On antimode graphs

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Abstract. The term mode graph was introduced by Boland, Kauffman and Panrong [2] to define a connected graph G such that, for every pair of vertices v, w in G, the number of vertices with eccentricity $e(v)$ is equal to the number of vertices with eccentricity $e(w)$. As a natural extension to this work, the concept of an antimode graph was introduced to describe a graph for which if $e(v) \neq e(w)$ then the number of vertices with eccentricity $e(v)$ is not equal to the number of vertices with eccentricity $e(w)$. In this paper we determine the existence of some classes of antimode graphs, namely equisequential and (a, d)-antimode graphs.

Keywords: *Mode graph, eccentricity, antimode graph.*

1 Introduction

All graphs considered in this paper are finite, simple and undirected. The set of vertices and edges of a graph G will be denoted by $V(G)$ and $E(G)$, respectively. The number of vertices in a graph G is known as the *order* of G, denoted $|V(G)|$. The *distance* between two vertices u and v in a graph G, denoted $d(u, v)$, is equal to the shortest uv-path in the graph. The maximum distance from u to any other vertex in the graph is called the *eccentricity* of u, written $e(u)$. The maximum of all eccentricities is the *diameter* while the minimum of all eccentricities is called the *radius*, denoted r or $rad(G)$.

Vertices having eccentricities equal to the diameter make up the *periphery* of the graph; those vertices having eccentricities equal to the radius are collectively known as the *centre* of the graph. The *eccentric sequence* of a graph, denoted $ES(G) = (e_1, e_2, \ldots, e_n)$, is the non-decreasing sequence of integers representing all the eccentricities in the graph. Since most graphs have more than one vertex with the same eccentricity, it is common to adopt a short hand notation for the eccentric sequence, namely $ES(G) = (e_1^{m_1}, e_2^{m_2}, \ldots)$, where m_i is the number of vertices having eccentricity e_i and the sequence is strictly increasing in e_i.

A graph is said to be *self-centered* if all vertices have the same eccentricity. Every vertex in a disconnected graph has eccentricity ∞, hence, all disconnected graphs are self-centered. Boland, Kauffman and Panrong [2] define a *mode vertex* of a graph G as a vertex whose eccentricity occurs at least as often in the eccentric sequence of G as the occurrence of the eccentricity of any other vertex. Furthermore, the authors define the *mode of a graph* as the subgraph induced by the mode vertices and a *mode graph* to be a graph in which all vertices are mode vertices. In other words, a mode graph is a graph having $m_i = m_j$ for all i, j in the eccentric sequence. For example, the eccentric sequence of an even path P_n is $ES(P_n) = ((n-1)^2, (n-2)^2, \ldots, (n/2)^2)$, hence all even paths are mode graphs. Any other graph terms are consistent with those in [3].

Inspired by Hartsfield and Ringel [4] defining *antimagic* graphs, Ryan [7] defined an *antimode graph* to be a graph having $m_i \neq m_j$ for all i, j in the eccentric sequence. Furthermore, following the lead of Bodendiek and Walther [1], who defined (a, d)-antimagic graphs, Ryan [7] introduced the term (a, d)-*antimode graph* to refer to an antimode graph having the eccentric sequence $(e_1^a, e_2^{a+d}, \ldots, e_s^{a+(s-1)d})$, where d is an integer and a is a positive integer. Note that when $d = 0$ the graph G is a mode graph, hence, (a, d)-antimode graphs can be considered to be a generalisation of mode graphs.

In some cases it may be important to identify the length of the eccentric sequence. In such cases, if the number of unique eccentricities in a graph, that is, the length of the eccentric sequence, is N, where $1 \leq N \leq$ diameter of G we may refer to a graph as an $(a, d; N)$-*antimode graph.*

In the next section we consider antimode graphs in which the number of vertices having eccentricity $e(v)$ is equal to $e(v)$ for all $v \in V(G)$. In Section 3 we examine the more general (a, d)-antimode graphs.

