Edge-antimagic total labeling of disjoint union of caterpillars

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Abstract. Let $G = (V, E)$ be a finite graph, where $V(G)$ and $E(G)$ are the (non-empty) sets of vertices and edges of G. An (a,d)-*edge-antimagic total labeling* is a bijection β from $V(G) \cup E(G)$ to the set of consecutive integers $\{1, 2, \ldots, |V(G)| + |E(G)|\}$ with the property that the set of all the edge-weights, $w(uv) = \beta(u) + \beta(uv) + \beta(v), uv \in E(G)$, is $\{a, a + d, a + 2d, \ldots, a + (|E(G)| - 1)d\}$, for two fixed integers $a > 0$ and $d \geq 0$. Such a labeling is *super* if the smallest possible labels appear on the vertices. In this paper we investigate the existence of super (a,d)-edge-antimagic total labelings for disjoint union of multiple copies of a regular caterpillar.

1 Introduction

All graphs considered in this paper are finite, simple and undirected. The graph G has vertex set $V(G)$ and edge set $E(G)$ and we let $p = |V(G)|$ and $q = |E(G)|$. The general references for graph theoretic notions are [16] and [17].

A *vertex labeling* of G is a bijection $\alpha : V(G) \to \{1, 2, \ldots, p\}$ and the associated edge-weight $w_\alpha(uv)$ of an edge uv in G is $w_\alpha(uv) = \alpha(u) + \alpha(v)$. A *total labeling* of G is a bijection $\beta : V(G) \cup E(G) \to \{1, 2, \ldots, p+q\}$ and the associated edge-weight $w_\beta(uv)$, $uv \in E(G)$, is $w_\beta(uv) = \beta(u) + \beta(uv) + \beta(v)$.

The vertex labeling α of G is (a,d)-*edge-antimagic vertex labeling* if the set of all the edge-weights, $\{w_\alpha(uv) : uv \in E(G)\}$, is $\{a, a+d, a+2d, \ldots, a+(q-1)d\}$, for two integer $a > 0$ and $d \geq 0$. In his Ph.D thesis, Hegde called this labeling a *strongly (a,d)-indexable* (see Acharya and Hegde [1]).

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An (a,d)-*edge-antimagic total labeling* of G is the total labeling β with the property that the edge-weights $w_\beta(uv)$, $uv \in E(G)$, form an arithmetic progression $a, a+d, a+2d, \dots, a+(q-1)d$, where $a > 0$ and $d \geq 0$ are two fixed integers. Such a labeling is *super* if $\beta(V) = \{1, 2, \dots, p\}$.

These labelings, introduced by Simanjuntak *et al.* in [12], are natural extensions of the concept of *magic valuation*, studied by Kotzig and Rosa [9] (see also [11],[15]), and the concept of *super edge-magic labeling*, defined by Enomoto *et al.* in [5] (see also [6],[7]).

Assume that the graph G has a super (a,d)-edge-antimagic total labeling $\beta : V(G) \cup E(G) \to \{1, 2, \dots, p+q\}$. The minimum possible edge-weight under the labeling β is at least $1 + 2 + (p+1) = p+4$. On the other hand, the maximum possible edge-weight is at most $(p-1) + p + (p+q) = 3p + q - 1$. So we have $a \geq p+4$ and $a + (q-1)d \leq 3p + q - 1$. The last inequality gives an upper bound for the difference d, i.e.

$$d \leq \frac{2p+q-5}{q-1}. \tag{1}$$

In this paper we investigate the existence of super edge-antimagicness for disconnected graphs. Ivančo and Lučkaničová [8] described some constructions of super edge-magic (super $(a,0)$-edge-antimagic total) labelings for $nC_k \cup mP_k$ and $K_{1,m} \cup K_{1,n}$. Baskoro and Ngurah [4] described super edge-magic total labelings for nP_3. The super (a,d)-edge-antimagic total labelings for $P_n \cup P_{n+1}$, $nP_2 \cup P_n$ and $nP_2 \cup P_{n+2}$ have been described by Sudarsana *et al.* in [13] and (a,d)-edge-antimagic total labeling of mC_n can be found in [10].

