How many graphs are super edge-magic? An asymptotic approach.

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Abstract. We study the number of super edge-magic (bipartite) graphs from an asymptotic point of view

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1 Introduction

Unless otherwise specified, we will follow Chartrand and Lesniak notation [2].

In 1966, Alexander Rosa [9] introduced graceful labelings and graceful graphs as follows.

Given a graph G of size q, an injective function $f : V(G) \to \{1, 2, \ldots, q\}$ is called a graceful labeling of G, if the function $\bar{f} : E(G) \to \mathbb{Z}$ defined by the rule $\bar{f} = |f(u) - f(v)|$ assigns different labels to the edges of G. If a graph G admits a graceful labeling then G is called a graceful graph.

In an unpublished paper, Erdős (see [6, 8] for further information) proved that almost all graphs are not graceful.

Later, in 1980, Graham and Sloan [8] defined the concepts of harmonious labelings and harmonious graphs as follows.

Let G be a graph of size q. An injective function $f : V(G) \to \mathbb{Z}_q$ is called a harmonious labeling of G if the function $\bar{f} : E(G) \to \mathbb{Z}_q$ defined by the rule

$$\bar{f} = (f(u) + f(v)) \bmod q.$$

assigns different labels to the edges of G. If G is a tree then the condition that f is injective is relaxed and exactly two vertex labels are allowed to be equal. If a graph admits a harmonious labeling then it is said to be a harmonious graph. Once again using a similar idea to the one introduced by Erdős, Graham and Sloan [8] proved that almost no graphs are harmonious.

In 1986, Acharga and Hedge [1] introduced the concept of strong indexable graphs, which was much later reintroduced, in 1996, under the name of super edge-magic labelings by Enomoto et al. [3]. Since this name has become the most popular one, we will choose this terminology for the rest of the paper.

Let G be any graph of order p and size q. Any bijective function

$$f : V(G) \cup E(G) \to \{1, 2, \ldots, p+q\}$$

with the properties that

$$f(V(G)) = \{1, 2, \ldots, p\} \text{ and } f(u) + f(uv) + f(v) = k$$

for every edge uv in $E(G)$ is called a super edge-magic labeling of G. If a graph admits a super edge-magic labeling we call the graph super edge-magic.

Another definition that will be very useful for the rest of the paper is the one of super edge-magic deficiency of a graph G by Figueroa-Centeno et al. in [5]. Let $M = \{n \in \mathbb{N} \cup \{0\} \mid G \cup nk, \text{ is super edge-magic}\}$. Then the super edge-magic deficiency of G, denoted by $\mu_s(G)$, is defined to be

$$\mu_s(G) = \begin{cases} min\ (M), & \text{if } M \neq \emptyset; \\ \infty, & \text{if } M = \emptyset. \end{cases}$$

In 1998, Figueroa-Centeno et al. [8] proved that almost no graphs are super edge-magic following the steps outlined next. First they proved in [8] that if a graph is either a tree or its size is at least as large as its order and is super edge-magic, then such graph is harmonious. Then using a result found by Gilbert [7] which guarantees that almost all graphs are connected, and the result that states that almost all graphs are not harmonious, they concluded that almost no graph is super edge-magic.

Here, we will provide a different approach in order to show that the following ratio

$$\frac{\text{number of non-isomorphic super edge-magic graphs}}{\text{number of non-isomorphic graphs}}$$

is asymptotically zero. We will also go a little bit further, and study the "conditional probability" that a graph is super edge-magic provided that it is bipartite. In order to conduct this study, we will recall some well known results about enumerative graph theory and mathematical analysis, that we state next. It is worthwhile to mention that we found out Lemmas 1 and 2 through personal communication with A. J. Schwenk and R. Stanley respectively. Lemma 3 is the well known Stolz's Theorem.

