On multiplicative labelings of a graph

S.M. Hegde

Department of Mathematical and Computational Sciences
National Institute of Technology Karnataka
Surathkal, Srinivasanagar-575025, INDIA
smhegde@nitk.ac.in

Abstract. A (p, q)-graph G is said to be *multiplicative* if its vertices can be assigned distinct positive integers so that the values of the edges, obtained as the products of the numbers assigned to their end vertices are all distinct. Such an assignment is called a *multiplicative labeling* of G. A multiplicative labeling is said to be (a, r)*-geometric* if the values of the edges, can be arranged as a geometric progression $a, ar, ar^2, \ldots, ar^{q-1}$. In this paper we prove that some well known classes of graphs are geometric for certain values of a,r and also initiate a study on the structure of finite (a,r)-geometric graphs.

Keywords: Multiplicative labelings/graphs, geometric labelings/graphs
M. R. Classification: 05C78

1 Introduction

We consider only finite graphs. For all notations in graph theory we follow Harary [4] and West [5].

Several practical problems in real life situations have motivated the study of labelings of a graph G =(V, E), which are required to obey a variety of conditions. There is an enormous literature built up on several kinds of labelings of graphs over the past four decades or so. An interested reader can refer to Gallian [3].

Given a graph $G = (V, E)$, the set R of real numbers, a subset N of R and a commutative binary operation $* : R \times R \to R$, every vertex function $f : V(G) \to N(N$ is the set of positive integers) induces an edge function $f^* : E(G) \to R$ such that $f^*(xy) = f(x) \times f(y)$ for all $xy \in E(G)$. One is interested in determining the vertex functions f having a specified property P such that the induced edge function f^* have a specified property Q, where P and Q need not necessarily be distinct.

In this paper we are interested in the study of vertex functions f:$V(G)\to N$, for which the induced edge function f^*:$E(G)\to N$ is defined as $f^*(xy) =$

$f(x) \times f(y), \forall xy \in E(G)$. Such vertex functions are said to be *multiplicative vertex functions* and henceforth this induced map f^* of f is denoted as $f^\times$.

The following result gives a general property of multiplicative vertex functions, which can be proved by easy counting arguments.

Theorem 1. (*Acharya and Hegde* [1]): *For any graph G and for any multiplicative vertex function* $f: V(G) \to N$

$$\prod_{e \in E(G)} f^\times(e) = \prod_{u \in V(G)} f(u)^{d(u)} \tag{1}$$

Corollary 1.1: If G is an r-regular graph then for any multiplicative vertex function f of G,

$$\prod_{e \in E(G)} f^\times(e) = \prod_{u \in V(G)} f(u)^r$$

2 Multiplicative Labelings

A *multiplicative labeling* of a graph G is an injective multiplicative vertex function f such that the induced edge function $f^\times$ is also injective.

We adopt the following notations throughout this paper:

$M(G)$ = The set of all multiplicative labelings of G.

$f(G) = \{f(u) : u \in V(G)\}$ $f^\times(G) = \{f^\times(e) : e \in E(G)\}$

$f_{\min}(G) = \min_{u \in V(G)} f(u)$ $\quad f_{\max}(G) = \max_{u \in V(G)} f(u)$

$f^\times_{\max}(G) = \max_{e \in E(G)} f^\times(e)$ $\quad \theta(G) = \min_{f \in M(G)} f_{\max}(G)$

Figure 1 shows two multiplicative labelings of K_6

Remark 1. Figure $1(a)$ shows that $\theta(K_6)$ can be very small compared to $f_{max}(K_6)$ of Figure $1(b)$. One can also verify that $\theta(K_6)$ is 7. Thus, finding a suitable upper bound for $\theta(K_n)$ is an interesting problem.

Theorem 2. *Every graph admits a multiplicative labeling.*

proof. We observe that if a graph G admits a multiplicative labeling f then each of its subgraphs does so - in fact, if H is a subgraph of G then the restriction map f/H is an multiplicative labeling of H. Therefore, to prove the theorem, it is enough to consider complete graphs.

