P_h-supermagic labelings of some trees

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Abstract. Let P_h be a path on h vertices. A simple graph $G = (V, E)$ admits a P_h-*covering* if every edge in E belongs to a subgraph of G that is isomorphic to P_h. G is called P_h-*magic* if there is a total labeling $f : V \cup E \rightarrow \{1, 2, \ldots, |V| + |E|\}$ such that for each subgraph $H' = (V', E')$ of G that is isomorphic to P_h, $\sum_{v \in V'} f(v) + \sum_{e \in E'} f(e)$ is constant. When $f(V) = \{1, 2, \ldots, |V|\}$, we say that G is P_h-*supermagic*.
In this paper, we study some P_h-supermagic trees. We give some sufficient or necessary conditions for a tree being P_h-supermagic. We also consider the P_h-supermagicness of special type of trees, namely shrubs and banana trees.

Keywords: H-covering, H-supermagic, Total labeling

1 Introduction

Let H be a graph. A graph $G = (V, E)$ admits an H-*covering* if every edge in E belongs to a subgraph of G that is isomorphic to H. A total labeling of G is an injective function $f : V \cup E \rightarrow \{1, 2, \ldots, |V| + |E|\}$. We define that G is H-*magic* if there exists a total labeling $f : V \cup E \rightarrow \{1, 2, \ldots, |V| + |E|\}$ such that for each subgraph $H' = (V', E')$ of G isomorphic to H, $\sum_{v \in V'} f(v) + \sum_{e \in E'} f(e)$ is constant. Additionally, if $f(V) = \{1, 2, \ldots, |V|\}$, then G is called H-*supermagic*. The sum of all label vertices and edges on G is denoted by $\sum f(G)$.

The H-(super)magic coverings was first studied by Gutiérrez and Lladó [4] in 2005. They proved that the star S_n and the complete bipartite graph

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$K_{n,m}$ are S_h-supermagic graphs for some h. They also proved that the path P_n and the cycle C_n are P_h-supermagic graphs for some h. In 2006, Lladó and Moragas [7] studied some C_n-magic graphs. They proved that the wheel W_n, the prim $C_n \times K_2$, the book $K_{1,n} \times K_2$, and the windmill $W(r, k)$ are C_h-magic graphs for some h.

In case $H = K_2$, the H-supermagic graph is also called super edge-magic graph. The notion of super edge-magic graph was introduced by Enomoto et al [1] as an extension of the magic valuation given by Rosa [10], Kotzig and Rosa [5]. There are many graphs that are super edge magic graphs, for instance: C_n and $K_{m,n}$ [1], the fan $\hat{F}_n$ and the ladder L_n [2], any tree with at most 17 vertices [6], the generalized Petersen graph $P(n, m)$ [2] and [8], the fire cracker, the banana tree, and the unicyclic graph [11], a subdivision of $K_{1,3}$ [9]. For further information, see [12] and [3].

In this paper, we study P_h-*supermagic* labelings of some trees. We derive a sufficient condition for a tree being P_h-supermagic. We also present a necessary condition for a graph being P_3-supermagic. Besides that, we characterize the P_h-supermagicness of shrubs and its subdivisions, and banana trees.

2 Main Results

Theorem 1. *Let $G = (V, E)$ be a tree that admits a P_h-covering, for some fixed h. If every subgraph P_h of G contains a fixed vertex c, then G is P_h-supermagic.*

Proof. Let $V = \{c, v_i | 1 \le i \le n\}$. For any $v_i \in V$ and $v_i \neq c$, denote by e_i the edge which in a (c, v_i)-path and incident to v_i. Let $t = |V \cup E|$. Partition the set $\{2, 3, 4, ..., t\}$ into 2−sets such that the sum of the elements of each 2−set is $t + 2$.

Define a total labeling f as follows. $f(c) = 1$. Use all these 2−sets to label all $\{v_i, e_i\}$ in any order so that $f(v_i) + f(e_i) = t + 2$ and $f(v_i) < f(e_i)$ for $i = 1, 2, ..., n$. Hence, all vertices receive the smallest labels.