Lemma 1. *Consider the function*

$$g : \{i\}_{i=1}^{p} \times \{j\}_{j=0}^{\binom{p}{2}} \to \mathbb{N}$$

defined by the rule $g(i,j)=$ number of non isomorphic graphs of order i and size j. Then there exists $k \in \mathbb{N}$ such that, when $p > k$, the function

$$g\Big|_{\{p\}\times\{j\}_{j=0}^{\binom{p}{2}}} \longrightarrow \mathbb{N}$$

is unimodal, reaching its maximum at $\frac{1}{2}\binom{p}{2}$ if $\binom{p}{2}$ is even or at $\left\lfloor\frac{1}{2}\binom{p}{2}\right\rfloor, \left\lceil\frac{1}{2}\binom{p}{2}\right\rceil$ if $\binom{p}{2}$ is odd.

Lemma 2. *Let $p \in \mathbb{N}$, and let $p_1, p_2 \in \mathbb{N}$ such that $p = p_1 + p_2$. Also assume that G is a bipartite graph with vertex set $V = A \cup B$ where $|V| = p$, $|A| = p_1$ and $|B| = p_2$. Then the function*

$$f^V_{A,B} : \{0, 1, \dots, p_1p_2\} \longrightarrow \mathbb{N}$$

defined by the rule $f^V_{A,B}(n)=$ number of bipartite graphs with vertex set $V = A\cup B$ and size n is unimodal, reaching its maximum at $\frac{1}{2}p_1p_2$ if p_1p_2 is even or at $\left\lfloor\frac{1}{2}p_1p_2\right\rfloor, \left\lceil\frac{1}{2}p_1p_2\right\rceil$ if p_1p_2 is odd.

Lemma 3 (Stolz's Theorem). *Let a_n and b_n be two sequences such that*

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \infty \text{ and } \lim_{n\to\infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = L,$$

then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L.$$

To conclude this section, we will recall the following two results found in [3] and [5] respectively; and we will introduce Lemma 6.

Lemma 4. *If $G$is a super edge-magic (p,q)-graph then $q \leq 2p - 3$.*

Lemma 5. *If G is a super edge-magic bipartite (p,q)-graph then $q \leq 2p - 5$.*

Lemma 6. *Let G be a (p,q)-graph such that $\mu_s(G) = n$ then $q \leq 2p + 2n - 3$.*

Proof. If G is a super edge-magic (p,q)-graph, then we know by Lemma 4 that $q \leq 2p - 3$. Since $\mu_s(G) = n$, it follows that the graph $H \cong G \cup nK$, is super edge-magic. Now, $|V(H)| = p + n$, and $|E(H)| = q$. Therefore, $q \leq 2p + 2n - 3$.

2 Almost no graph is super edge-magic

In this section, we prove that almost no graphs are super edge-magic by computing the ratio

$$\frac{\text{number of super edge-magic non-isomorphic graphs}}{\text{number of non-isomorphic graphs}}$$

in an asymptotic way.

However, in order to accomplish this goal, we need to introduce the following definitions.

Let $p \in \mathbb{N}$ and define S_p to be the set that contains all non-isomorphic graphs of order less than or equal to p. Also define the set S_p^* to be the set that contains all non-isomorphic graphs of order less than or equal to p and size upper bounded by two times the order minus 3.

Theorem 1.

$$\lim_{p\to\infty}\frac{|S_p^*|}{|S_p|}=0$$

Proof. Through this proof, we let $g(1,-1)=0$.

Fix $p\in\mathbb{N}$. We compute $|S_i^*|$ for each $i\in\{1,2,\ldots,p\}$.

Fix $i,j\in\mathbb{N}$ and denote the number of non isomorphic graphs of order i and size j by $g(i,j)$ when i, j are both non negative.