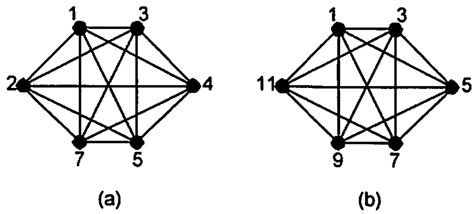

Fig. 1. Multiplicative labelings of K_6

Hence, let $G = K_n$ and $V(G) = \{v_1, v_2, \ldots, v_n\}$. Then assign 1 to the vertex v_1 and first n-1 prime numbers to the remaining n-1 vertices of K_n at random in a one-to-one manner. Then it is not hard to verify that K_n is multiplicative. ◇

3 Geometric Labelings (Graphs)

Given a (p, q)-graph G and an $f \in M(G)$, we say that f is (a, r)-*geometric*, if $f^{\times}(G) = \{a, ar, ar^2, ..., ar^{q-1}\}$ where a and r are positive integers both at least 2. Let $M_{a,r}(G)$ denote the set of all (a, r)-geometric labelings of G. Further, G is said to be (a, r)-geometric if $M_{a,r}(G) \neq \emptyset$, then $M_{a,r}(G)$ is finite as there are only finite number of factors of a. Thus, one may conceive of two parameters for any $(a,\ r)$-geometric graph G, viz.,
$\theta_{a,r}(G) = \min\limits_{f \in M_{a,r}(G)} f_{\max}(G) \quad \theta'_{a,r}(G) = \max\limits_{f \in M_{a,r}(G)} f_{\max}(G).$
Clearly, for any (a, r)-geometric graph G,

$$p \leq \theta(G) \leq \theta_{a,r}(G) \leq \theta'_{a,r}(G) \leq ar^{q-1}. \tag{2}$$

where the last inequality is attained by the star $K_{1,p-1}$ by assigning 1 to its central vertex and $a, ar, \ldots, ar^{q-1}$ to its pendant vertices at random in a one-to-one manner.

We say that a graph is geometric if it is (a, r)-geometric at least for one value of the positive integers $a \geq 2$ and $r \geq 2$.
Remark 2. If a graph G is (a, r)-geometric then it is $(at^2, r)-$geometric, where t is a positive integer.

Theorem 3. *Let G be a connected (a, a)-geometric (p, q)-graph. Then for any $f \in M_{a,a}(G)$, $1 \in f(G)$ if and only if $a | f(u), \forall u,\ f(u) \neq 1$.*

Proof. Let $f \in M_{a,a}(G)$. Then $f^{\times}(G) = \{a^j : 1 \leq j \leq q\}$. Hence if $a|f(u)$ $\forall u \in V(G) - \{v\}$ then for the edge $xy \in E(G)$ with $f^{\times}(xy) = a$, we must have either $f(x)$ or $f(y)$ to be 1, whence $1 \in f(G)$.

For the converse, let $v \in V(G)$ be such that $f(v)$ =1. Then for all $w \in N(v) = \{u \in V(G) : vu \in E(G)\}$ we must have $f(v).f(w) = f^{\times}(vw) = a^t$ where t is a positive integer. This yields $f(w) = a^t$ as $f(v) = 1$. Thus $a|f(w)$ $\forall\, w \in N(v)$. Now, fix any $w_0 \in N(v)$. Then for any $w_1 \in N(w_0) - N(v)$ we have $f(w_0).f(w_1) = f^{\times}(w_0 w_1) = a^j$ for the positive integer $j \geq 1$, so that $f(w_1) = a^{|j-t|}$, whence $a|f(w_1)$. Since w_1 was arbitrary by choice we get that $a|f(w)$ for all $w \in N(v) \cup (\cup N(u), u \in N(v))$. Continuing in this way, we exhaust all the vertices of G with the conclusion that the vertex values except v are factors of a. ◇

From the above proof one can see that (a, r)-geometric graph G with a, a prime number or the square of a prime number, $1 \in f(G)$ for every $f \in M_{a,r}(G)$.

Theorem 4. *Let G be a connected bipartite graph which is not a star. If G is an (a, a)-geometric graph, then for any $f \in M_{a,a}(G)$, $1 \notin f(G)$.*

