# On antimagic labelings of disjoint union of complete $s$-partite graphs

Dafik[1,2], Mirka Miller[1,3], Joe Ryan[1] and Martin Bača[1,4]

[1] School of Information Technology and Mathematical Sciences
University of Ballarat, Australia
[2] Department of Mathematics Education
Universitas Jember, Indonesia
[3] Department of Mathematics
University of West Bohemia, Plzeň, Czech Republic
Department of Appl. Mathematics
[4] Technical University, Košice, Slovak Republic
d.dafik@students.ballarat.edu.au
{m.miller, joe.ryan}@ballarat.edu.au
Martin.Baca@tuke.sk

**Abstract.** By an $(a,d)$-edge-antimagic total labeling of a graph $G(V,E)$ we mean a bijective function $f$ from $V(G)\cup E(G)$ onto the set $\{1,2,\dots,|V(G)|+|E(G)|\}$ such that the set of all the edge-weights, $w(uv)=f(u)+f(uv)+f(v), uv\in E(G)$, is $\{a,a+d,a+2d,\dots,a+(|E(G)|-1)d\}$, for two integers $a>0$ and $d\geq 0$.
In this paper we study the edge-antimagic properties for the disjoint union of complete $s$-partite graphs.

*Keywords: complete s-partite graph, $(a,d)$-edge-antimagic total labeling, super $(a,d)$-edge-antimagic total labeling.*

## 1 Introduction and Definitions

We assume that $G(V,E)$ is a finite, simple, and undirected graph with $p$ vertices and $q$ edges. We refer the reader to [12] or [13] for all other terms and notation not provided in this paper.

By a *labeling* we mean any mapping that carries a set of graph elements onto a set of numbers, called *labels*. In this paper, we deal with labelings with domain the set of all vertices and edges. This type of labeling belongs to the class of *total* labelings. We define the *edge-weight* of an edge $uv\in E(G)$ under a total labeling to be the sum of the vertex labels corresponding to vertices $u$, $v$ and edge label corresponding to edge $uv$.

An *$(a,d)$-edge-antimagic total labeling* on a graph $G$ is a bijective function $f: V(G)\cup E(G)\to\{1,2,\dots,p+q\}$ with the property that the edge-weights $w(uv)=f(u)+f(uv)+f(v), uv\in E(G)$, form an arithmetic progression $\{a,a+d,a+2d,\dots,a+(q-1)d\}$, where $a>0$ and $d\geq 0$ are two fixed integers. If such a labeling exists then $G$ is said to be an *$(a,d)$-edge-antimagic total graph.* Such a graph $G$ is called *super* if the smallest possible labels appear on the vertices. Thus, a *super $(a,d)$-edge-antimagic total graph* is a graph that admits a super $(a,d)$-edge-antimagic total labeling.

The concept of $(a,d)$-edge-antimagic total labeling, introduced by Simanjuntak *et al.* in [9], is a natural extension of the notion of an *edge-magic* labeling, defined by Kotzig and Rosa [7] (see also [1], [5], [8] and [11]). The super $(a,d)$-edge-antimagic total labeling is a natural extension of the notion of a *super edge-magic* labeling, which was defined by Enomoto *et al.* in [4].

In this paper we investigate the existence of super $(a,d)$-edge-antimagic total labelings for disconnected graphs. Some constructions of super $(a,0)$-edge-antimagic total labelings for $nC_k \cup mP_k$ and $K_{1,m} \cup K_{1,n}$ have been given by Ivančo and Lučkaničová in [6], and super $(a,d)$-edge-antimagic total labelings for $P_n \cup P_{n+1}$, $nP_2 \cup P_n$ and $nP_2 \cup P_{n+2}$ have been described by Sudarsana *et al.* in [10].