Since each P_h contains c, we have that $\sum f(P_h) = (h - 1)(t + 2) + 1$. Therefore, G is P_h-supermagic. □

Theorem 2. *If G is P_3-magic then G contains no subgraph of Figure 1.*

Proof. By a contradiction, assume G is a P_3-magic and G contains the subgraph of Figure 1. So, there exists a total labeling f such that $\sum f(P_3)$ is constant for every subgraph P_3. Let

$$f(v_i) = a_i, 1 \le i \le 6 \text{ and } f(e_i) = b_i, 1 \le i \le 5.$$

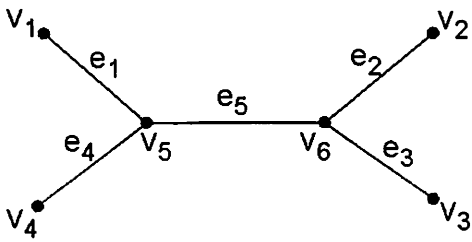

Fig. 1.

Then, $a_1+b_1+a_5+b_5+a_6 = a_2+b_2+a_6+b_5+a_5$. Hence, $a_1+b_1 = a_2+b_2$.

By similar ways, we obtain $a_2 + b_2 = a_3 + b_3 = a_4 + b_4$. Therefore, $a_5 = a_6$. We get a contradiction. Therefore, the theorem holds. □

2.1 Shrubs

A *shrub* $\check{S}(m_1, m_2, ..., m_n)$ is a graph obtained from a star S_n by connecting each leaf v_i to m_i new vertices, $i = 1, 2, ..., n$. In this subsection we study the P_h-supermagicness of shrubs and their subdivisions. By applying Theorem 1 and Theorem 2, we obtain the following corrolary.

Corollary 1. *$\check{S}(m_1, m_2, ..., m_n)$ is P_h-supermagic for $4 \leq h \leq 5$, but it is not P_3-supermagic for $n \geq 3$.*

As an illustration, in Figure 2, we present a P_h-supermagic labeling of $\check{S}(2,2,3,3,4)$ for $4 \leq h \leq 5$.

Theorem 3. *$\check{S}(m_1, m_2)$ is P_3-supermagic.*

Proof. Let $G \simeq \check{S}(m_1, m_2)$, with

$$V(G) = \{c\} \cup \{v_i, v_i^j | 1 \leq i \leq 2, 1 \leq j \leq m_i\},$$
$$E(G) = \{cv_i, v_iv_i^j | 1 \leq i \leq 2, 1 \leq j \leq m_i\}.$$

Then, let $t = 2(m_1 + m_2) + 5$ and define the labeling f on G as follows.

$$f(c) = m_1 + 2;$$
$$f(v_1) = f(c) + 1, \quad f(cv_1) = (t+2) - f(v_1);$$
$$f(v_2) = f(c) - 1, \quad f(cv_2) = (t+1) - f(v_2);$$
$$f(v_1^j) = j, \quad f(v_1v_1^j) = (t+1) - f(v_1^j) \quad \text{for } j \in [1, m_1];$$
$$f(v_2^j) = f(v_1) + j, \quad f(v_2v_2^j) = (t+2) - f(v_2^j) \quad \text{for } j \in [1, m_2].$$

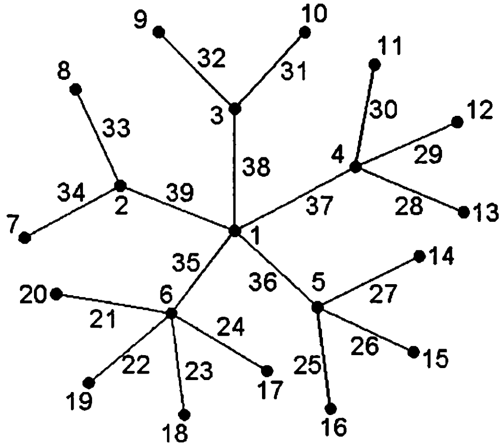

Fig. 2. A P_h-supermagic labeling of $\check{S}(2,2,3,3,4)$ for $4 \leq h \leq 5$.

Clearly that $f(V(G)) \in \{1,2,3,...,|V(G)|\}$ and every $\sum f(P_3) = 2t + m_1 + 5$. Therefore, G is P_3-supermagic. □

For illustration, we present a P_3-supermagic labeling of $\check{S}(4,3)$ in Figure 3.

In the next theorem we consider the P_h-supermagicness of the balanced k-subdivision of a shrub. Let $\check{S}_k(m_1, m_2)$ be a subdivision of $\check{S}(m_1, m_2)$ by inserting k new vertices, for $k \geq 1$, in each edge of $\check{S}(m_1, m_2)$.