Then,

$$|S_p^*|=\sum_{i=1}^{p}\sum_{j=0}^{2i-3}g(i,j)\text{ and }|S_p|=\sum_{i=1}^{p}\sum_{j=0}^{\frac{i^2-i}{2}}g(i,j).$$

Then,

$$\begin{aligned}\lim_{p\to\infty}\frac{|S_p^*|}{|S_p|} &= \lim_{p\to\infty}\frac{\sum_{i=1}^{p}\sum_{j=0}^{2i-3}g(i,j)}{\sum_{i=1}^{p}\sum_{j=0}^{\frac{i^2-i}{2}}g(i,j)}\\ &= \lim_{p\to\infty}\frac{\sum_{i=1}^{k}\sum_{j=0}^{2i-3}g(i,j)}{\sum_{i=1}^{p}\sum_{j=0}^{\frac{i^2-i}{2}}g(i,j)}+\lim_{p\to\infty}\frac{\sum_{i=k+1}^{p}\sum_{j=0}^{2i-3}g(i,j)}{\sum_{i=1}^{p}\sum_{j=0}^{\frac{i^2-i}{2}}g(i,j)}\\ &= \lim_{p\to\infty}\frac{\sum_{i=k+1}^{p}\sum_{j=0}^{2i-3}g(i,j)}{\sum_{i=1}^{p}\sum_{j=0}^{\frac{i^2-i}{2}}g(i,j)}.\end{aligned}$$

Where k is the minimum integer for which the function $g(k,j)$ is unimodal (see Lemma 1), hence $\sum_{i=1}^{k}\sum_{j=0}^{2i-3}g(i,j)$ is constant.

Since the function $g\Big|_{\{l\}\times\{j\}_{j=0}^{\binom{i}{2}}}$ is unimodal when $l>k$ and its maximum appears either at $\frac{1}{2}\binom{i}{2}$ if $\binom{i}{2}$ is even or at $\lfloor\frac{1}{2}\binom{i}{2}\rfloor,\lceil\frac{1}{2}\binom{i}{2}\rceil$ if $\binom{i}{2}$ is odd, it follows that

$$g(i,j)\le g(i,2i-2)\text{ when }i\ge k+1\text{ and }j\in\{0,1,\ldots,2i-2\}.$$

Therefore,

$$\sum_{j=0}^{2i-3}g(i,j)\le(2i-2)g(i,2i-2),$$

hence

$$\sum_{i=k+1}^{p}\sum_{j=0}^{2i-3}g(i,j)\le\sum_{i=k+1}^{p}(2i-2)g(i,2i-2). \tag{1}$$

Also,

$$\sum_{j=0}^{\frac{i^2-i}{2}}g(i,j)\ge\sum_{j=2i-2}^{\frac{i^2-5i+6}{2}}g(i,j)$$

where $\frac{i^2-5i+6}{2}$ is equal to $\frac{i^2-i}{2}-2i-3$.

Now, if $i \geq k+1$, by unimodality, we have that

$$\sum_{j=2i-2}^{\frac{i^2-5i+6}{2}} g(i,j) \geq \frac{i^2-9i+8}{2} g(i,2i-2)$$

then

$$\sum_{i=k+1}^{p} \sum_{j=0}^{\frac{i^2-i}{2}} g(i,j) \geq \sum_{i=k+1}^{p} \frac{i^2-9i+8}{2} g(i,2i-2). \qquad (2)$$

Thus, by (1) and (2)

$$\lim_{p\to\infty} \frac{\sum_{i=k+1}^{p} \sum_{j=0}^{2i-3} g(i,j)}{\sum_{i=1}^{p} \sum_{j=0}^{\frac{i^2-i}{2}} g(i,j)} \leq \lim_{p\to\infty} \frac{\sum_{i=k+1}^{p} (2i-2) g(i,2i-2)}{\sum_{i=k+1}^{p} \frac{i^2-9i+8}{2} g(i,2i-2)} \equiv \frac{\infty}{\infty}.$$