2 Equisequential antimode graphs

Graphs with eccentric sequences of the form $(e_1^{e_1}, e_2^{e_2}, \ldots, e_s^{e_s})$ are referred to as *equisequential antimode* graphs. In other words, the eccentricity of the centre (respectively, periphery) determines the size of the centre (respectively, periphery) and likewise for the intervening eccentricities. The unique vertex of K_1 has eccentricity zero and so it is not an equisequential antimode graph. Our first result shows that there are no (finite) graphs of the form $(e_1^{e_1})$.

Lemma 1. *There are no self-centered equisequential antimode graphs.*

Proof. Consider a graph with x vertices each having eccentricity x, then there must be at least one path of length x. However the longest path on a connected graph of x vertices has length $x - 1$.

Boland, Kauffman and Panrong [2] proved that the only trees that are mode graphs are the even paths. Seeking a corresponding result for antimode graphs, we investigated equisequential antimode graphs and discovered that $K_{1,2}$ and the unique tree on five vertices with maximum degree $\Delta = 3$, denoted T_5, pictured in Figure 1, are the only equisequential antimode trees.

Lemma 2. *There are no equisequential antimode trees except for $K_{1,2}$ and T_5. (See Figure 1).*

Proof. Applying the result by Jordan [6], that is, that any tree T has either one or two vertices in the centre, the only possible equisequential antimode tree, apart from $K_{1,2}$ and T_5, must have eccentric sequence $(2^2, 3^3, 4^4)$. Since the centre of a tree is always the same as the centre of the longest path in the tree [6], the path on five vertices, P_5, must be have two vertices in the centre, which is impossible.

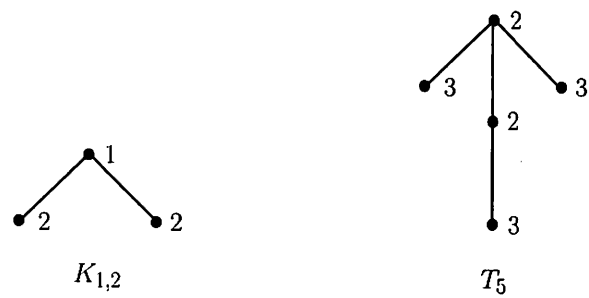


Fig. 1. Equisequential antimode trees.

In Theorem 2 we prove that the equisequential antimode trees given in Lemma 2 are the only equisequential antimode graphs with eccentric sequence length equal to two. To do this we require Theorem 1 by Haviar, Hrnčiar and Monoszová [5].

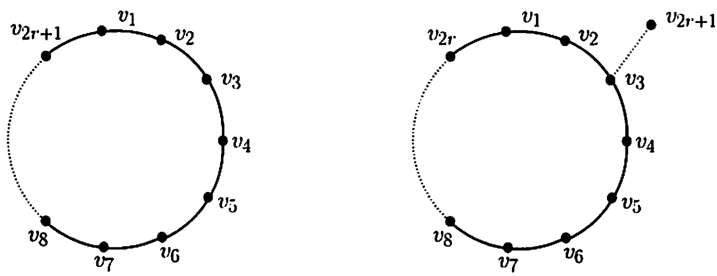
Theorem 1. *[5] If a graph G satisfies the conditions*
1. $rad(G) = r$, where $rad(G)$ is the radius of G,
2. $diam(G) \leq 2r - 2$, where $diam(G)$ is the diameter of G,
3. $|V(G)| \leq 3r - 2$,
then G contains a geodesic cycle of length $2r$ or $2r + 1$.