We will concentrate on the disjoint union of $m$ copies of complete $s$-partite graph, denoted by $mK\underbrace{n,n,\ldots,n}_{s}$. For $m \geq 2$, $n \geq 1$ and $s \geq 2$ let

$$V(mK\underbrace{n,n,\ldots,n}_{s}) = \bigcup_{j=1}^{m}\bigcup_{t=1}^{s}\left\{x_{t,i}^{j} : 1 \leq i \leq n\right\}$$

be the vertex set and

$$E(mK\underbrace{n,n,\ldots,n}_{s}) = \bigcup_{j=1}^{m}\bigcup_{t=1}^{s-1}\bigcup_{i=1}^{n}\left\{x_{t,i}^{j}x_{t+k,r}^{j} : 1 \leq k \leq s-t, 1 \leq r \leq n\right\}$$

be the edge set of $mK\underbrace{n,n,\ldots,n}_{s}$.

Thus, let $p = |V(mK\underbrace{n,n,\ldots,n}_{s})| = mns$ and $q = |E(mK\underbrace{n,n,\ldots,n}_{s})| = \frac{mn^2s(s-1)}{2}$.

# 2 Main Result

Before beginning this section, we note that super $(a,d)$-edge-antimagic total labelings for $mK_{n,n}$ are studied in [2], and properties of super $(a,d)$-edge-antimagic total labelings of disjoint union of multiple copies of complete threepartite graph $mK_{n,n,n}$ are investigated in [3]. Therefore, we will study the super edge-anti magicness of $mK\underbrace{n,n,\ldots,n}_{s}$ for $s \geq 4$.

If the graph $mK\underbrace{n,n,\ldots,n}_{s}$ admits a super $(a,d)$-edge-antimagic total labeling

$$f: V(m\underbrace{K_{n,n,\ldots,n}}_{s}) \cup E(m\underbrace{K_{n,n,\ldots,n}}_{s}) \to \left\{1, 2, \ldots, \frac{mns}{2}(n(s-1)+2)\right\}$$

then $W = \left\{ w(uv) = f(u) + f(uv) + f(v) : uv \in E(m\underbrace{K_{n,n,\ldots,n}}_{s}) \right\} =$ $\left\{a, a+d, a+2d, \ldots, a + \left(\frac{mn^2 s(s-1)}{2} - 1\right) d\right\}$ is the set of the edge-weights, and the sum of all the edge-weights in $W$ is

$$\sum_{uv \in E(m\underbrace{K_{n,n,\ldots,n}}_{s})} w(uv) = \frac{mn^2 s(s-1)}{8}\left[4a + \left(mn^2 s(s-1) - 2\right) d\right]. \quad (1)$$

In the computation of the edge-weights of $m\underbrace{K_{n,n,\ldots,n}}_{s}$, each edge label is used once and the label of each vertex is used $(s-1)n$ times. The sum of all the vertex labels and the edge labels used to calculate the edge-weights is thus equal to

$$(s-1)n \sum_{u \in V(m\underbrace{K_{n,n,\ldots,n}}_{s})} f(u) + \sum_{uv \in E(m\underbrace{K_{n,n,\ldots,n}}_{s})} f(uv) =$$

$$\frac{mns+1}{2} mn^2 s(s-1) + \frac{mn^2 s(s-1)}{8}\left[4mns + mn^2 s(s-1) + 2\right]. \quad (2)$$

The sum of all the vertex labels and the edge labels used to calculate the edge-weights is equal to the sum of the edge-weights in the set $W$, under the labeling $f$. Thus combining (1) and (2) gives the following equation

$$4a + (mn^2 s(s-1) - 2)d = 8mns + mn^2 s(s-1) + 6. \quad (3)$$

At this point, we are ready to establish an upper bound on the parameter $d$.

**Lemma 1** *For the graph $m\underbrace{K_{n,n,\ldots,n}}_{s}$, $m \geq 2$, $n = 1$ and $s = 4$, there is no super $(a,d)$-edge-antimagic total labeling with $d \geq 3$.*

**Proof.** Since the minimum possible edge-weight under the labeling $f$ is at least $mns + 4$, then from Equation (3) it follows that

$$d \leq 1 + \frac{4mns - 8}{mn^2 s(s-1) - 2}. \quad (4)$$

It is easy to verify that $1 < \frac{4mns-8}{mn^2 s(s-1)-2} < 2$ only when $m \geq 2$, $n = 1$ and $s = 4$, which completes the proof. □

Since $\frac{4mns-8}{mn^2 s(s-1)-2} < 1$ for $m \geq 2$, $n \geq 2$ and $s \geq 4$, it follows that (4) gives $d < 2$ and we have the following lemma.