Proof. Consider any $f \in M_{a,a}(G)$ and let $A = \{u_1, u_2, ..., u_m\}$ and $B = \{v_1, v_2, ..., v_n\}$ constitute the parts of a bipartition of G. Suppose that $1 \in f(G)$. Without loss of generality, assume that $f(u_1) = 1$. Since $f^{\times}(G) = \{a, a^2, a^3, ..., a^q\}$ we must have $f(v_j) = a$ for some $v_j \in N(u_1)$. Without loss of generality, let $j = 1$. Next let $xy \in E(G)$ be such that $a^2 = f^{\times}(xy) = f(x).f(y)$. By Theorem 3, a divides both $f(x)$ and $f(y)$, or one of $f(x)$ and $f(y)$. Thus $f(x)$ =1 and $x = u_1$. Also, $f(y) = a^2$ and, without loss of generality, we may take $y = v_2$. Next; let $xy \in E(G)$ be such that $a^3 = f^{\times}(xy) = f(x).f(y)$. Again by Theorem 3, a divides both $f(x)$ and $f(y)$ or one of $f(x)$ and $f(y)$. Therefore, we see that either one of $f(x)$ and $f(y)$ is a and the other is a^2 or that one of $f(x)$ and $f(y)$ is 1 and the other is a^3. Clearly, the first possibility cannot arise as f is injective and vertices labeled a and a^2 have already occurred in B. So, the latter case must hold. Then, without loss of generality we may assume that $f(x) = 1$ and $f(y) = a^3$. But then injectivity of f forces $x = u_1$, so that $y = v_3$. Continuing this way, we see that if $N(u_1) = \{v_1, v_2, ..., v_t\}$ where t is the degree of u_1 in G then $f(v_j) = a^j$ for each j, $1 \leq j \leq t$. Since G is not a star and connected, it follows that $|A| \geq 2$ so that $a^{t+1} \in f^{\times}(G)$. Let $xy \in E(G)$ be such that $a^{t+1} = f^{\times}(xy) = f(x).f(y)$. This yields $x \neq u_1$ and $f(x) = (a^{t+1})/ f(y) \leq a^t$ or $f(y) = (a^{t+1})/f(x) \leq a^t$ as $f(x) \geq a$ and $f(y) \geq a$, a contradiction to the fact that $f(N(u_1)) = \{a^j : 1 \leq j \leq t\}$ and that f is injective. ◇

Corollary 4.1 : No connected bipartite graph, except the star is (a, a)-geometric when a is a prime number or square of a prime number.

Corollary 4.2 : Any connected (a, a)-geometric graph when a is a prime number or square of a prime number, is either a star or has a triangle.

Remark 3. In Corollary 4.1 it is possible to relax the condition of connectedness of G if a = 2,3,5. However, if a= 4 then it is not possible to do so as $G = K_2 \cup K_{1,3}$ has (4, 4)-geometric labeling, which can be easily verified.

Our next Theorem gives a method to recursively enlarge a given (a,r)-geometric graph G to a (a, r)-geometric graph H of an arbitrarily high order.

Theorem 5. *Let G be a (p,q)-graph having an (a, r)-geometric labeling f such that the elements of $f(G')\subseteq f(G)$ where $G'\subseteq G$, can be arranged as a subsequence P, of the geometric progression $Q= \{a,ar,ar^2,\ldots,ar^k = f_{\max}(G)\}$. Let $X = \{x_1,x_2,...,x_t\} = V(\bar{K}_t)$ and let $Y = \{a_1,a_2,....,a_w\}$ be a set with $V(G)\cap Y = \emptyset$, where w is the number of terms in Q which are not in P. Then the $(a,r)-$geometric labeling f of G can be extended to the newly added edges of $(G' \cup Y) + \bar{K}_t$, where t is a positive integer.*

Proof. Let G be (a,r)-geometric with a geometric labeling f. Let G'be a subgraph of G such that $f(G')$ can be arranged as a subsequence P of the geometric progression $Q= \{a,\ ar,\ ar^2,\ldots,ar^k = f_{\max}(G)\}$. Let $Y = \{a_1,\ a_2,\ ....a_w\}$ be a set of vertices with $V(G)\cap Y = \emptyset$, where w is the number of terms in Q which are not in P and $X = \{x_1,\ x_2,\ ....x_t\} = V(\bar{K}_t)$. Construct the graph H from G and Y by joining the vertices of G' and Y to the vertices of X. Let $\gamma : Y \to (Q-P)$ be a bijection. Then define a map F: $V(H) \to N$ by

$$F(u) = \begin{cases} f(u) & if \quad u \in V(G) \\ \gamma(a_i) & if \quad u = a_i \in Y \\ a^{q+1+(|f(G')|+|Y|)(i-1)} & if \quad u = x_i \in X, 1 \le i \le t, \quad a = r\,. \\ ar^{(|f(G')|+|Y|)i+1}, & \quad 1 \le i \le t, \quad a \neq r. \end{cases}$$

Then one can verify that F is an extension of the (a,r)-geometric labeling f of G to H. ◊

Consider the (2, 2)-geometrically labeled graph G shown in Figure 2(a), where $f(G')= \{1, 2, 4, 16\}$, $P= \{1, 2, 4, 16\}$, $Q =\{1, 2, 4, 8, 16\}$, $Y = \{a_1\}$ and let $X = \{x_1,\ x_2\}$. The augmented graph $H = G \cup Y$, geometrically labeled, is shown in Figure 2(b) and the (2, 2)-geometrically labeled graph is shown in Figure 2(c).