Theorem 2. *There are no equisequential antimode graphs with eccentric sequence of length two except for $(1, 2)$ and $(2, 3)$.*

Proof. Assume that there exists an equisequential antimode graphs, G, with $|ES(G)| = 2$. The eccentric sequence can be written $(r^r, (r+1)^{r+1})$, note that the graph has radius r and diameter $r + 1$ and $|V(G)| = 2r + 1$. Applying Theorem 1, G contains a cycle of length $2r$ or $2r + 1$, when $r \geq 3$.

The cycles C_{2r} and C_{2r+1} have the eccentric sequences of length one, namely $ES(C_{2r}) = (r^{2r})$ and $ES(C_{2r+1}) = (r^{2r+1})$. If the graph contains the cycle C_{2r+1} all vertices are contained in the cycle. If the graph contains the cycle C_{2r} one vertex, v_{2r+1}, remains. This vertex can be added to the cycle to create the cycle C_{2r+1}, or added as an isolated vertex in which case $ES(G) = (\infty^{2r+1})$ or alternatively we can add a pendent vertex, as shown in Figure 2. This will induce two vertices to have eccentricity $r + 1$, thus resulting in an eccentric sequence of length two, namely $(r^{2r-1}, (r + 1)^2)$. This sequence is not equisequential.

Therefore, equisequential antimode graphs with sequence length two, exist only in the case $r < 3$, which are the trees in Lemma 2.



$e(v_i) = r,\ 1 \leq i \leq 2r + 1$ $\qquad$ $e(v_i) = r,\ 1 \leq i \leq 2r$

Fig. 2. Sketch of proof for Theorem 2.

Since there are no equisequential antimode graphs with sequence length one or two, other than $K_{1,2}$ and T_5 (Lemma 2), we consider the existence of equisequential antimode graphs with sequence length three. Figure 5 depicts an example equisequential antimode graphs for $r = 2$.

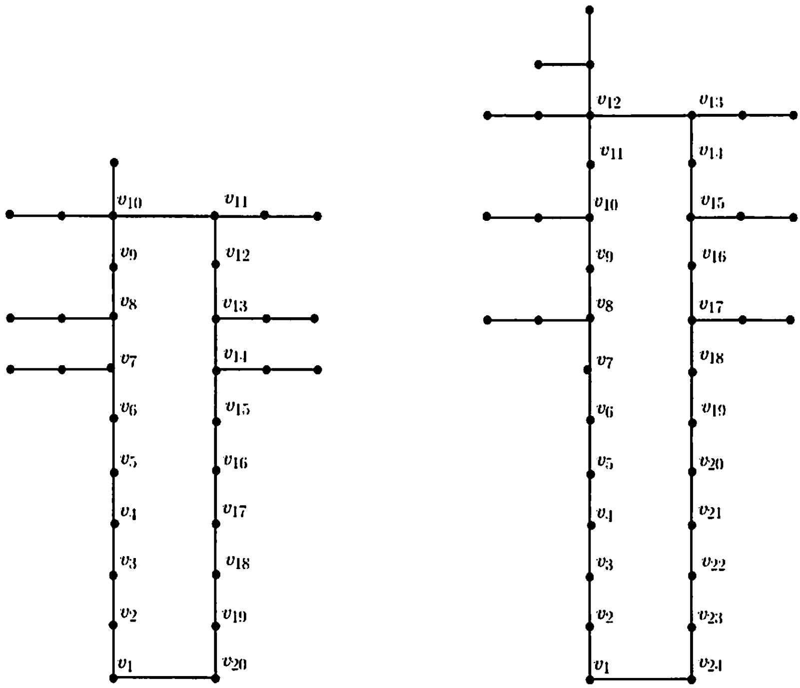
Since there are no trees which are equisequential antimode graphs excepting those in Lemma 2 we consider the next sparsest set of graphs, namely unicyclic graphs. For larger values of r and the additional constraint that the graph be unicyclic, we found the following constructions for the even and odd cases. Examples of these constructions are shown in Figure 3 and Figure 4.