**Lemma 2** *For the graph $mK\underbrace{n,n,\ldots,n}_{s}$, $m \geq 2$, $n \geq 2$ and $s \geq 4$, there is no super $(a,d)$-edge-antimagic total labeling with $d \geq 2$.*

First, we start to deal with super $(a,0)$-edge-antimagic total labeling for the disjoint union of $m$ copies of complete $s$-partite graph.

**Theorem 1** *If either $s \equiv 0,1 \pmod 4$, $s \geq 4$, $m \geq 2$, $n \geq 1$, or $mn$ is even, $m \geq 2$, $n \geq 1$, $s \geq 4$, then there is no super $(a,0)$-edge-antimagic total labeling for $mK\underbrace{n,n,\ldots,n}_{s}$.*

**Proof.** Assume that $mK\underbrace{n,n,\ldots,n}_{s}$ admits a super $(a,0)$-edge-antimagic total labeling

$$f : V(mK\underbrace{n,n,\ldots,n}_{s}) \cup E(mK\underbrace{n,n,\ldots,n}_{s}) \to \{1,2,\ldots,\frac{mns}{2}(n(s-1)+2)\}.$$

From Equation (3) we have

$$a = 2mns + \frac{mn^2s(s-1)}{4} + \frac{3}{2}. \tag{5}$$

If either $s \equiv 0,1 \pmod 4$, $s \geq 4$, $m \geq 2$, $n \geq 1$, or $mn$ is even, $m \geq 2$, $n \geq 1$ and $s \geq 4$, then from Equation (5) it is easy to see that the value $a$ is not an integer, which is a contradiction. □

The minimum edge-weight in Equation (5) is an integer if and only if $mn$ is odd and $s \equiv 2,3 \pmod 4$. In this case we do not have any answer for super $(a,0)$-edge-antimagicness of $mK\underbrace{n,n,\ldots,n}_{s}$. Therefore, we propose the following open problem.

**Open Problem 1** *For the graph $mK\underbrace{n,n,\ldots,n}_{s}$, $mn$ odd, $m \geq 3$, $n \geq 1$ and $s \equiv 2,3 \pmod 4$, $s \geq 6$, determine if there is a super $\left(2mns + \frac{mn^2s(s-1)+6}{4}, 0\right)$-edge-antimagic total labeling.*

From Lemma 1, it follows that the graph $mK\underbrace{n,n,\ldots,n}_{s}$ may be super $(a,2)$-edge-antimagic total only when $m \geq 2$, $n = 1$ and $s = 4$. Our next result gives a negative answer.

**Theorem 2** *If $m \geq 2$, $n = 1$ and $s = 4$, then there is no super $(a,2)$-edge-antimagic total labeling for the graph $mK\underbrace{n,n,\ldots,n}_{s}$.*

**Proof.** Assume to the contrary that for $m \geq 2$, $n = 1$ and $s = 4$, the graph $m\underbrace{K_{n,n,\ldots,n}}_{s}$ has a super $(a, 2)$-edge-antimagic total labeling $f : V(mK_{n,n,n,n}) \cup E(mK_{n,n,n,n}) \to \{1, 2, \ldots, 10m\}$. From Equation (3) we get that $2a = 10m + 5$. This contradicts the fact that $a$ is an integer. □

Now, we will concentrate on the existence of super $(a, 1)$-edge-antimagic total labelings of the disjoint union of $m$ copies of complete 4-partite graph.