Remark 4. The above example shows that the number 1 can be included in Q in some cases (when $a = r$) to construct a bigger (a,r)-geometric graph as an application of Theorem 5, making a slight modification in the function in F to the vertices of X. Also, one can verify that it is not always possible to include the numbers less than a in Q.

Henceforth, unless mentioned otherwise, any bipartite graph G with bipartition $\{A, B\}$ will be assumed to be accompanied with a labeling of its vertices given by $A = \{u_1, u_2, ..., u_m\}$ and $B = \{v_1, v_2, ..., v_n\}$ where $m \le n$.

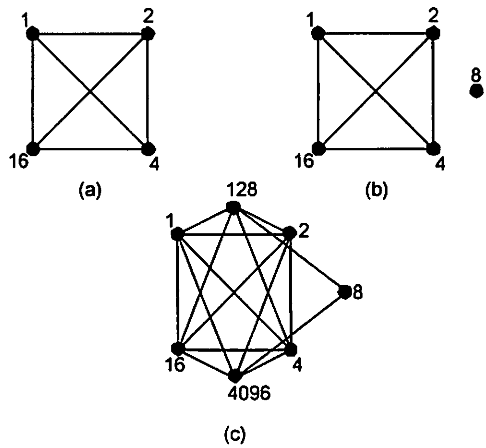

Fig. 2. Illustration of Theorem 5.

Theorem 6. *For the positive integers a and r both at least 2 , and for any two factors a_1 and a_2 of a with $1 \leq a_1 < a_2$, the star $K_{1,n}$ has an (a, r)-geometric labeling f such that a_1, a_2, $\in f(K_{1,n})$.*

Proof. Let $A = \{ u\}$ (i.e. $u_1 = u$) and f: $A \cup B \rightarrow N$ be a map defined by

$f(u) = a_1$

$f(v_i) = a_2r^{i-1}$, $1 \leq i \leq b$.

It can be easily seen that f is a required (a, r)-geometric labeling of $K_{1,n}$. ◊

Thus, $K_{1,n}$ is "*arbitrarily geometric*", in the sense that it is an (a, r)-geometric graph for all values of a and r. According to Corollary 4.1 there are no other connected bipartite graphs with this property. Thus, it follows that the star $K_{1,n}$ is the only arbitrarily geometric graph.

Theorem 7. *The complete bipartite graph $K_{m,n}$, $2 \leq m \leq n$ is (k, k)-geometric if and only if k is neither a prime number nor the square of a prime number.*

Proof. Suppose, $K_{m,n}$ is (k, k)-geometric. It follows from Theorem 4 that $1 \notin f(G)$. This means the number $k$is obtained as the product of two distinct numbers a_1 and a_2 such that neither of them is one. As a prime number has only two factors 1 and itself, k cannot be a prime. Similarly, as the square of a prime number p has three factors 1, p and p^2, k cannot be the square of any prime number.

Conversely, suppose that k is a positive integer which is not a prime number or square of any prime number. Let $k = a_1.a_2, 2 \leq a_1 < a_2$.

Define a map $f : A \cup B \to N$by

$$f(u_i) = a_1 k^{i-1}, \quad 1 \leq i \leq m$$

$$f(v_j) = a_2 k^{a(j-1)}, \quad 1 \leq j \leq n.$$

Then one can verify that the labeling f and $f^{\times}$ defined above are injective and that f is an (k, k)-geometric labeling of $K_{m,n}$. ◇

The labeling mentioned in the proof of Theorem 7 is illustrated in Figure 3 and 4 below.

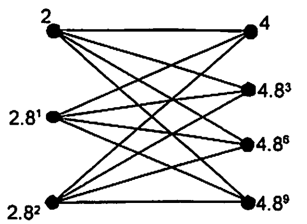

Fig. 3. An (8,8)-geometric labeling of $K_{3,4}$

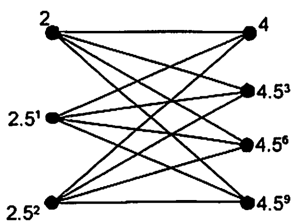

Fig. 4. An (8,5)-geometric labeling of $K_{3,4}$

Remark 5. From Figure 4 one can see that $K_{m,n}$ is $(8, 5)$-geometric. This means that $K_{m,n}$ is geometric for some other values of k and r,when $k \neq r$. We note that the labeling is obtained by using the labeling mentioned in the proof of Theorem 7. Hence it is interesting to investigate the other values of k, r for which $K_{m,n}$ is (k,r)-geometric.

