1)^D-1}{\Delta-2} & \text{if } \Delta > 2 \\ 2D+1 & \text{if } \Delta = 2 \end{cases} \end{aligned} \tag{1}$$

The right hand side of this expression is known as the *Moore bound*, and is denoted by $M_{\Delta,D}$. A graph whose order is equal to the Moore bound is called a Moore graph.

Moore graphs exist only for certain special values of maximum degree and diameter. For diameter $D = 1$ and degree $\Delta \geq 1$, Moore graphs are the complete graphs $K_{\Delta+1}$. For diameter $D = 2$, Hoffman and Singleton [5] proved that Moore graphs exist for $\Delta = 2, 3, 7$ and possibly $\Delta = 57$, but not for any other degrees. Finally, for $D \geq 3$ and $\Delta = 2$, Moore graphs are the cycles on $2D + 1$ vertices. The fact that Moore graphs do not exist for $D \geq 3$ and $\Delta \geq 3$ was shown by Damerell [3] and, independently, also by Bannai and Ito [1].

Therefore, we are interested in studying the existence of large graphs of given maximum degree Δ, diameter D and order $M(\Delta, D)$ - δ for $\delta > 0$, that is, (Δ, D, δ)-graphs, where δ is called *the defect*. In this paper we particularly restrict ourselves to the case when $\Delta = 3$.

A cubic graph, that is, a regular graph of degree 3, must have an even number of vertices. It is then clear that $(3, D, \delta)$-graphs cannot exist whenever δ is odd. Therefore, the next interesting case is when the order is $M_{3,D} - 2$. This case was analyzed by Jorgensen [6] in 1992. Jorgensen proved that for $D \geq 4$ there are no $(3, D, 2)$-graphs and showed the uniqueness of the two known $(3, 2, 2)$-graphs and of the known $(3, 3, 2)$-graph. Therefore, $n_{3,D} \leq M_{3,D} - 4$, for $D \geq 4$.

We consider graphs of maximum degree 3, diameter $D \geq 2$ and order $M_{3,D}$-4.

If a $(3, D, 4)$-graph has a vertex of degree at most 2, then the order of such a graph would be at most $\frac{2}{3}M_{3,D} + \frac{1}{3}$. Therefore, for $D \geq 3$, a $(3, D, 4)$-graph is regular.

For diameters 2 and 3, the catalogue of the cubic graphs on 6 and 18 vertices can be found in [9]. However, for $D = 2$, a $(3, 2, 4)$-graph does not have to be regular and so we include all the non-regular $(3, 2, 4)$-graph as well as all the regular ones from [9].

The cubic graph on 18 vertices, i.e., $(3, 3, 4)$-graph, was constructed first by Faradzhev [4]; however, his catalogue was not widely available. For this reason, McKay and Royle [8] in 1986 constructed, among other graphs, again the cubic graph with diameter 3 on 18 vertices, and proved its uniqueness.

The nonexistence of $(3, 4, 4)$-graphs was proved by Jorgensen [7] in 1993.

The case of $(3, D, 4)$-graphs, is especially interesting because it is the first time that we deal with (Δ, D, δ)-graphs, where $\delta > \Delta$.

By a counting argument, it can be showed that a $(3, D, 4)$-graph has girth at least $2D-2$. In this paper, we state that the girth of such a graph is at least $2D-1$, giving, in this way, the first steps to completely characterize $(3, D, 4)$-graphs.

When referring to paths, we shall always mean shortest paths.

The rest of this paper is structured as follows: in Section 2, we settle the notation and terminology used throughout this paper; in Section 3, we present all the $(3, D, 4)$-graphs with $D = 2, 3$. Section 4 is dedicated to proving that the girth of a $(3, D, 4)$-graph is at least $2D - 1$, for $D > 4$. In Section 5,we summarize the obtained results.

2 Notation and terminology

We denote an edge with endvertices u and v by uv. A path from a vertex x to a vertex y is denoted by $x - y$ or by the sequence of its vertices. A cycle of length k is called a k-cycle. A cycle could be denoted by a sequence of paths, for instance, $x - yzt - x$.

The set of vertices at distance k from a vertex x is denoted by $N_k(x)$ and $N_k(x)$ is called the k – *distance class* of x. The set of neighbors of a vertex x in G is simply denoted by $N(x)$.

3 Enumeration of $(3, D, 4)$-graphs of diameters 2 and 3

3.1 Diameter 2

As mentioned earlier, all the cubic graphs on 6 vertices were already known. However, a $(3, 2, 4)$-graph G needs not be regular. In particular, it is possible for G to contain some vertices of degree 2 (but none of degree 1). In our construction of $(3, 2, 4)$-graphs, we have obtained the two (already known) regular $(3, 2, 4)$-graphs as well as the three new non-regular graphs.