**Theorem 3** *The graph $mK_{n,n,n,n}$ has a super $(8mn + 2, 1)$-edge-antimagic total labeling for every $m \geq 2$ and $n \geq 1$.*

**Proof.** If $s = 4$ and $d = 1$, then from (3) it follows that $a = 8mn + 2$. Consider the following bijective function
$g : V(mK_{n,n,n,n}) \cup E(mK_{n,n,n,n}) \to \{1, 2, \ldots, 2mn(3n + 2)\}$, where
$g(x^j_{t,i}) = m(4i + t - 5) + j$ for $1 \leq t \leq 4$, $1 \leq i \leq n$, $1 \leq j \leq m$.

$$g(x^j_{1,i}x^j_{2,r}) = \begin{cases} 2mn(3n + 8 - 6i) + 6m\sum_{k=0}^{i-2}(1 + 2k) + 2m(1 - r) - j + 1 \\ \quad \text{for } 1 \leq i \leq n, \ 1 \leq r \leq n - i + 1 \text{ and } 1 \leq j \leq m, \\ 2mn(6i - 3n - 2) + 4m\sum_{k=0}^{n-i}(1 + 3k) - 2m(r - 2 + i) - j + 1 \\ \quad \text{for } 2 \leq i \leq n, \ n + 2 - i \leq r \leq n \text{ and } 1 \leq j \leq m. \end{cases}$$

$$g(x^j_{1,i}x^j_{3,r}) = \begin{cases} 6mn(n + 2 - 2i) + 2m\sum_{k=1}^{i-1}(6k - 1) + m(3 - 2r) - j + 1 \\ \quad \text{for } 1 \leq i \leq n, \ 1 \leq r \leq n - i + 1 \text{ and } 1 \leq j \leq m, \\ 6mn(2i - n) + 12m\sum_{k=0}^{n-1-i}(1 + k) - m(2r - 5 + 2i) - j + 1 \\ \quad \text{for } 2 \leq i \leq n, \ n + 2 - i \leq r \leq n \text{ and } 1 \leq j \leq m. \end{cases}$$

$$g(x^j_{2,i}x^j_{3,r}) = \begin{cases} 2mn(3n + 4 - 6i) + 2m\sum_{k=1}^{i-1}(6k + 1) + m(5 - 2r) - j + 1 \\ \quad \text{for } 1 \leq i \leq n, \ 1 \leq r \leq n - i + 1 \text{ and } 1 \leq j \leq m, \\ \alpha + 4m\sum_{k=0}^{n-1-i}(2 + 3k) - m(2r - 3 + 2i) - j + 1 \\ \quad \text{for } 2 \leq i \leq n, \ n + 2 - i \leq r \leq n \text{ and } 1 \leq j \leq m. \end{cases}$$

$$g(x^j_{3,i}x^j_{4,r}) = \begin{cases} 2mn(3n + 4 - 6i) + 2m\sum_{k=1}^{i-1}(6k + 1) + m(4 - 2r) - j + 1 \\ \quad \text{for } 1 \leq i \leq n - 1, \ 1 \leq r \leq n - i \text{ and } 1 \leq j \leq m, \\ \alpha + 4m\sum_{k=0}^{n-1-i}(2 + 3k) - 2m(r - 1 + i) - j + 1 \\ \quad \text{for } 1 \leq i \leq n, \ n + 1 - i \leq r \leq n \text{ and } 1 \leq j \leq m, \end{cases}$$

where $\alpha = 2mn(6i - 3n + 2)$.

If $1 \le i \le n, 1 \le r \le n - i + 1$ and $1 \le j \le m$ then

$$g(x^j_{1,i}x^j_{4,r}) = g(x^j_{1,i}x^j_{2,r}) - m \quad \text{and}$$
$$g(x^j_{2,i}x^j_{4,r}) = g(x^j_{1,i}x^j_{3,r}) - m.$$

If $2 \le i \le n, n + 2 - i \le r \le n$ and $1 \le j \le m$ then

$$g(x^j_{1,i}x^j_{4,r}) = g(x^j_{1,i}x^j_{2,r}) - m \quad \text{and}$$
$$g(x^j_{2,i}x^j_{4,r}) = g(x^j_{1,i}x^j_{3,r}) - m.$$