For even $r > 4$, a unicyclic $(r, r+1, r+2)$-equisequential antimode graph on $3r + 3$ vertices can be constructed using the following algorithm:

1. Construct a cycle of length $2r$.
2. Label the vertices on the cycle consecutively from v_1 to v_{2r}.
3. Append K_2 graphs to the vertices labeled v_{r-2i} and v_{r+1+2i} on the cycle, for $i = 0, 1, 2, \ldots, \lfloor \frac{r}{4} \rfloor - 1$ by creating an edge between the vertex on the cycle and one of the vertices of the K_2 graph.
4. If $r \equiv 2 \pmod 4$ connect additional K_2 graphs to the vertices $v_{\frac{r}{2}+2}$ and $v_{\frac{3r}{2}-1}$. Append the remaining isolated vertex to v_r.
5. If $r \equiv 0 \pmod 4$ append a P_2 graph by adding an edge from the central vertex in the path to v_r on the cycle.

For odd $r > 5$, a unicyclic $(r, r+1, r+2)$-equisequential antimode graph on $3r+3$ vertices can be constructed using the following algorithm:

1. Construct a cycle of length $2r + 1$.
2. Label the vertices on the cycle consecutively from v_1 to v_{2r+1}.



(10, 11, 12) (12, 13, 14)

Fig. 3. Unicyclic $(r, r+1, r+2)$-equisequential antimode graphs, $r = 10, 12$.

3. Append K_2 graphs to the vertices labeled v_{r+1-2i} and v_{r+1+2i} on the cycle, for $i = 0, 1, 2, \ldots, \lfloor \frac{r+1}{4} \rfloor - 2$ by creating an edge between the vertex on the cycle and one of the vertices of the K_2 graph.
4. If $r \equiv 1 \pmod 4$ connect K_1 graphs to the vertices $v_{\lfloor \frac{r}{2} \rfloor + 2i}$ and $v_{2r-2\lfloor \frac{r}{4} \rfloor - 2i+2}$ for $i = 1, 2, \ldots, \lfloor \frac{r}{4} \rfloor - 2$.
5. If $r \equiv 3 \pmod 4$ connect K_1 graphs to the vertices $v_{\lfloor \frac{r}{2} \rfloor + 2i}$ and $v_{2r-2\lfloor \frac{r}{4} \rfloor - 2i+1}$ for $i = 1, 2, \ldots, \lfloor \frac{r}{4} \rfloor - 1$.
6. Append one of the remaining vertices to the vertex labeled v_{r+1} on the cycle by adding an edge.
7. Create edges between any remaining vertices and the vertex that is not on the cycle and is at distance one from the vertex labeled v_{r+1}.

Having found all equisequential antimode graphs with eccentric sequence lengths two and constructions for equisequential antimode graphs with eccentric sequence lengths three, we consider the existence of equisequential antimode graphs with the longest possible eccentric sequence. Figure 5 includes a $(2^2, 3^3, 4^4)$ and a $(3^3, 4^4, 5^5, 6^6)$-equisequential antimode graph which can be used as a basis for the construction of equisequential antimode graphs having eccentric sequence of the form $(r, r+1, \ldots, 2r)$ as demonstrated in Theorem 3.

