Magic Digraphs

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Abstract

In this paper we extend the idea of magic labeling to directed graphs. In particular, a *magic labeling of a digraph* is the directed analog of a vertex-magic total labeling. Some elementary results are obtained and some infinite families of magic digraph labelings are exhibited.

1 Definitions and notation

Digraphs

We assume the standard ideas of graph theory; see, for example, [10]. A *digraph* D can be described as a collection $V = V(D)$ of *vertices* together with a collection $E = E(D)$ of ordered pairs of vertices called *arcs* or *directed edges*; in the arc (x, y), vertices x and y are called the *tail* and *head* respectively. If there is an arc (x, y), we may write $x \mapsto y$. For our purposes digraphs are finite; the definition implies that there cannot be two arcs with the same tail and the same head ("there are no repeated arcs").

For a given vertex x, $A(x)$ denotes the set of all vertices y such that (x, y) is an arc ("*after* x"), while $B(x)$ denotes the set of all vertices z such that (z, x) is an arc ("*before* x"); in symbols,

$$A(x) = \{y : x \mapsto y\}, \quad B(x) = \{z : z \mapsto x\}.$$

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$|A(x)|$ and $|B(x)|$ are respectively the *outdegree* and *indegree* of x; common notations are $od(x)$ and $id(x)$ respectively. A vertex is a *source* if $B(x)$ is empty, and a *sink* if $A(x)$ is empty; an isolate is both a source and a sink. (Caution: some writers reserve "source" and "sink" for non-isolates).

It may happen that the digraphs formed from D by deleting an arc are all isomorphic, whichever arc is deleted. (This will be true if the automorphism group of D is transitive on the arcs.) In that case, we write $D-a$ for the resulting digraph.

We shall write $v = |V(D)|$, $e = |E(D)|$ and $n = |V(D) \cup E(D)| = v + e$.

Labels

A *labeling* or *valuation* λ of a graph or digraph D is a one-to-one mapping from $V(D) \cup E(D)$ onto the set of integers $\{1, 2, \ldots n\}$. Graceful labelings — those for which the differences $\lambda(x) - \lambda(y)$ are different for different edges xy — were introduced by Rosa [7] in 1966, and several other labelings and their arithmetical properties were studied shortly thereafter (a good survey is [4]). In particular, *magic* denotes the requirement that certain sums of labels have the same value; this idea was introduced (with labels ranging over the real numbers) by Sedláček [8] in 1963; see also [5], [6] and the survey [9].

Nearly all work on labelings has concentrated on the undirected case. *Graceful* labelings of directed graphs were introduced in 1985 [3] and have been studied in a few papers. Ahmed [2] discusses mappings with a magic property, but the labels are unrestricted non-negative integers and only arcs are labeled, so those structures are more closely related to integer network flows. In this note we apply the idea of magic labeling to digraphs.

Magic labelings of digraphs

We define a *magic labeling of a digraph D* to be a labeling in which all the sums

$$m_A(x) = \left[\lambda(x) + \sum_{x \mapsto y} \lambda(x, y)\right]$$

and all the sums

$$m_B(x) = \left[\lambda(x) + \sum_{z \mapsto x} \lambda(z, x)\right]$$

are constant, independent of the choice of x. A digraph with a magic labeling will be called a *magic digraph*.

2 Necessary conditions

The divisibility condition

If you add the values $m_A(x)$ (or the values $m_B(x)$) for all vertices x, you will add every label exactly once. So

$$\begin{aligned}\sum_{x\in V(D)} m_A(x) = \sum_{x\in V(D)} m_B(x) &= \sum_{x\in V(D)} \lambda(x) + \sum_{(x,y)\in E(D)} \lambda(x,y) \\ &= \sum_{i=1}^{n} i = \tfrac{1}{2}n(n+1).\end{aligned}$$

That is, the two constants are equal. We shall write $m(\lambda)$, or simply m, for this common constant sum. As m must be an integer, we deduce:

Theorem 1 *In any magic digraph,* $2v$ *must divide* $(v+e)(v+e+1)$.

Small degrees

Suppose x were an isolated vertex in a magic digraph. Necessarily $\lambda(x) = m_\lambda$ for any magic labeling λ. This must be the largest label, so $n = m$. We immediately see from the uniqueness of labels that there can be no more than one isolate.

Now consider the element with label $m-1$. If it were an arc, both endpoints cannot be labeled 1, so one or other endpoint must have label 2 or greater, yielding at least one sum greater than m, which is impossible. So it is a vertex, y say. This vertex cannot be isolated. The sum of labels into y must be 1, so y is the head of exactly one arc, which receives label 1. But the same reasoning shows that y is the tail of an arc labeled 1. This repetition is impossible. So there are no isolates.