We denote by $C_{m,n}$ a caterpillar with bipartition $\{A, B\}$, with $m = |A|$ and $n = |B|$. For given integers $m, n, C_{m,n}$ is not necessarily unique.

Theorem 8. *The caterpillar $C_{m,n}$, $2 \le m \le n$, is (k,k)-geometric if and only if k is neither a prime number nor the square of a prime number.*

Proof. Suppose, $C_{m,n}$ is (k,k)-geometric. Then by Theorem 4 we get $1 \notin f(G)$. This means the number k is obtained as the product of two distinct numbers a_1 and a_2 such that neither of them is one. As a prime number has only two factors 1 and itself, k cannot be a prime. Similarly, as the square of a prime number p has three factors 1, p and p^2, k cannot be the square of any prime number.

Conversely, suppose that k is a positive integer which is not a prime number or square of any prime number. Let $k = a_1.a_2, 2 \le a_1 < a_2$.

Define a map $f : A \cup B \to N$ by

$$f(u_i) = a_1 k^{i-1}, \quad 1 \le i \le m$$

$$f(v_j) = a_2 k^{(j-1)}, \quad 1 \le j \le n.$$

Then one can easily verify that f so defined is a (k,k)-geometric labeling of $C_{m,n}$ with $2 \le a_1 < a_2$. ◇

The labeling mentioned in the proof of Theorem 8 is illustrated in Figure 5 and 6 below.

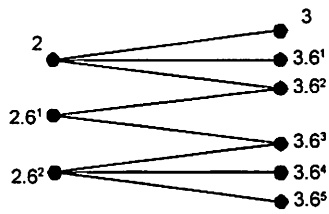

Fig. 5. A (6,6)-geometric labeling of $C_{3,6}$

Remark 6. From Figure 6 one can see that $C_{m,n}$ is (6,7)-geometric. This means that $C_{m,n}$ is geometric for some other values of k and r,when $k \neq r$. We note that the labeling is obtained by using the labeling mentioned in the proof of Theorem 8. Hence it is interesting to investigate the other values of k,r for which $C_{m,n}$ is (k,r)-geometric.

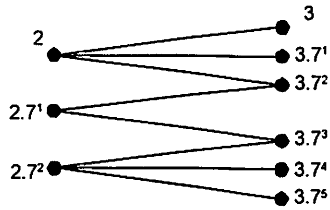

Fig. 6. A (6,7)-geometric labeling of $C_{3,6}$

Proposition 1. The graph $K_{m,n,1}$, $m \leq n$, is (k,k)-geometric for all integers $k \geq 2$.

Proof. Let $\{A, B, C\}$, where $A = \{u_1, u_2, ..., u_m\}$, $B= \{v_1, v_2, ..., v_n\}$ and $C = \{ w\}$, be the tripartition of the complete tripartite graph $K_{m,n,1}$. Define the map $f : A \cup B \cup C \to N$ by

$f(w) = 1$

$f(u_i) = k^i, \quad 1 \leq i \leq m,$

$f(v_j) = k^{(m+1)j}, \quad 1 \leq j \leq n,$

can be easily verified to be a required (k,k)-geometric labeling of $K_{m,n,1}$. ◊

Lemma 1. If a complete graph $K_n, n \geq 3$ is (a,r)-geometric then a is a multiple of r.

Proof. Suppose that $K_n, n \geq 3$ is (a, r)-geometric. Without loss of generality, let $f(v_1) = a_1, f(v_2) = a_2, f(v_3) = a_3$ ...., such that $a_1 < a_2 < a_3 <$ .... $< a_n$. Since $a_1 < a_2 < a_3$, we can take

$$a_1\ a_2 = a \tag{3}$$

$$a_1\ a_3 = ar^i \tag{4}$$

$$a_2\ a_3 = ar^j \tag{5}$$

Solving (3)-(5) for a_1, a_2, a_3 we get

$$a_1 = \sqrt{ar^{i-j}}, \quad a_2 = a/\sqrt{ar^{i-j}}, \quad a_3 = ar^i/\sqrt{ar^{i-j}}.$$

Since a_1, a_2, a_3 are positive integers ≥ 1, ar^{i-j} must be a perfect square i.e.,

$a = r^{j-i}\ t^2 = (r^{j-i-1}\ t^2)r = kr$ for some$k \geq 1$.