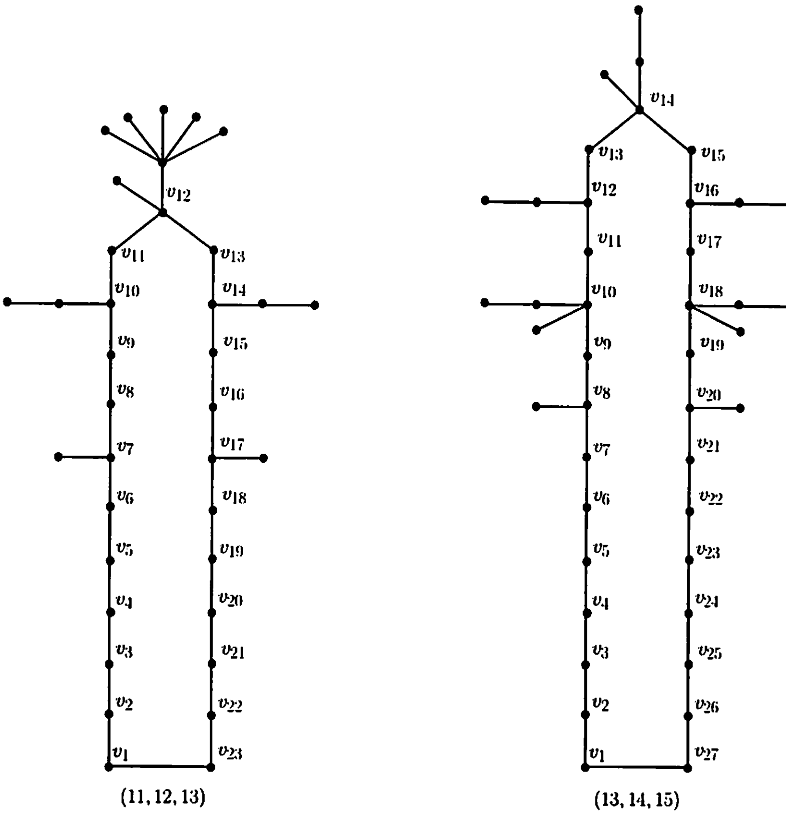


Fig. 4. Unicyclic $(r, r+1, r+2)$-equisequential antimode graphs, $r = 11, 13$.

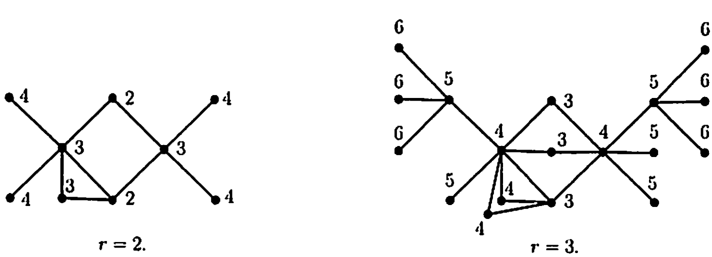


Fig. 5. Equisequential antimode graphs.

Theorem 3. *There exists an equisequential antimode graph for any eccentric sequence of the form* $(r, r+1, \ldots, 2r)$.

Proof. The proof is by construction. Figure 5 gives the constructions for $r = 2$ and $r = 3$. We can add as many degree two vertices to the central C_4 as we like and, likewise as many degree two vertices to the inner C_3 as we like. The remainder of the eccentric sequence is made up of the required number of pendant vertices, such that there is at least one vertex from each of the eccentricities $(r + 2, \ldots, 2 \times r)$ on each side of the centre (see Figure 6).

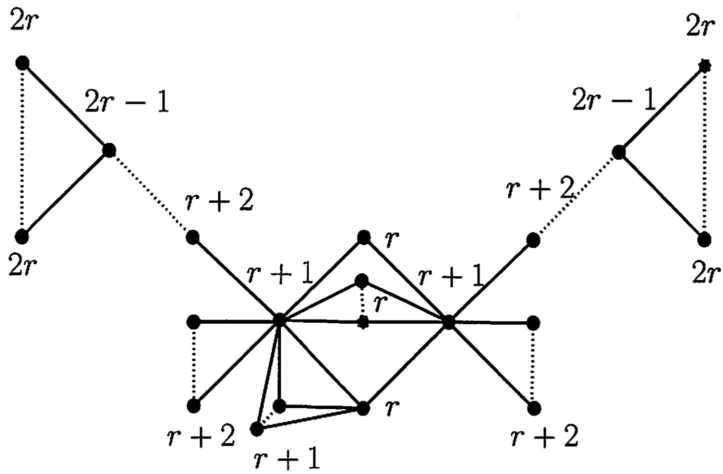


Fig. 6. Maximal equisequential antimode graphs.