Then, using Stolz's Theorem, we obtain that the previous limit is 0. Thus

$$\lim_{p\to\infty} \frac{|S_p^*|}{|S_p|} \leq 0.$$

On the other hand, it is obvious that

$$\lim_{p\to\infty} \frac{|S_p^*|}{|S_p|} \geq 0.$$

Therefore,

$$\lim_{p\to\infty} \frac{|S_p^*|}{|S_p|} = 0$$

Now, notice that by Lemma 4, the set of super edge-magic graphs of order less than or equal to p, denoted by SEM_p, is a subset of S_p^*. Hence,

$$0 \leq \lim_{p\to\infty} \frac{|SEM_p|}{|S_p|} \leq \lim_{p\to\infty} \frac{|S_p^*|}{|S_p|} = 0.$$

Therefore,

$$\lim_{p\to\infty} \frac{|SEM_p|}{|S_p|} = 0$$

A similar argument to the one used in the proof of Theorem 1, together with Lemma 6, provides the following corollary.

Corollary 1. *Let $n \in \mathbb{N}$ and let SEM_p^n to be the set of graphs with super edge-magic deficiency less than or equal to n and order at most p. Then,*

$$\lim_{p\to\infty} \frac{|SEM_p^n|}{|S_p|} = 0.$$

3 Bipartite Graphs

Let p, p_1, p_2 be positive integers such that $p = p_1 + p_2$ and $p_1p_2 = o(p^2)$. Denote by S_{p,p_1,p_2} the set of all non isomorphic bipartite graphs of order p with vertex set V and fixed partition sets V_1 and V_2 such that $|V_1| = p_1$ and $|V_2| = p_2$ and let

$$S^*_{p,p_1,p_2} = \{G \in S_{p,p_1,p_2} : |E(G)| \leq 2p - 5\}$$

Theorem 2. *For every $\epsilon > 0$, there exists $k(\epsilon) \in \mathbb{N}$ such that if G is a bipartite graph with vertex set V, and bipartite sets V_1, V_2, with the property that*

$$|V| = p \geq K(\epsilon),\ |V_i| = p_i\ (i = 1, 2)\ \text{and}\ p_1p_2 = o(p^2).$$

Then

$$\frac{|S^*_{p,p_1,p_2}|}{|S_{p,p_1,p_2}|} < \epsilon.$$

Proof. The possible sizes of the graphs in S_{p,p_1,p_2} range in the set $\{0, 1, 2 \ldots, p_1p_2\}$. Now, define the function

$$f : \{0, 1, 2, \ldots, p_1p_2\} \to \mathbb{N}$$

by the rule $f(i)$ = number of graphs in S_{p,p_1,p_2} of size i. Then, by Lemma 2, this function is unimodal, and has its maximum at $\frac{p_1p_2}{2}$ if p_1p_2 is even or at $\lfloor \frac{p_1p_2}{2} \rfloor$, $\lceil \frac{p_1p_2}{2} \rceil$ if p_1p_2 is odd. Also,

$$|S^*_{p,p_1,p_2}| = \sum_{i=0}^{2p-5} f(i) \text{ and } |S_{p,p_1,p_2}| = \sum_{i=0}^{p_1p_2} f(i)$$

thus,

$$\begin{aligned}
\frac{|S^*_{p,p_1,p_2}|}{|S_{p,p_1,p_2}|} &= \frac{\sum_{i=0}^{2p-5} f(i)}{\sum_{i=0}^{p_1p_2} f(i)} \\
&= \frac{\sum_{i=0}^{2p-5} f(i)}{2\sum_{i=0}^{2p-5} f(i) + \sum_{i=2p-4}^{p_1p_2-(2p-5)+1} f(i)} \\
&= \left(\frac{2\sum_{i=0}^{2p-5} f(i) + \sum_{i=2p-4}^{p_1p_2-(2p-5)+1} f(i)}{\sum_{i=0}^{2p-5} f(i)} \right)^{-1} \\
&= \left(2 + \frac{\sum_{i=2p-4}^{p_1p_2-(2p-5)+1} f(i)}{\sum_{i=0}^{2p-5} f(i)} \right)^{-1}.
\end{aligned}$$