Theorem 9. *For the cycle C_n, $n \geq 4$, the following statements hold:*

(A) The cycle C_{4t} is is (a, a)-geometric if and only if a is neither a prime number nor the square of a prime number.

(B) For any integer positive integer $t \geq 1$ and $r \geq 2, C_{4t+1}$ is (r^{2t}, r)-geometric.

(C) C_{4t+2} is not geometric for any positive integer $t \geq 1$.

(D) For any positive integer $t \geq 1$ and $r \geq 2, C_{4t+3}$ is (r^{2t+1}, r)-geometric.

Proof. (A): Suppose, C_{4t} is (a, a)-geometric. Then by Theorem 4 we get $1 \notin f(G)$. This means the number a is obtained as the product of two distinct numbers a_1 and a_2 such that neither of them is one. As a prime number has only two factors 1 and itself, a cannot be a prime. Similarly, as the square of a prime number p has three factors 1, p and p^2, a cannot be the square of any prime number.

Conversely, suppose that a is a positive integer which is not a prime number or square of any prime number. Let $a = a_1.a_2,\ 2 \leq a_1 < a_2$.

Define a function $f : V(C_{4t}) \to N$ by

$$f(u_i) = \begin{cases} a_1 a^{(i-1)/2} & if\ i\ is\ odd \\ a_2 a^{(i-2)/2} & if\ i\ is\ even\ and\ 2 \leq i \leq 2t \\ a_2 a^{i/2} & if\ i\ is\ even\ and\ 2t+2 \leq i \leq 4t. \end{cases}$$

One may then easily verify that f turns out to be a required geometric labeling of C_{4t} in both the cases.

(B) Under the hypothesis, the map $f : V(C_{4t+1}) \to N$ defined by

$$f(u_i) = \begin{cases} r^{(i-1)/2} & for\ odd\ i'\ s \\ r^{(4t+i)/2} & for\ even\ i'\ s, \end{cases}$$

can be easily verified to be a geometric labeling of C_{4t+1}.

(C) Suppose that C_{4t+2} has an (a, r)-geometric labeling f. Let $f(u_i) = x_i, 1 \leq i \leq 4t+2$. Without loss of generality, we may assume that $x_1 . x_2 = a$. Then by Theorem 1 we get

$$(x_1 x_2)^2 \prod_{i=3}^{4t+2} x_i^2 = a^{4t+2} r^{(2t+1)(4t+1)}$$

i.e.,

$$a^2((x_3x_4)(x_5x_6)...(x_{4t+1}\ x_{4t+2}))^2 = a^{4t+2} r^{(2t+1)(4t+1)}$$

i.e.,

$$a^2(a^{2t} r^m)^2 = a^{4t+2} r^{(2t+1)(4t+1)}$$

i.e.,

$$r^{2m} = r^{(2t+1)4t+1)}$$

where m is a positive integer. Hence $2m$ = $(2t+1)$ $(4t+1)$, a preposterous statement, both sides being positive integers.

(D) Under the hypothesis the map $f : V(C_{4t+3}) \to N$ defined by

$$f(u_i) = \begin{cases} r^{(i-1)/2} & for\ odd\ i'\ s \\ r^{(4t+2+i)/2} & for\ even\ i'\ s, \end{cases}$$

can be verified to be a required geometric labeling of C_{4t+3}.

One can also verify that C_{4t} is (a, r)-geometric when $a \neq r$, using the same labeling given in the proof above for some a and r.◇

4 TRANSFORMED TREES (T_P-TREES)

In this section we prove that a class of trees called T_p-*trees* (*transformed trees*) (see Acharya [2]) are geometric. Also, we prove that the subdivision $S(T)$ of a T_p-tree T, obtained by subdividing every edge of T exactly once is geometric. Note that the subdivision $S(T)$ of a T_p-tree T is not necessarily a T_p-tree.

Let T be a tree and u_0 *and* v_0 be two adjacent vertices in T. Let there be two pendant vertices u and v in T such that the length of $u_0 - u$ path is equal to the length of $v_0 - v$ path. If the edge u_0v_0 is deleted from T and u , v are joined by an edge uv, then such a transformation of T is called an elementary parallel transformation (or an *ept*) and the edge u_0v_0 is called a *transformable edge.*

If by a sequence of ept's T can be reduced to a path then T is called a T_p-tree (transformed tree) and any such sequence regarded as a composition of mappings (ept's) denoted by P, is called a parallel transformation of T. The path, the image of T under P, is denoted as $P(T)$.