If x is any vertex, the sum of labels on arcs into x must equal the sum of labels on arcs out of x. It follows that a magic digraph cannot contain a source (it cannot be an isolate; but if it is not, $m_A(x) > \lambda(x) = m_B(x)$) or a sink.

Finally, if x had indegree and outdegree each equal to 1, the label on the two arcs touching x would be equal. This is also impossible. In summary:

Lemma 2 *In any magic digraph, each vertex lies on at least three arcs, not all directed into or all directed out of the vertex.*

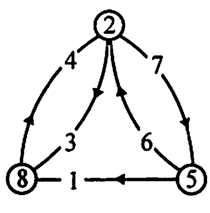
Theorem 3 *In any magic digraph,* $e \geq 3v/2$.

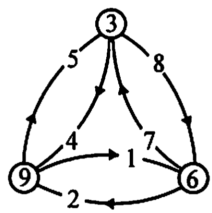
3 Some small cases

The one-vertex digraph is trivially magic, and Theorem 3 excludes 2-vertex cases.

Three vertices

From Theorem 3, a three-vertex magic digraph must have at least five arcs. As there are no repeated arcs, the maximum number of arcs is six. So there are two possible digraphs, and both have magic labelings:





These labelings are conveniently represented by arrays: assuming the vertices to have been labeled x_1, x_2, x_3 in some order, the (i, i) entry shows the label of x_i, while the (i, j) entry shows the label of arc (x_i, x_j) when $i \neq j$. Entry 0 means there is no link. We shall refer to this array as a *labeling matrix*.

2	7	3
6	5	1
4	0	8

3	8	4
7	6	2
5	1	9

Four vertices

From Theorem 3, a four-vertex magic digraph must have at least six arcs; the maximum is 12. Taking Theorem 1 into account, there must be 11 or 12 arcs. Again there are two possible digraphs, and again both are magic. Possible arrays are

6	3	12	9
0	5	10	15
11	14	1	4
13	8	7	2

7	4	13	10
1	6	11	16
12	15	2	5
14	9	8	3

Five vertices

In this case, a magic digraph could have 9, 10, 14, 15, 19 or 20 arcs. All cases are possible:

10	4	0	0	7
9	11	1	0	0
0	6	12	3	0
0	0	8	13	0
2	0	0	5	14

11	5	0	0	8
10	12	2	0	0
0	7	13	4	0
0	0	9	14	1
3	0	0	6	15

9	10	19	0	0
0	6	1	14	17
15	0	13	8	2
11	4	0	16	7
3	18	5	0	12

10	11	20	1	8
0	7	2	15	18
16	0	14	9	3
12	5	0	17	8
4	19	6	0	13

5	3	11	19	22
26	6	4	12	15
16	24	7	0	13
14	17	20	8	1
2	10	18	21	9

6	4	12	20	23
24	7	5	13	16
17	25	8	1	14
15	18	21	9	2
3	11	19	22	1

4 Some families of magic digraphs

We conclude with some infinite families of magic digraphs.

A digraph is *regular* if every vertex has the same indegree and every vertex has the same outdegree. Obviously the two common degrees must be equal; if the common value is d, we have a *regular digraph of degree d*. Such a digraph satisfies $e = dk$.

Suppose D is a regular digraph with a magic labeling, and suppose label 1 appears on an arc: say $\lambda(x_i, x_j) = 1$. Subtract 1 from each label. The result is a magic labeling of the digraph $D - (x_i, x_j)$ obtained from D by deleting arc (x_i, x_j). Several of the examples in the preceding section were obtained by this method, which we shall call the *subtraction technique*.

Complete digraphs

The complete digraph DK_v has v vertices and all $v(v-1)$ possible arcs. The labeling matrix of a magic labeling of DK_v is a $v \times v$ array with entries $1, 2, \ldots, v^2$ in which all row and column sums are equal. Such arrays exist if and only if $v > 2$; in fact, one can use a magic square of order v. (The condition here is in fact a little less stringent, as it is not required that the diagonal sums equal the magic constant; we only need constant row and column sums.) For more on magic squares, see, for example, [1].

Suppose A is a labeling matrix for DK_v. Permute rows and columns until the entry 1 is in row i, column j, where $i \neq j$. The row and column sums are still constant. So DK_v has a magic labeling with label 1 on an arc, and we can apply the subtraction technique to obtain a magic labeling of $DK_v - a$.

Double cycles

The *double cycle* DC_v is constructed from a v-cycle by replacing each edge by a pair of arcs: edge xy gives rise to arcs (x, y) and (y, x). It follows from Theorem 1 that DC_v is not magic when v is even.