It is not difficult to verify by a routine procedure that a system of sets
$\bigcup_{j=1}^{m} \bigcup_{t=1}^{3} \bigcup_{i=1}^{n} \left\{g(x^j_{t,i}) + g(x^j_{t,i}x^j_{t+k,r}) + g(x^j_{t+k,r}) \ : \ 1 \le k \le 4 - t, 1 \le r \le n\right\}$
consists of consecutive integers of the form $8mn+2, 8mn+3, 8mn+4, \dots, 6n^2m+8mn, 6n^2m + 8mn + 1$, which are the required edge-weights of $mK_{n,n,n,n}$ under the total labeling $g$. Thus $g$ is a super $(8mn+2, 1)$-edge-antimagic total labeling. □

Figure 1 shows a vertex labeling of the graph $4K_{3,3,3,3}$.

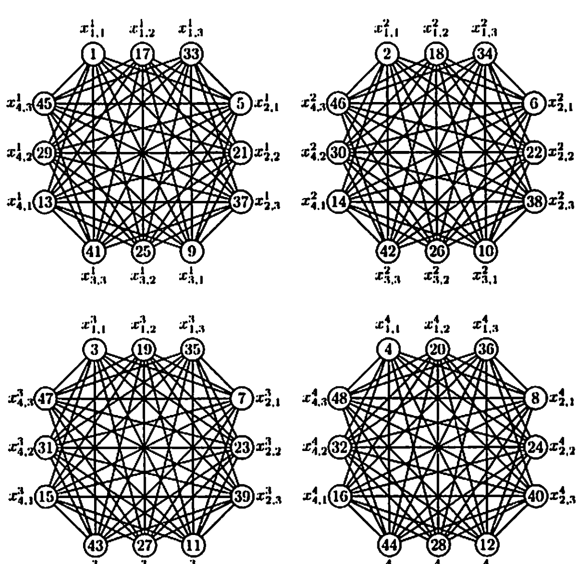


**Fig. 1.** Vertex labeling of the graph $4K_{3,3,3,3}$

Under the labeling $g$ defined as above we describe in the following tables the edge labels and the edge-weights of the graph $4K_{3,3,3,3}$. The values in cells of the tables are in the format of $\frac{\frac{x}{y}}{z}$, where the value $x$ is the sum of labels of two end vertices of the corresponding edge, the value $y$ determines the corresponding edge label and the value $z = x + y$ determines the corresponding total edge-weight.

| | $x^1_{2,1}$ | $x^1_{2,2}$ | $x^1_{2,3}$ |
|---|---|---|---|
| $x^1_{1,1}$ | 6 / 264 / 270 | 22 / 256 / 278 | 38 / 248 / 286 |
| $x^1_{1,2}$ | 22 / 144 / 166 | 38 / 136 / 174 | 54 / 80 / 134 |
| $x^1_{1,3}$ | 38 / 72 / 110 | 54 / 160 / 214 | 70 / 152 / 222 |

| | $x^1_{3,1}$ | $x^1_{3,2}$ | $x^1_{3,3}$ |
|---|---|---|---|
| $x^1_{1,1}$ | 10 / 220 / 230 | 26 / 212 / 238 | 42 / 204 / 246 |
| $x^1_{1,2}$ | 26 / 116 / 142 | 42 / 108 / 150 | 58 / 100 / 158 |
| $x^1_{1,3}$ | 42 / 60 / 102 | 58 / 196 / 254 | 74 / 188 / 262 |

| | $x^1_{4,1}$ | $x^1_{4,2}$ | $x^1_{4,3}$ |
|---|---|---|---|
| $x^1_{1,1}$ | 14 / 260 / 274 | 30 / 252 / 282 | 46 / 244 / 290 |
| $x^1_{1,2}$ | 30 / 140 / 170 | 46 / 132 / 178 | 62 / 76 / 138 |
| $x^1_{1,3}$ | 46 / 68 / 114 | 62 / 156 / 218 | 78 / 148 / 226 |