A T_p-tree and a sequence of two ept's reducing it to a path are illustrated in Figure 7.

Theorem 10. *Every* T_p*-tree* T *is geometric.*

Proof. Let T be a T_p-tree with $n + 1$ vertices, where n is a positive integer. By the definition of a T_p-tree there exists a parallel transformation P of T such that for the path $P(T)$, we have

$V(P\ (T)) = V(T)$

$E(P(T)) = (E(T) - E_d) \cup E_p$,

where E_d is the set of edges deleted from T and E_p is the set of edges newly added through the sequence $P = (P_1, P_2, ...P_k)$ of the *ept*'s P_i used to arrive at the path $P(T)$. Clearly E_d and E_p have the same number of

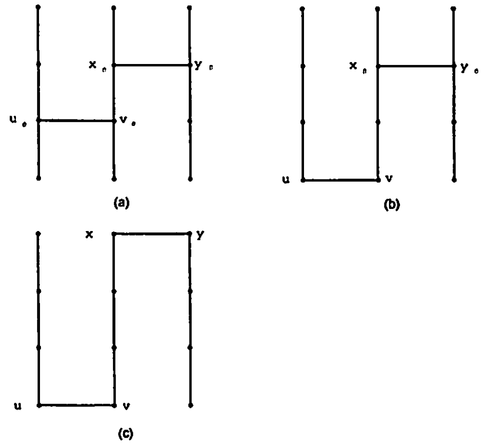

Fig. 7. T_p trees

edges. Denote the vertices of $P(T)$ successively as $v_1, v_2, ..., v_{n+1}$ starting from one pendant vertex of $P(T)$ right up to the other.

Define a function $f : V(P(T)) \to N$ by

$$f(v_i) = \begin{cases} a^{[(i-1)/2]d} & for\ odd\ i, \quad 1 \le i \le n+1 \\ a^{k+(q-1)d+[(i-2)/2]d} & for\ even\ i, \quad 2 \le i \le n+1 \end{cases}$$

where k and d are positive integers and q is the number of edges of T.

Let $v_i v_j$be an edge in T for some indices i and j, $1 < i < j \le n+1$ and let P_1 be the *ept* obtained by deleting this edge and adding the edge $v_{i+t}v_{j-t}$ where t is the distance of v_i from v_{i+t} and the distance of v_j from v_{j-t} . Let P be a parallel transformation of T that contains P_1 as one of the constituent *ept*'s. Since $v_{i+t}v_{j-t}$ is an edge in the path $P(T)$ it follows that $i + t + 1 = j - t \Rightarrow j = i + 2t + 1$. Therefore i and j are of opposite parity.

The value of the edge $v_i v_j$ is

$$\begin{aligned} f^{\times}(v_i v_j) &= f^{\times}(v_i v_{i+2t+1}) \\ &= f(v_i).f(v_{i+2t+1}) \qquad (6) \end{aligned}$$

If i is odd and $1 \le i \le n$, then

$$\begin{aligned} f(v_i).f(v_{i+2t+1}) &= a^{[(i-1)/2]d}.a^{k+(q-1)d+[(i+2t+1-2)/2]d} \\ &= a^{k+(q-1)d+(i+t-1)d}. \qquad (7) \end{aligned}$$

If i is even and $2 \leq i \leq n$, then

$$f(v_i).f\left(v_{i+2t+1}\right) = a^{k(q-1)d+[(i-2)/2]d}.a^{[(i+2t+1)/2]d}$$
$$= a^{k+(q-1)d+(i+t-1)d} \qquad (8)$$

Therefore, from (6), (7), (8), we get

$$f^{\times}(v_i v_j) = a^{k+(q-1)d+(i+t-1)d.}\ \forall\, i. \qquad (9)$$

The value of the edge $v_{i+t}v_{j-t}$ is given by

$$f^{\times}(v_{i+t}v_{j-t}) = f(v_{i+t}).f(v_{j-t}).$$
$$= f(v_{i+t}).f(v_{i+t+1}) \qquad (10)$$

If $i+t$ is odd, then

$$f\left(v_{i+t}\right).f\left(v_{i+t+1}\right) = a^{[(i+t-1)/]d}.a^{k+(q-1)d+[(i+t+1-2)/2]d}$$
$$= a^{k+(q-1)d+(i+t-1)d}. \qquad (11)$$