We are studying super (a,d)-edge-antimagic total labeling for the disjoint union of multiple copies of a regular caterpillar. The paper concludes with a list of open problems and conjecture.

2 Disjoint union of caterpillars

A caterpillar is a graph derived from a path by hanging any number of leaves from the vertices of the path. We denote the caterpillar as $S_{t_1,t_2,\dots,t_n}$, where $x_1, x_2, \dots, x_n$ are vertices of the path and $x_{i,k}$, $1 \leq k \leq t_i$, are leaves from the vertex x_i, for $1 \leq i \leq n$. In [14], it is proved that all caterpillars $S_{t_1,t_2,\dots,t_n}$ have a super (a,d)-edge-antimagic total labeling for $d \in \{0,1,2\}$.

Any caterpillar is bipartite; thus we denote by $\{A, B\}$ the bipartition of its vertex set. As a consequence of the results, which have been proved for edge-antimagicness of trees in [3], it follows that any caterpillar with $|A| - |B| \leq 1$ admits a super $(a,3)$-edge-antimagic total labeling.

Now, we will study the super edge-antimagicness of a disjoint union of m copies of a caterpillar denoted by $mS_{t_1,t_2,\dots,t_n}$. It is the disconnected graph with vertex set

$$V(mS_{t_1,t_2,\dots,t_n}) = \bigcup_{j=1}^{m}\bigcup_{i=1}^{n}\left\{x_i^j\right\} \cup \bigcup_{j=1}^{m}\bigcup_{i=1}^{n}\left\{x_{i,k}^j : 1 \leq k \leq t_i\right\}$$

and edge set

$$E(mS_{t_1,t_2,\dots,t_n}) = \bigcup_{j=1}^{m}\bigcup_{i=1}^{n-1}\left\{x_i^j x_{i+1}^j\right\} \bigcup \bigcup_{j=1}^{m}\bigcup_{i=1}^{n}\left\{x_i^j x_{i,k}^j : 1 \le k \le t_i\right\}$$

. If every vertex of the path of caterpillar $S_{t_1,t_2,\dots,t_n}$ has the same number of leaves, i.e. $t_1 = t_2 = \cdots = t_n$, then the caterpillar is said to be a *regular*.

If the disjoint union of m copies of a regular caterpillar $mS_{t_1,t_2,\dots,t_n}$, $t_1 = t_2 = \cdots = t_n = t$, has a super (a,d)-edge-antimagic total labeling then, for $p = mn(t+1)$ and $q = mn(t+1)-m$, it follows from (1) that $d \le 3 + \frac{2m-2}{mn(t+1)-m-1}$. If $m \ge 2$, $n \ge 2$ and $t \ge 1$ then $\frac{2m-2}{mn(t+1)-m-1} < 1$ and thus $d < 4$.

The following theorem describes an $(a,1)$-edge-antimagic vertex labeling for the disjoint union of m copies of a regular caterpillar.

Theorem 4 *If mn is odd, $m,n \ge 3$, then the graph $mS_{t_1,t_2,\dots,t_n}$, for $t_1 = t_2 = \cdots = t_n = t \ge 1$, has a $(\frac{mn+2m+3}{2}, 1)$-edge-antimagic vertex labeling.*

Proof. For mn odd, $m \ge 3$, $n \ge 3$ and $t_1 = t_2 = \cdots = t_n = t \ge 1$, we define a vertex labeling $\alpha_1 : V(mS_{t_1,t_2,\dots,t_n}) \to \{1, 2, \dots, mn(t+1)\}$ as follows

$$\alpha_1(x_i^j) = \begin{cases} \frac{m(i-1)+j+1}{2}, & \text{if } 1 \le i \le n-2 \text{ is odd, and } j \text{ is odd} \\ \frac{mi+1+j}{2}, & \text{if } 1 \le i \le n-2 \text{ is odd, and } j \text{ is even} \\ \frac{(n+i+1)m}{2} - j + 1, & \text{if } i \text{ is even, and } 1 \le j \le m \\ \frac{mn+j}{2}, & \text{if } i = n, \text{ and } j \text{ is odd} \\ \frac{m(n-1)+j}{2}, & \text{if } i = n, \text{ and } j \text{ is even} \end{cases}$$