| | $x^1_{3,1}$ | $x^1_{3,2}$ | $x^1_{3,3}$ |
|---|---|---|---|
| $x^1_{2,1}$ | 14 / 180 / 194 | 30 / 172 / 202 | 46 / 164 / 210 |
| $x^1_{2,2}$ | 30 / 92 / 122 | 46 / 84 / 130 | 62 / 124 / 186 |
| $x^1_{2,3}$ | 46 / 52 / 98 | 62 / 236 / 298 | 78 / 228 / 306 |

| | $x^1_{4,1}$ | $x^1_{4,2}$ | $x^1_{4,3}$ |
|---|---|---|---|
| $x^1_{2,1}$ | 18 / 216 / 234 | 34 / 208 / 242 | 50 / 200 / 250 |
| $x^1_{2,2}$ | 34 / 112 / 146 | 50 / 104 / 154 | 66 / 96 / 162 |
| $x^1_{2,3}$ | 50 / 56 / 106 | 66 / 192 / 258 | 82 / 184 / 266 |

| | $x^1_{4,1}$ | $x^1_{4,2}$ | $x^1_{4,3}$ |
|---|---|---|---|
| $x^1_{3,1}$ | 22 / 176 / 198 | 38 / 168 / 206 | 54 / 64 / 118 |
| $x^1_{3,2}$ | 38 / 88 / 126 | 54 / 128 / 182 | 70 / 120 / 190 |
| $x^1_{3,3}$ | 54 / 240 / 294 | 70 / 232 / 302 | 86 / 224 / 310 |

| | $x^2_{2,1}$ | $x^2_{2,2}$ | $x^2_{2,3}$ |
|---|---|---|---|
| $x^2_{1,1}$ | 8 / 263 / 271 | 24 / 255 / 279 | 40 / 247 / 287 |
| $x^2_{1,2}$ | 24 / 143 / 167 | 40 / 135 / 175 | 56 / 79 / 135 |
| $x^2_{1,3}$ | 40 / 71 / 111 | 56 / 159 / 215 | 72 / 151 / 223 |

| | $x^2_{3,1}$ | $x^2_{3,2}$ | $x^2_{3,3}$ |
|---|---|---|---|
| $x^2_{1,1}$ | 12 / 219 / 231 | 28 / 211 / 239 | 44 / 203 / 247 |
| $x^2_{1,2}$ | 28 / 115 / 143 | 44 / 107 / 151 | 60 / 99 / 159 |
| $x^2_{1,3}$ | 44 / 59 / 103 | 60 / 195 / 255 | 76 / 187 / 263 |

| | $x^2_{4,1}$ | $x^2_{4,2}$ | $x^2_{4,3}$ |
|---|---|---|---|
| $x^2_{1,1}$ | 16 / 259 / 275 | 32 / 251 / 283 | 48 / 243 / 291 |
| $x^2_{1,2}$ | 32 / 139 / 171 | 48 / 131 / 179 | 64 / 75 / 139 |
| $x^2_{1,3}$ | 48 / 67 / 115 | 64 / 155 / 219 | 80 / 147 / 227 |

| | $x^2_{3,1}$ | $x^2_{3,2}$ | $x^2_{3,3}$ |
|---|---|---|---|
| $x^2_{2,1}$ | 16 / 179 / 195 | 32 / 171 / 203 | 48 / 163 / 211 |
| $x^2_{2,2}$ | 32 / 91 / 123 | 48 / 83 / 131 | 64 / 123 / 187 |
| $x^2_{2,3}$ | 48 / 51 / 99 | 64 / 235 / 299 | 80 / 227 / 307 |

| | $x^2_{4,1}$ | $x^2_{4,2}$ | $x^2_{4,3}$ |
|---|---|---|---|
| $x^2_{2,1}$ | 20 / 215 / 235 | 36 / 207 / 243 | 52 / 199 / 251 |
| $x^2_{2,2}$ | 36 / 111 / 147 | 52 / 103 / 155 | 68 / 95 / 163 |
| $x^2_{2,3}$ | 52 / 55 / 107 | 68 / 191 / 259 | 84 / 183 / 267 |