If $i+t$ is even, then

$$f\left(v_{i+t}\right).f\left(v_{i+t+1}\right) = a^{[(i+t+1-1)/]d}.a^{k+(q-1)d+[(i+t-2)/2]d}$$
$$= a^{k+(q-1)d+(i+t-1)d}. \qquad (12)$$

Therefore, from (10), (11), (12) , we get

$$f^{\times}(v_{i+t}v_{j-t}) = a^{k+(q-1)d+(i+t-1)d}. \qquad (13)$$

Thus, from (9) and (13), we get

$$f^{\times}(v_i v_j) = f^{\times}(v_{i+t}v_{j-t}).$$

Also, one can verify that the labeling as defined by f is a $(a^{(k+(q-1)d},a)$-geometric labeling of T for all positive integers k, d.◇

Figure 8 shows a $(3^{14}$, 3)-geometric labeling of a T_p-tree, using the labeling given in the proof of the theorem 10.

Theorem 11. *The subdivision tree $S(T)$ of a T_p- tree is geometric.*

Proof. Let T be a T_p-tree with n vertices and q edges. By the definition of a T_p- tree there exists a parallel transformation P of T so that we get P(T). Denote the succession vertices of $P(T)$ as $v_1, v_2, ..., v_n$ starting from one pendant vertex of $P(T)$ right up to other and preserve the same for T. Now construct the subdivision tree $S(T)$ of T by introducing exactly one vertex between every edge $v_i v_j$ of T and denote the vertex as $v_{i,j}$. Let

$v_m x\ v_h x,\ x = 1,\ 2, ..., z$ be the z transformable edges of T with $m^x < h^x + 1$ for all x. Let t_x be the path length from the vertex $v_m x$ to the corresponding pendant vertex decided by the transformable edge $v_m x\ v_h x$ of T.

Define a function $f : V(S(T)) \to N$ by

$f(v_i) = t^{k+(q-1)d}$ if $i = 1, 2, \ldots, n$.

$f(v_{i,j}) = t^{(i-1)d}$ if $j \neq i + 1$.

$f(v_{i,j}) = t^{id}$ if $j = i + 1$ and $i = m^c, m^c + 1, \ldots, m^c + t_c - 1,$

$c = 1, 2, \ldots, z$

$f(v_{i,j}) = t^{(i-1)d}$ if $j = i + 1$ and $i \neq m^c, \neq (m^c + 1), ...,$

$\neq (m^c + t_c - 1), c = 1, 2, \ldots, .z$

where k and d are positive integers and $2q$ is the number of edges of $S(T)$. Then one can verify that f is a $(a^{k+(q-1)d}, a)$ geometric labeling of $S(T)$ for all nonnegative integers k, d with k, d not simultaneously zero. $\diamond$

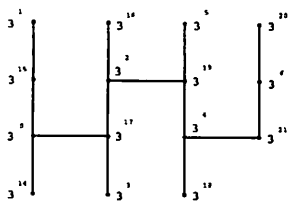

Fig. 8. A $(3^{14}, 3)$-geometric labeling of a T_p tree.

Figure 9 given below is an illustrative example of a $(2^{14}, 2)$ – geometric labeling of a $S(T)$ using the labeling given in the proof of the theorem 11.

We strongly believe that a tree admits a geometric labeling for at least one value of a , r and hence propose the following:

Conjecture : All trees are geometric.

References

1. B.D. Acharya and S.M. Hegde, *On certain vertex valuations of graphs*, Indian J. Pure and Applied Math., 22(7) (1991) 553-560.

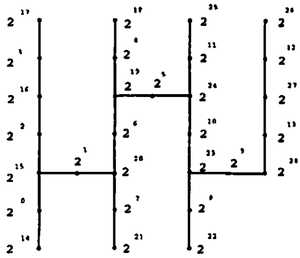

Fig. 9. A $(2^{14}, 2)$ -geometric labeling of a $S(T)$

2. B.D. Acharya, Elementary parallel transformations of graphs, AKCE International Journal of Graphs and Combinatorics, Vol. 1, 2004.

3. J.A. Gallian, *A dynamic Survey of Graph Labeling*, The Electronic Journal of Combinatorics, 6(2007), #DS6, 1-180.

4. F. Harary, *Graph Theory*, Addison Wesley, Massachusetts, 1972.

5. D.B. West, *Introduction to Graph Theory*, Second Edition, Prentice Hall, 2001.