3 *(a, d)-antimode graphs*

Having found equisequential antimode graphs with maximal and minimal eccentric sequence lengths, we consider (a, d)-antimode graphs, where the cardinalities of eccentricities form a more general arithmetic sequence, namely $ES(G) = (e_1^a, e_2^{a+d}, \ldots, e_s^{a+(s-1)d})$, where d is an integer and a is a positive integer. Note that an equisequential antimode graph is an (a, d)-antimode graph having $e_1 = a$ and $d = 1$. Even when the superscripts are restricted to a consecutive sequence of integers, the freedom to choose a allows for trees and self-centered graphs to be included.

One of the first considerations, when exploring (a, d)-antimode graphs is to find any forbidden configurations. That is, to find any subgraph whose existence would prevent the construction of an (a, d)-antimode graph. Our first theorem of this section shows that no such configuration exists.

Theorem 4. *Any graph G can appear as an induced subgraph of an (a,d)-antimode graph M, with the set of eccentricities in G the same as the set of eccentricities in M.*

Proof. For $d = 1$ let G be a graph with $ES(G) = \{e_1^{p_1}, e_2^{p_2}, \ldots, e_N^{p_N}\}$. Choose $c = \max_j\{p_j - j\}$ (or the first such j that attains this maximum), then for each e_i, add $i + c - p_i$ vertices so that the superscripts form a consecutive sequence of numbers beginning from $1+c$. To demonstrate the method of adding new vertices, choose e_i such that $p_i - i < c$. Then add a vertex v to G by creating an edge between v and a vertex u with eccentricity e_i and edges between v and all other vertices adjacent to u. Now $e(v) = i$ and the eccentricities of all other vertices remain the same. This process can be repeated $i + c - p_i$ times for each i until we obtain a $(a,1)$-antimode graph.

For $d > 1$ choose $c = \max_j\{p_j - dj\}$ and repeat the above process adding the required number of vertices.

As a consequence of Theorem 4, there are no subgraphs that would impede the construction of an (a,d)-antimode graph. The technique in Theorem 4 can also be used to construct $(a,d;2)$-antimode graphs by simply adding $|d|$ vertices to the periphery (or centre if $d < 0$) of any bimodal graph.

In Theorem 6 we give conditions under which the Cartesian product of two graphs forms an (a,d)-antimode graph. In order to do this we require Theorem 5 by Boland, Kauffman and Panrong [2] which employs the Cartesian product of two graphs. The *Cartesian product* $G = G_1 \times G_2$, has $V(G) = V(G_1) \times V(G_2)$, and two vertices (u_1, u_2) and (v_1, v_2) of G are adjacent if and only if either $u_1 = v_1$ and $u_2v_2 \in E(G_2)$ or $u_2 = v_2$ and $u_1v_1 \in E(G_1)$.

Theorem 5. *[2] If $e_G(u) = a$ and $e_H(v) = b$, then the eccentricity of the vertex in $G \times H$ corresponding to u in $V(G)$ and v in $V(H)$ is $a + b$.*

Theorem 6. *The Cartesian product of two graphs, $G_1 \times G_2$ is an (a,d)-antimode graph if and only if G_1 is an (a',d)-antimode graph and G_2 is self-centered.*

Proof. ($\Leftarrow$) Let G_1 be a graph with eccentric sequence $ES(G_1) = e_1^{p_1}, \ldots, e_n^{p_n}$ such that the p_i's form an arithmetic progression with common difference d, and let G_2 be a self-centered graph with $ES(G_2) = f^k$. Then using Theorem 5, $ES(G_1 \times G_2) = (e_1 + f)^{p_1+k}, \ldots, (e_n + f)^{p_n+k}$ so that the superscripts of the eccentricities form an arithmetic progression beginning with $p_1 + k$ and with common difference d.