| | $x^2_{4,1}$ | $x^2_{4,2}$ | $x^2_{4,3}$ |
|---|---|---|---|
| $x^2_{3,1}$ | 24 / 175 / 199 | 40 / 167 / 207 | 56 / 63 / 119 |
| $x^2_{3,2}$ | 40 / 87 / 127 | 56 / 127 / 183 | 72 / 119 / 191 |
| $x^2_{3,3}$ | 56 / 239 / 295 | 72 / 231 / 303 | 88 / 223 / 311 |

| | $x^3_{2,1}$ | $x^3_{2,2}$ | $x^3_{2,3}$ |
|---|---|---|---|
| $x^3_{1,1}$ | 10 / 262 / 272 | 26 / 254 / 280 | 42 / 246 / 288 |
| $x^3_{1,2}$ | 26 / 142 / 168 | 42 / 134 / 176 | 58 / 78 / 136 |
| $x^3_{1,3}$ | 42 / 70 / 112 | 58 / 158 / 216 | 74 / 150 / 224 |

| | $x^3_{3,1}$ | $x^3_{3,2}$ | $x^3_{3,3}$ |
|---|---|---|---|
| $x^3_{1,1}$ | 14 / 218 / 232 | 30 / 210 / 240 | 46 / 202 / 248 |
| $x^3_{1,2}$ | 30 / 114 / 144 | 46 / 106 / 152 | 62 / 98 / 160 |
| $x^3_{1,3}$ | 46 / 58 / 104 | 62 / 194 / 256 | 78 / 186 / 264 |

| | $x^3_{4,1}$ | $x^3_{4,2}$ | $x^3_{4,3}$ |
|---|---|---|---|
| $x^3_{1,1}$ | 18 / 258 / 276 | 34 / 250 / 284 | 50 / 242 / 292 |
| $x^3_{1,2}$ | 34 / 138 / 172 | 50 / 130 / 180 | 66 / 74 / 140 |
| $x^3_{1,3}$ | 50 / 66 / 116 | 66 / 154 / 220 | 82 / 146 / 228 |

| | $x^3_{3,1}$ | $x^3_{3,2}$ | $x^3_{3,3}$ |
|---|---|---|---|
| $x^3_{2,1}$ | 18 / 178 / 196 | 34 / 170 / 204 | 50 / 162 / 212 |
| $x^3_{2,2}$ | 34 / 90 / 124 | 50 / 82 / 132 | 66 / 122 / 188 |
| $x^3_{2,3}$ | 50 / 50 / 100 | 66 / 234 / 300 | 82 / 226 / 308 |

| | $x^3_{4,1}$ | $x^3_{4,2}$ | $x^3_{4,3}$ |
|---|---|---|---|
| $x^3_{2,1}$ | 22 / 214 / 236 | 38 / 206 / 244 | 54 / 198 / 252 |
| $x^3_{2,2}$ | 38 / 110 / 148 | 54 / 102 / 156 | 70 / 94 / 164 |
| $x^3_{2,3}$ | 54 / 54 / 108 | 70 / 190 / 260 | 86 / 182 / 268 |

| | $x^3_{4,1}$ | $x^3_{4,2}$ | $x^3_{4,3}$ |
|---|---|---|---|
| $x^3_{3,1}$ | 26 / 174 / 200 | 42 / 166 / 208 | 58 / 62 / 120 |
| $x^3_{3,2}$ | 42 / 86 / 128 | 58 / 126 / 184 | 74 / 118 / 192 |
| $x^3_{3,3}$ | 58 / 238 / 296 | 74 / 230 / 304 | 90 / 222 / 312 |