and for $1 \le j \le m$ and $1 \le k \le t$

$$\alpha_1(x_{i,k}^j) = \begin{cases} \frac{mn(2k+1)+j}{2} + \frac{m(i-1)}{2} + 1, & \text{if } 1 \le i \le n-2 \text{ is odd} \\ & \text{and } j \text{ is odd} \\ \frac{mn(2k+1)+mi+j}{2} + 1, & \text{if } 1 \le i \le n-2 \text{ is odd} \\ & \text{and } j \text{ is even} \\ kmn + \frac{m(i+1)+3}{2} - j, & \text{if } i \text{ is even} \\ kmn + \frac{j+1}{2}, & \text{if } i = n, \text{ and } j \text{ is odd} \\ mn(k+1) + \frac{j-m+1}{2}, & \text{if } i = n, \text{ and } j \text{ is even.} \end{cases}$$

The vertex labeling α_1 is a bijective function. The edge-weights, under the labeling α_1, constitute the sets

$W_{\alpha_1}^1 = \{w_{\alpha_1}^1(x_i^j x_{i+1}^j) = \frac{m(n+2i+1)+3-j}{2} :$ if $1 \le i \le n-2$, and j is odd$\}$,

$W_{\alpha_1}^2 = \{w_{\alpha_1}^2(x_i^j x_{i+1}^j) = \frac{m(n+2i+2)+3-j}{2} :$ if $1 \le i \le n-2$, and j is even$\}$,

$W_{\alpha_1}^3 = \{w_{\alpha_1}^3(x_{n-1}^j x_n^j) = \frac{3mn+2-j}{2} :$ if j is odd$\}$,

$W_{\alpha_1}^4 = \{w_{\alpha_1}^4(x_{n-1}^j x_n^j) = \frac{m(3n-1)+2-j}{2} :$ if j is even$\}$,

$W_{\alpha_1}^5 = \{w_{\alpha_1}^5(x_i^j x_{i,k}^j) = \frac{mn(2k+1)+3}{2} + m(i-1) + j :$ if $1 \le i \le n-2$ is odd, $1 \le k \le t$, and j is odd$\}$,

$W_{\alpha_1}^6 = \{w_{\alpha_1}^6(x_i^j x_{i,k}^j) = \frac{mn(2k+1)+3}{2} + mi + j :$ if $1 \le i \le n-2$ is odd, $1 \le k \le t$, and j is even$\}$,

$W_{\alpha_1}^7 = \{w_{\alpha_1}^7(x_i^j x_{i,k}^j) = \frac{mn(2k+1)+5}{2} + m(i+1) - 2j :$ if $2 \le i \le n-1$ is even, $1 \le k \le t$, and $1 \le j \le m\}$,

$W_{\alpha_1}^8 = \{w_{\alpha_1}^8(x_n^j x_{n,k}^j) = \frac{mn(2k+1)+1}{2} + j :$ if $1 \le k \le t$, and j is odd$\}$,

$W_{\alpha_1}^9 = \{w_{\alpha_1}^9(x_n^j x_{n,k}^j) = \frac{mn(2k+3)+1}{2} - m + j :$ if $1 \le k \le t$, and j is even$\}$.

It is not difficult to see that the set $\bigcup_{r=1}^9 W_{\alpha_1}^r = \left\{\frac{mn+2m+3}{2}, \frac{mn+2m+5}{2}, \ldots, \frac{mn(2t+3)+1}{2}\right\}$ contains an arithmetic sequence with the first term $\frac{mn+2m+3}{2}$ and common difference 1. Thus α_1 is a $\left(\frac{mn+2m+3}{2}, 1\right)$-edge-antimagic vertex labeling.

□

For extending edge-antimagic vertex labelings to edge-antimagic total labelings we can use the following theorem that appeared in [2].