Now,

$$\begin{aligned}
\frac{\sum_{i=2p-4}^{p_1p_2-(2p-5)+1} f(i)}{\sum_{i=0}^{2p-5} f(i)} &\geq \frac{[p_1p_2 - (2p - 5 + 1) - (2p - 4) + 1]f(2p - 4)}{(2p - 4)f(2p - 4)} \\
&\approx \frac{o(p^2)}{o(p)} \to \infty \text{ as } p \to \infty.
\end{aligned}$$

Hence,

$$\left(2+\frac{\sum_{i=2p-4}^{p_1p_2-(2p-5)+1} f(i)}{\sum_{i=0}^{2p-5} f(i)}\right)^{-1} \to 0 \text{ as } p \to \infty.$$

Therefore for every $\epsilon > 0$, there exist $k(\epsilon) \in \mathbb{N}$, such that if $p \geq k(\epsilon)$ then

$$\frac{|S^*_{p,p_1,p_2}|}{|S_{p,p_1,p_2}|} < \epsilon$$

As a corollary, and since the set of super edge-magic bipartite graphs of order p and fixed bipartite set V_1, V_2 such that $|V_i| = p_i$, $(i = 1,2)$ and $p_1p_2 = o(p^2)$, denoted by SEM_{p,p_1,p_2}, is a subset of S^*_{p,p_1,p_2}, we have that for all $\epsilon > 0$, there exists $k(\epsilon) \in \mathbb{N}$ such that $p > k(\epsilon)$ implies

$$\frac{|SEM_{p,p_1,p_2}|}{|S_{p,p_1,p_2}|} < \epsilon$$

4 Conclusions

In light of what it has been exposed in this paper, it seems clear that the correct sets to compare are not the set of graphs with the set of super edge-magic graphs, but the set of graphs with size upper bounded by two times its order minus 3, with the set of super edge-magic graphs.

We also want to point out that in Theorem 2, the hypothesis that $p_1p_2 = o(p^2)$ cannot be eliminated since if we let $p_1 = 1$, and $p_2 = n$, what we get is the set of stars and stars with isolated vertices, and all these graphs are super edge-magic.

Finally, we want to propose the following open questions.

- What is the probability that a graph is super edge-magic given that it is bipartite?
- What is the probability that a graph is super edge-magic given that the graph is a tree?
- What is the probability that a graph has finite super edge-magic deficiency?

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References

1. B.D. Acharga and S.M. Hedge: Strong by indexable graphs. Graph Theory Newsletters, **15-1** (1986).
2. G. Chartrand and L. Lesniak: Graphs and Digraphs. 3^{rd} ed. CRC Press (1996).

3. H. Enomoto, A.S. Lladó, T. Nakamigawa, G. Ringel: Super edge-magic graphs. SUT J. Math., **34** (1998) 105–109.
4. R.M. Figueroa-Centeno, R. Ichishima and F.A. Muntaner-Batle: The place of super edge-magic labelings among other classes of labelings. Discrete Math., **231** (2001) 153–168.
5. R.M. Figueroa-Centeno, R. Ichishima and F.A. Muntaner-Batle: On the super edge-magic deficiency of graphs. Discrete Math. (to appear).
6. J.Gallian: A dynamic survey of graph labeling. The electronic journal of combinatorics. **5** (1998) http://www.combinatorics.org.
7. E.N. Gilbert: Random graphs. Ann. Math. Stat. **30** (1959) 1141–1144.
8. R.L. Graham and N.J.A. Sloane: On additive bases and harmonious graphs. SIAM J.Alg.Discrete Math., **1** (1980) 382–404.
9. A. Rosa: On certain valuations of the vertices of graphs. Gordon and Breach, N.Y. and Dunod Paris (1967) 349–355.