Theorem 4. *$\check{S}_k(m_1, m_2)$ is P_{2k+3}-supermagic.*

Proof. Let denote all vertices and edges in $G \simeq \check{S}_k(m_1, m_2)$ as follows.

$$V(G) = \{c\} \cup \{v_{i,a}, v_i^{j,a} | 1 \leq i \leq 2, 1 \leq j \leq m_i, 0 \leq a \leq k\},$$
$$E(G) = \{cv_{i,0}, v_{i,(a-1)}v_{i,a}, v_{i,k}v_i^{j,0}, v_i^{j,(a-1)}v_i^{j,a} | 1 \leq i \leq 2, 1 \leq j \leq m_i, 1 \leq a \leq k\}.$$

Let us denote $e_{i,0} = cv_{i,0}$, $e_{i,a} = v_{i,(a-1)}v_{i,a}$, $e_i^{j,0} = v_{i,k}v_i^{j,0}$, and $e_i^{j,a} = v_i^{j,(a-1)}v_i^{j,a}$. Let t be the number of vertices and edges of G. Thus, $t = (2k+2)(m_1+m_2+2)+1$. Next, define the labeling f on G as follows.

- $f(c) = m_1 + km_1 + k + 2$;
- For $a \in [0,k]$:
  $f(v_{1,a}) = f(c) + (a+1)$, $f(e_{1,a}) = (t+2) - f(v_{1,a})$;

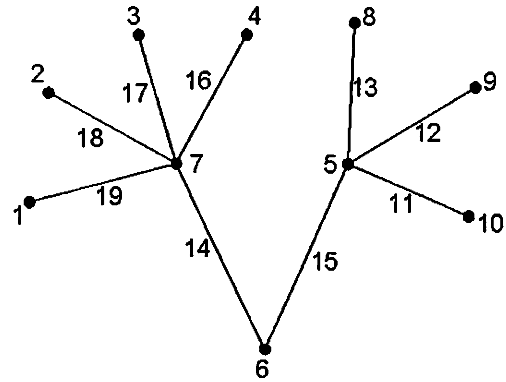

Fig. 3. A P_3-supermagic labeling of $\check{S}(4,3)$

$f(v_{2,a}) = f(c) - (a+1)$, $f(e_{2,a}) = (t+1) - f(v_{2,a})$;
$f(v_1^{j,a}) = j + m_1 a$, $f(e_1^{j,a}) = (t+1) - f(v_1^{j,a})$ for $j \in [1, m_1]$;
$f(v_2^{j,a}) = f(v_{1,k}) + (j + m_2 a)$, $f(e_2^{j,a}) = (t+2) - f(v_2^{j,a})$ for $j \in [1, m_2]$.

It is easy to see that $f(V(G)) \in \{1, 2, 3, ..., |V(G)|\}$ and $\sum f(P_{2k+3}) = (k+1)(2t + m_1 + 4) + 1$ for each subgraph P_{2k+3} of G. Therefore, G is P_{2k+3}-supermagic. □

In Figure 4 we present an example for P_7-supermagicness of shrub $\check{S}_2(3,2)$.

2.2 Banana Trees

In this subsection we study the P_h-supermagicness of banana trees. A *banana tree* $Bt(m_1, m_2, ..., m_n)$ is the tree obtained by joining a vertex c to one leaf vertex of each star in a family of disjoint stars $S_{m_1}, S_{m_2}, ..., S_{m_n}$. By applying Theorem 1, we have the following corollary.

Corollary 2. *For* $n \geq 1$, $Bt(m_1, m_2, ..., m_n)$ *is* P_h*-supermagic with* $4 \leq h \leq 7$.

As an illustration, in Figure 5, we present a P_h-supermagic labeling of $Bt(3,3,3,4)$ for $4 \leq h \leq 7$.

Swaminathan and Jeyanthi [11] proved that $Bt(m_1, m_2, ..., m_n)$ is P_2-supermagic. Next, we consider about a P_3-supermagicness of $Bt(m_1, m_2)$.