Theorem 5 [2] *If a graph G with p vertices and q edges has an (a,d)-edge-antimagic vertex labeling, then*

(i) G has a super $(a+p+1, d+1)$-edge-antimagic total labeling,

(ii) G has a super $(a+p+q, d-1)$-edge-antimagic total labeling.

If we consider the $\left(\frac{mn+2m+3}{2}, 1\right)$-edge-antimagic vertex labeling α_1 from Theorem 4 then applying Theorem 5 we obtain the following result.

Theorem 6 *If mn is odd, $m \ge 3$ and $n \ge 3$, then the graph $mS_{t_1,t_2,\ldots,t_n}$, for $t_1 = t_2 = \cdots = t_n = t \ge 1$, has a super $\left(\frac{mn(4t+5)+3}{2}, 0\right)$-edge-antimagic total labeling and a super $\left(\frac{mn(2t+3)+5}{2} + m, 2\right)$-edge-antimagic total labeling.*

The next Lemma, which appeared in [14], will be useful for proving the existence of a super $(a,1)$-edge-antimagic total labeling.

Lemma 3 [14] *Let $\mathfrak{U}$ be a sequence $\mathfrak{U} = \{c, c+1, c+2, \ldots c+k\}$, k even. Then there exists a permutation $\Pi(\mathfrak{U})$ of the elements of $\mathfrak{U}$ such that $\mathfrak{U} + \Pi(\mathfrak{U}) = \{2c + \frac{k}{2}, 2c + \frac{k}{2} + 1, 2c + \frac{k}{2} + 2, \ldots, 2c + \frac{3k}{2} - 1, 2c + \frac{3k}{2}\}$.*

For the disjoint union of m copies of a regular caterpillar we obtain the following result.

Theorem 7 *If the product mnt is odd, $m \ge 3$, $n \ge 3$ and $t \ge 1$, then the graph $mS_{t_1,t_2,\ldots,t_n}$, for $t_1 = t_2 = \cdots = t_n = t$, has a super $\left(\frac{mn(3t+4)+m}{2} + 2, 1\right)$-edge-antimagic total labeling.*

Proof. Consider the vertex labeling α_1 from Theorem 4, which is $\left(\frac{mn+2m+3}{2}, 1\right)$-edge-antimagic vertex, where the set of edge-weights gives the sequence $\mathfrak{U} = \{c, c+1, c+2, \ldots, c+k\}$ for $c = \frac{mn+2m+3}{2}$ and $k = mn(t+1) - m - 1$. If mnt is odd then k is even. According to Lemma 3, there exists a permutation $\Pi(\mathfrak{U})$ of the elements of $\mathfrak{U}$, such that

$$\mathfrak{U} + [\Pi(\mathfrak{U}) - c + mn(t+1) + 1]$$

$= \left\{ \frac{mn(3t+4)+m}{2} + 2, \frac{mn(3t+4)+m}{2} + 3, \ldots, \frac{mn(5t+6)-m}{2}, \frac{mn(5t+6)-m}{2} + 1 \right\}.$

If $[H(\mathfrak{U}) - c + mn(t+1) + 1]$ is an edge labeling of the considered disconnected graph with labels $p+1, p+2, \ldots, p+q$, then $\mathfrak{U} + [H(\mathfrak{U}) - c + mn(t+1) + 1]$ determines the set of the edge-weights under the resulting total labeling. This implies that the resulting total labeling is super $\left(\frac{mn(3t+4)+m}{2} + 2, 1 \right)$-edge-antimagic total. □

The next theorem gives a super $(a, 1)$-edge-antimagic total labeling for m even and n odd.

Theorem 8 *If m is even, $m \geq 2$, and n is odd, $n \geq 3$, then the graph $mS_{t_1,t_2,\ldots,t_n}$, for $t_1 = t_2 = \cdots = t_n = t \geq 1$, has a super $(a, 1)$-edge-antimagic total labeling.*

Proof. Let us consider three cases.

Case 1. $n = 3$.