($\Rightarrow$) **Case 1:** Suppose G_1 is not an (a',d)-antimode graph and G_2 is self-centered of order p. Then $ES(G_1) = e_1^{k_1}, e_2^{k_2}, \ldots, e_s^{k_s}$ such that the superscripts do not form an arithmetic progression, that is, there is some k_i where $k_{i+1} - k_i \neq d$. By Theorem 5, $ES(G_1 \times G_2) = f_1^{pk_1}, f_2^{pk_2}, \ldots, f_s^{pk_s}$ and $pk_{i+1} - pk_i \neq d$ so $G_1 \times G_2$ is not an antimode graph.

Case 2: Suppose neither G_1 nor G_2 is self-centered. Let the respective eccentric sequences be

$$ES(G_1) = e^{p_0}, (e+1)^{p_1}, \ldots, (e+i)^{p_i} \quad \text{and}$$

$$ES(G_2) = f^{q_0}, (f+1)^{q_1}, \ldots, (f+j)^{q_j}.$$

Consider the Cartesian product $G_1 \times G_2$. By Theorem 5, the set of vertices with eccentricity $e+f$ has cardinality $p_0 + q_0$ while the set of vertices with eccentricity $e+f+1$ has cardinality $p_0 + q_1 + q_0 + p_1$. Since all exponents are positive, d must also be positive and the sequence of cardinalities must be increasing. However, the cardinality of the set of vertices with eccentricity $(e+i+f+j-1)$ is $p_{i-1} + q_j + p_i + q_{j-1}$ and the cardinality of the set of vertices with eccentricity $(e+i+f+j)$ is $p_i + q_j$ indicating that the sequence is decreasing.

Clearly there can be no sequence that is both increasing and decreasing, so $G_1 \times G_2$ is not an (a,d)-antimode graph.

Boland *et al.* [2] gave conditions under which a graph is a $(1,2)$-mode graph. In Theorem 7 we use a similar technique to characterise $(a,1;2)$-antimode graphs with unit radius. For this we require the following definition of the *join* of a graph from [3]. The join $G = G_1 + G_2$ has $V(G) = V(G_1) \bigcup V(G_2)$ and $E(G) = E(G_1) \bigcup E(G_2) \bigcup \{uv | u \in V(G_1)$ and $v \in V(G_2)$.

Theorem 7. *A graph G is an $(a,1;2)$-antimode graph with unit radius if and only if it is the join of a complete graph, G_1, and a second graph G_2 with radius $rad(G_2) \geq 2$ and $|V(G_2)| = |V(G_1)| + 1$.*

Proof. ($\Rightarrow$) Suppose G is an $(a,1;2)$-antimode graph with unit radius. Then G has a vertices with eccentricity one and $a+1$ vertices with eccentricity two. Let G_1 be the induced subgraph on the centre, then G_1 must be a complete graph. Let G_2 be the induced subgraph on the periphery, then for $v \in V(G_2), e_{G_2}(v) \geq 2$ since $e_G(v) = 2$. It is easy to see that G is the join of G_1 and G_2.

($\Leftarrow$)Let G be the join of K_a and G_2 where $\text{rad}(G_2) \geq 2$ and the order of G_2 is $a+1$. So for every vertex $v \in V(K_a), e_{K_a}(v) = 1$ and for each vertex $u \in V(G_2), e_{G_2}(u) \geq 2$. Hence $e_{K_a}(v) = 1$ and $e_{G_2}(u) = 2$. Hence G is an $(a,1;2)$-antimode graph with unit radius.

Corollary 1. *A graph G is a $(a,-1;2)$-antimode graph with unit radius if and only if it is the join of a complete graph, G_1, and a second graph G_2 with radius $rad(G_2) \geq 2$ and $|V(G_2)| = |V(G_1)| - 1$.*

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