| | $x^4_{2,1}$ | $x^4_{2,2}$ | $x^4_{2,3}$ |
|---|---|---|---|
| $x^4_{1,1}$ | 12 / 261 / 273 | 28 / 253 / 281 | 44 / 245 / 289 |
| $x^4_{1,2}$ | 28 / 141 / 169 | 44 / 133 / 177 | 60 / 77 / 137 |
| $x^4_{1,3}$ | 44 / 69 / 113 | 60 / 157 / 217 | 76 / 149 / 225 |

| | $x^4_{3,1}$ | $x^4_{3,2}$ | $x^4_{3,3}$ |
|---|---|---|---|
| $x^4_{1,1}$ | 16 / 217 / 233 | 32 / 209 / 241 | 48 / 201 / 249 |
| $x^4_{1,2}$ | 32 / 113 / 145 | 48 / 105 / 153 | 64 / 97 / 161 |
| $x^4_{1,3}$ | 48 / 57 / 105 | 64 / 193 / 257 | 80 / 185 / 265 |

| | $x^4_{4,1}$ | $x^4_{4,2}$ | $x^4_{4,3}$ |
|---|---|---|---|
| $x^4_{1,1}$ | 20 / 257 / 277 | 36 / 249 / 285 | 52 / 241 / 293 |
| $x^4_{1,2}$ | 36 / 137 / 173 | 52 / 129 / 181 | 68 / 73 / 141 |
| $x^4_{1,3}$ | 52 / 65 / 117 | 68 / 153 / 221 | 84 / 145 / 229 |

| | $x^4_{3,1}$ | $x^4_{3,2}$ | $x^4_{3,3}$ |
|---|---|---|---|
| $x^4_{2,1}$ | 20 / 177 / 197 | 36 / 169 / 205 | 52 / 161 / 213 |
| $x^4_{2,2}$ | 36 / 89 / 125 | 52 / 81 / 133 | 68 / 121 / 189 |
| $x^4_{2,3}$ | 52 / 49 / 101 | 68 / 233 / 301 | 84 / 225 / 309 |

| | $x^4_{4,1}$ | $x^4_{4,2}$ | $x^4_{4,3}$ |
|---|---|---|---|
| $x^4_{2,1}$ | 24 / 213 / 237 | 40 / 205 / 245 | 56 / 197 / 253 |
| $x^4_{2,2}$ | 40 / 109 / 149 | 56 / 101 / 157 | 72 / 93 / 165 |
| $x^4_{2,3}$ | 56 / 53 / 109 | 72 / 189 / 261 | 88 / 181 / 269 |

| | $x^4_{4,1}$ | $x^4_{4,2}$ | $x^4_{4,3}$ |
|---|---|---|---|
| $x^4_{3,1}$ | 28 / 173 / 201 | 44 / 165 / 209 | 60 / 61 / 121 |
| $x^4_{3,2}$ | 44 / 85 / 129 | 60 / 125 / 185 | 76 / 117 / 193 |
| $x^4_{3,3}$ | 60 / 237 / 297 | 76 / 229 / 305 | 92 / 221 / 313 |

We can see that the edge labels (values $y$) are $49, 50, \ldots, 263, 264$ and the edge-weights under the total labeling (values $z$) constitute the set $\{98, 99, 100, \ldots, 312, 313\}$. Thus the vertex labeling and the edge labeling of $4K_{3,3,3,3}$ combine to super $(98, 1)$-edge-antimagic total labeling.

A natural question to ask is whether we can say anything about super $(a, 1)$-edge-antimagic total labeling for disjoint union of complete $s$-partite graphs for $s \geq 5$. Although we have not yet found the general formulas for vertex and edge labelings of $mK_{\underbrace{n, n, \ldots, n}_{s}}$ that will produce a required super $(a, 1)$-edge-antimagic total labeling, the observed antimagic properties of $mK_{\underbrace{n, n, \ldots, n}_{s}}$ lead us to suggest the following

**Conjecture 1** *There is a super $(a, 1)$-edge-antimagic total labeling for the graph $mK_{\underbrace{n, n, \ldots, n}_{s}}$ for $s \geq 5$ and for every $m \geq 2$ and $n \geq 1$.*

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