Let G be a $(3, 2, 4)$-graph and let x be a vertex of G. Let a be the number of edges joining vertices at $N_2(x)$, b be the number of edges from $N(x)$ to $N_2(x)$, and c be the number of edges joining vertices at $N(x)$; see Figure 1(a).

As $M_{3,2}$=10, the order of G is 6. We distinguish two cases; the case of cubic graphs and the case of graphs with at least one vertex of degree 2.

Cubic graphs In a cubic $(3, 2, 4)$-graph G there are 9 edges. We can start with some 3, as in Figure 1(a). Then 6 further edges need to be inserted in Figure 1(a). Let us consider the following system of equations:

$$a + b + c = 6 \qquad 2a + b = 6 \qquad 2c + b = 6 \tag{2}$$

From System 2, we see that $a = c$ and $b \geq 4$. Therefore, the solutions are

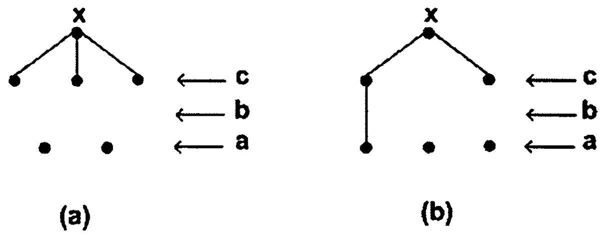

Fig. 1. Auxiliary figures for the case of diameter 2.

(a, b, c) = (1,4,1) or (0,6,0).

Case a) (a, b, c) = (1,4,1).

Up to isomorphism, the graph H_1 in Figure 2(a) is the only cubic graph that can be constructed with these parameters.

Case b) (a, b, c) = (0,6,0).

Up to isomorphism, the cubic complete bipartite graph $K_{3,3}$, depicted in Figure 2(b), is the only such cubic graph.

Non-regular graphs As there should be at least one vertex of degree 3, the only possible degree sequences are

$(2,2,2,2,3,3)$, or $(2,2,3,3,3,3)$.

Case 1. $(2,2,2,2,3,3)$.

In this case let us refer to Figure 1(a) again. We then have the following system, where a, b and c have the same meaning as before.

$$a + b + c = 4 \qquad a \leq 1 \qquad b \geq 2 \qquad c \leq 1 \tag{3}$$

The solutions of System 3 are:

(a, b, c) = (1,2,1) or (0,3,1) or (0,4,0) or (1,3,0).

It is not difficult to see that the first three cases do not correspond to any suitable graph.

If $a = 1$, $b = 3$ and $c = 0$ then, up to isomorphism, the only possible graph is the graph depicted in Figure 2(c), a subgraph of H_1.

Case 2. $(2,2,3,3,3,3)$.

If the vertices with degree 2 are not neighbors, then we can add one edge between them and we will obtain the graph H_1. Therefore, the possible graphs are subgraphs of H_1 and up to isomorphism the only possible graph is the one depicted in Figure 2(d).

Let us next suppose the vertices of degree 2 are neighbors. In this case, if we consider Figure 1(b) and that a, b and c have the same meaning as before, we have the following system of equations:

$$a + b + c = 5 \qquad 2a + b = 8 \qquad c = 0 \tag{4}$$

whose solution is (a, b, c) = (3,2,0) and the only possible graph is the one depicted in Figure 2(e).

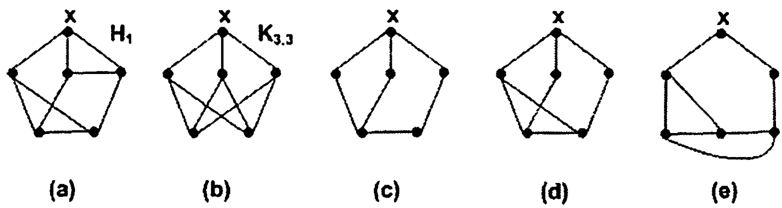

Fig. 2. All $(3, 2, 4)$-graphs.

3.2 Diameter 3

In this case, we restrict ourselves to show the unique $(3, 3, 4)$-graph in Figure 3. For a description of a method to construct cubic graphs on up to 20 vertices, we refer the reader to [8] and for the complete catalogue of cubic graphs on up to 24 vertices, see [9].

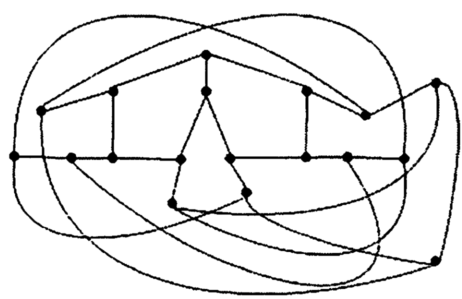

Fig. 3. The unique $(3, 3, 4)$-graph.

4 On the girth of a $(3, D, 4)$-graph with $D \geq 5$

In this section, we analyze the girth of $(3, D, 4)$-graphs with $D \geq 5$.

By a counting argument, we prove the following proposition.