Suppose v is odd: say $v = 2k + 1$. Consider the $(2k+1) \times (2k+1)$ array A defined by

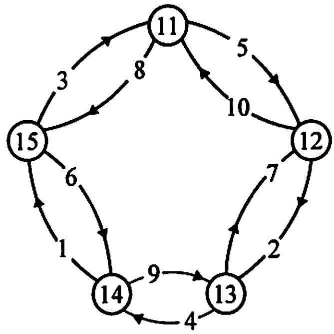
$$\begin{aligned} a_{ii} &= 4k + i, \\ a_{i,i+1} &= 2k + 2 - \tfrac{i+1}{2}, i \text{ odd}, \\ &= k + 1 - \tfrac{i}{2}, i \text{ even}, \\ a_{i+1,i} &= 4k + 3 - \tfrac{i+1}{2}, i \text{ odd}, \\ &= 3k + 2 - \tfrac{i}{2}, i \text{ even} \end{aligned}$$

(with $a_{1,2k+1} = 3k + 2$ and $a_{2k+1,1} = k + 1$), that is

$$A = \begin{array}{|ccccccc|} 4k+3 & 2k+1 & 0 & 0 & \cdots & 0 & 3k+2 \\ 4k+2 & 4k+4 & k & 0 & \cdots & \cdots & 0 \\ 0 & 3k+1 & 4k+5 & 2k & 0 & \cdots & 0 \\ 0 & 0 & 4k+1 & 4k+6 & k-1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 3k+3 & 6k+2 & 1 \\ k+1 & 0 & \cdots & \cdots & 0 & 2k+2 & 6k+3 \end{array}$$

This array contains all the positive integers from 1 to $6k + 3$ once each, and has constant row and column sums $9k + 6$. So it is the labeling matrix of a DC_{2k+1} with magic constant $9k + 6$. Therefore all double cycles of odd order are magic. The label 1 appears on an arc, so $DC_{2k+1} - a$ is also magic.

Here is an example of this construction:



Double cycles plus cycles

We have investigated two infinite families of regular digraphs that derive from double cycles. The first are formed from double cycles of odd order by adding a directed spanning cycle. If the double cycle has vertices $x_1, x_2, \dots, x_n$, and the directed cycle is $(x_1, x_3, x_5, \dots, x_n, x_2, x_4, \dots, x_{n-1})$, a suitable labeling matrix is

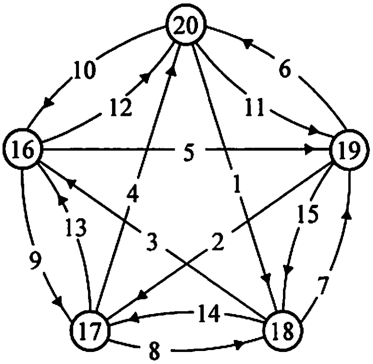
$4n$	$2n+1$	1	0	$\cdots$	$\cdots$	0	$2n$
$n+1$	$4n-1$	$3n$	2	0	$\cdots$	$\cdots$	0
0	$n+2$	$4n-2$	$3n-1$	3	0	$\cdots$	$\cdots$
0	0	$n+3$	$4n-3$	$3n-2$	4	$\cdots$	$\cdots$
$\vdots$	$\ddots$	$\ddots$	$\ddots$	$\ddots$	$\ddots$	$\ddots$	$\vdots$
$2n+2$	n	0	$\cdots$	$\cdots$	0	$2n-1$	$3n+1$

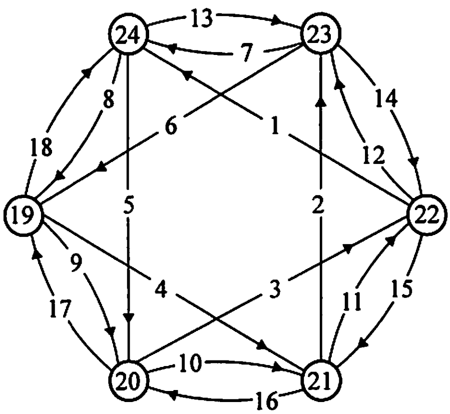
For even n with vertices $x_1, x_2, \dots, x_n$, we form a digraph by adding cycles $(x_{n-1}, x_{n-3}, \dots, x_3, x_1)$ and $(x_n, x_{n-2}, \dots, x_4, x_2)$. The matrix

$4n$	$2n+1$	0	$\cdots$	$\cdots$	0	$n-1$	$n+2$
$n+1$	$4n-1$	$2n+2$	0	$\cdots$	$\cdots$	0	n
1	$2n$	$4n-2$	$2n+3$	0	$\cdots$	$\cdots$	0
0	2	$2n-1$	$4n-3$	$2n+4$	0	$\cdots$	$\cdots$
$\vdots$	$\ddots$	$\ddots$	$\ddots$	$\ddots$	$\ddots$	$\ddots$	$\vdots$
$3n$	0	$\cdots$	$\cdots$	0	$n-2$	$n+3$	$3n+1$

provides a labeling. In both cases the subtraction technique may be applied; the resulting magic digraph is formed by deleting one of the new edges.

Here are examples of these two families:





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