Theorem 5. $Bt(m_1, m_2)$ *is* P_3*-supermagic.*

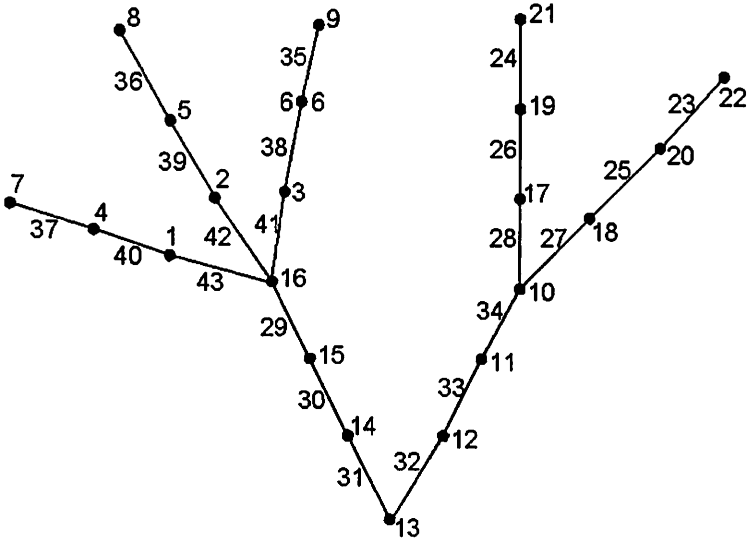

Fig. 4. A P_7-supermagic labeling of $\check{S}_2(3,2)$

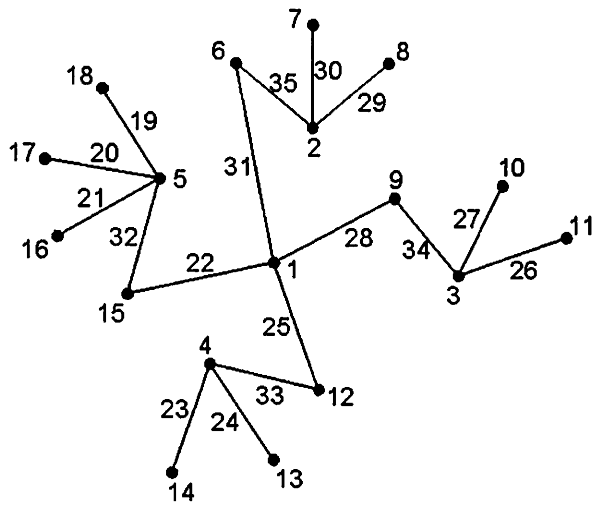

Fig. 5. A P_h-supermagic labeling of $Bt(3,3,3,4)$ for $4 \leq h \leq 7$

Proof. Let $G \simeq Bt(m_1, m_2)$ with

$$V(G) = \{c\} \cup \{v_i, v_i^j | 1 \le i \le 2, 1 \le j \le m_i\},$$
$$E(G) = \{cv_i^1, v_i v_i^j | 1 \le i \le 2, 1 \le j \le m_i\}.$$

Let $t = 2(m_1 + m_2) + 5$. Next, we define a total labeling f on G as follows.

$$f(c) = m_1 + 2, \;\; f(v_1) = f(c) + 1, \;\; f(v_2) = f(c) - 1;$$
$$f(cv_i^1) = (t + i) - f(c) \;\; \text{for } i \in [1, 2];$$
$$f(v_1^j) = f(v_2) - j, \;\; f(v_1 v_1^j) = (t + 1) - f(v_1^j) \;\; \text{for } j \in [1, m_1];$$
$$f(v_2^j) = f(v_1) + j, \;\; f(v_2 v_2^j) = (t + 2) - f(v_2^j) \;\; \text{for } j \in [1, m_2].$$

Clearly that $f(V(G)) \in \{1, 2, 3, ..., |V(G)|\}$ and $\sum f(P_3) = 2t + m_1 + 5$ for every subgraph P_3 of G. Therefore, G is P_3-supermagic. □

For illustration, we present a P_3-supermagic labeling of $Bt(5, 4)$ in Figure 6.

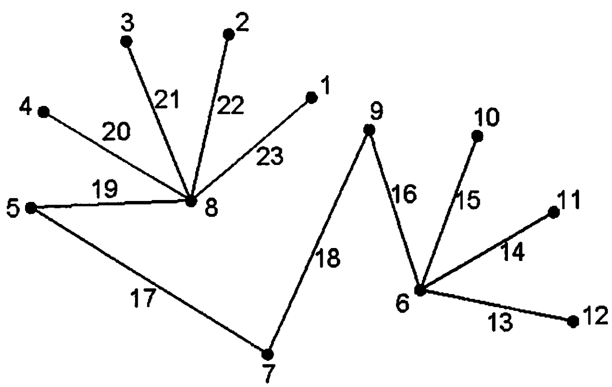

Fig. 6. A P_3-supermagic labeling of $Bt(5, 4)$.

3 Open Problems

We conclude the paper with two open problems.

Open Problem 1 *Is there a P_2-supermagic labeling on $\check{S}(m_1, m_2, ..., m_n)$?*

Open Problem 2 *For $n \geq 3$, is there a P_3-supermagic labeling on $Bt(m_1, m_2, ..., m_n)$?*

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