Proposition 1. *The girth of a $(3, D, 4)$-graph is at least $2D-2$. Furthermore, if x is a vertex contained in a $(2D-2)$-cycle then x is not contained in any other cycle of length at most $2D$.*

Proof. Let $N_i(x)$ be independent sets and let $|N_i(x)| = 3 \times 2^{i-1}$, for $i \in [1...D-3]$. If $N_{D-2}(x)$ has at most $3 \times 2^{D-3} - 1$ vertices then $|N_{D-1}(x)| \leq 3 \times 2^{D-2} - 3$ and $|N_D(x)| \leq 3 \times 2^{D-1} - 6$. Then $|G| \leq M_{3,D} - 10$, a contradiction. Thus, $N_{D-2}(x) = 3 \times 2^{D-1}$.

If there is an edge between any two vertices at distance $D-2$ from x, then $|N_{D-1}(x)| \leq 3 \times 2^{D-2} - 2$ and $|N_D(x)| \leq 3 \times 2^{D-1} - 4$. Therefore, $|G| \leq M_{3,D} - 6$, a contradiction. Thus, $N_{D-2}(x)$ is an independent set and G has girth at least $2D-2$. □

Theorem 1. *A $(3, D, 4)$-graph has girth at least $2D-1$.*

Proof. Let C be a cycle of length $2D-2$. Let x and y be two vertices in C at distance $D-1$ and let x_1 and y_1 be the respective neighbors of x and y not contained in C.

By Proposition 1, a path $P_1 = x_1 - y_1$ is a D-path and $P_1 \cap C = \emptyset$.

Let $y_2 \in N(y_1)$, $y_2 \neq y$ and $y_2 \notin P_1$. By Proposition 1, a path $P_2 = y_2 - x$ is a D-path and $P_2 \cap C = \{x\}$, so $x_1 \in P_2$. Let x_2 and x_3 be the two neighbors of x_1 different from x, such that $x_2 \in P_1$. Let us suppose that $|P_2 \cap P_1| > 1$. Then $|P_2 \cap P_1| = \{x_1, x_2\}$ and y_1 lies on a $2D-2$-cycle, say C_1. By Proposition 1, a path $y - x_3$ is a D-path and intersects C in y. Therefore, $y - x_3$ is going through y_1 but in this case some vertices of C_1 will be contained in another cycle of length at most $2D$, which contradicts Proposition 1. Thus, $|P_2 \cap P_1| = \{x_1\}$ and $x_3 \in P_2$; see Figure 4.

Let $y_3 \in N(y_2)$, $y_3 \neq y_1$ and $y_3 \notin P_2$. Let us now consider a path $P_3 = y_3 - x$. Then $\{y_2, x_3\} \notin P_3$ and $P_3 \cap C = \{x\}$. Therefore, $x_2 \in P_3$ and if P_3 is a $D-1$-path, then y_1 is contained in a cycle of length at most $2D-2$ and, as above, by considering a path $y - x_3$, we obtain a contradiction to Proposition 1. Therefore, P_3 is a D-path.

Let us now denote the neighbors of x in C by u and v. By Proposition 1, a path $P_4 = y_3 - u$ is D-path, $y_2 \notin P_4$ and $P_4 \cap P_3 = \{y_3\}$, otherwise either y or x is contained in a further cycle of length at most $2D-1$, a contradiction. Let us finally denote by z the neighbor of y_3 on P_4. Analogously, a D-path $P_5 = y_3 - v$ is also going through z, but in this case, x is contained in the $2D$-cycle $xu - z - vx$, contradicting Proposition 1. □

5 Conclusions

In this paper, we have given the first steps towards a characterization of $(3, D, 4)$-graphs for $D \geq 2$ as follows.

Diameters 2 and 3. There are two regular and three non-regular $(3, 2, 4)$-graphs. When the diameter is 3, there is a unique $(3, 3, 4)$-graph.

Diameter 4. The nonexistence of the particular case of $D = 4$ was proved by Jorgesen.

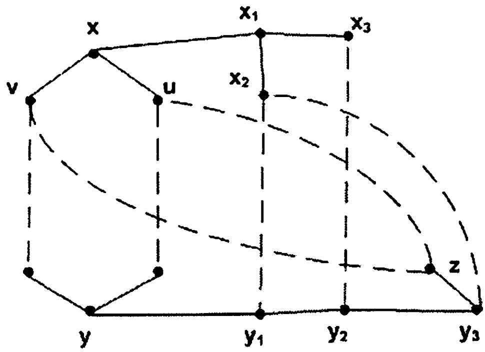

Fig. 4. Auxiliary figure for Theorem 1

Diameters greater than 4. We proved that if such a graph exists, then its girth is at least $2D - 1$. Furthermore, we propose the following conjecture.

Conjecture 1. The girth of a $(3, D, 4)$-graph is $2D$, for $D \geq 5$.

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