For m even, $n = 3$ and t odd, define the bijection $\beta_1 : V(mS_{t,t,t}) \cup E(mS_{t,t,t}) \to \{1, 2, \ldots, m(6t+5)\}$ in the following way:

$$\beta_1(x_i^j) = \begin{cases} \frac{m(i-1)}{2} + j, & \text{if } i = 1, 3 \text{ and } 1 \leq j \leq m \\ 2m + j, & \text{if } i = 2 \text{ and } 1 \leq j \leq m \end{cases}$$

and for $1 \leq j \leq m$ and $1 \leq k \leq t$, as follows

$\beta_1(x_{i,k}^j) = (3k - i + 3)m + j$, if $i = 1, 2$ and 3.

$$\beta_1(x_i^j x_{i+1}^j) = \begin{cases} (6t+5)m + 1 - j, & \text{if } i = 1 \\ \frac{(9t+7)m}{2} + 1 - j, & \text{if } i = 2 \end{cases}$$

$$\beta_1(x_i^j x_{i,k}^j) = \begin{cases} \frac{(9t-3k+8)m}{2} + 1 - j, & \text{if } i = 1 \text{ and } k \text{ odd} \\ \frac{(12t-3k+2i+5)m}{2} + 1 - j, & \text{if } i = 2, 3 \text{ and } k \text{ odd} \\ \frac{(12t-3k+10)m}{2} + 1 - j, & \text{if } i = 1 \text{ and } k \text{ even} \\ \frac{(9t-3k+2i+3)m}{2} + 1 - j, & \text{if } i = 2, 3 \text{ and } k \text{ even.} \end{cases}$$

The edge-weights under the labeling β_1 constitute the sets

$W_{\beta_1}^1 = \{w_{\beta_1}^1(x_i^j x_{i+1}^j) = (6t+7)m + j + 1 : \text{ if } i = 1 \text{ and } 1 \leq j \leq m\}$,

$W_{\beta_1}^2 = \{w_{\beta_1}^2(x_i^j x_{i+1}^j) = \frac{(9t+13)m}{2} + j + 1 : \text{ if } i = 2 \text{ and } 1 \leq j \leq m\}$,

$W_{\beta_1}^3 = \{w_{\beta_1}^3(x_i^j x_{i,k}^j) = \frac{(9t+3k+12)m}{2} + j + 1 : \text{ if } i = 1,\ k \text{ odd}$ and $1 \leq j \leq m\}$,

$W_{\beta_1}^4 = \{w_{\beta_1}^4(x_i^j x_{i,k}^j) = \frac{(12t+3k+14)m}{2} + j + 1 : \text{ if } i = 1,\ k \text{ even}$ and $1 \leq j \leq m\}$,

$W_{\beta_1}^5 = \{w_{\beta_1}^5(x_i^j x_{i,k}^j) = \frac{(12t+3k-2i+19)m}{2} + j + 1 : \text{ if } i = 2, 3,\ k \text{ odd}$ and $1 \leq j \leq m\}$,

$W_{\beta_1}^6 = \{w_{\beta_1}^6(x_i^j x_{i,k}^j) = \frac{(9t+3k-2i+17)m}{2} + j + 1 : \text{ if } i = 2, 3,\ k \text{ even}$ and $1 \leq j \leq m\}$.

Hence, the set $\bigcup_{r=1}^{6} W_{\beta_1}^r = \left\{ \frac{(9t+13)m}{2} + 2, \ldots, \frac{(15t+17)m}{2} + 1 \right\}$ consists of consecutive integers. Thus the labeling β_1 is a super $\left(\frac{(9t+13)m}{2} + 2, 1 \right)$-edge-antimagic total labeling.

When t is even and $n = 3$, we consider the total labeling $\beta_2 : V(mS_{t,t,t}) \cup E(mS_{t,t,t}) \to \{1, 2, \ldots, m(6t+5)\}$ such that

$\beta_2(x_i^j) = \beta_1(x_i^j)$, $\beta_2(x_{i,k}^j) = \beta_1(x_{i,k}^j)$, $\beta_2(x_i^j x_{i+1}^j) = \beta_1(x_i^j x_{i+1}^j)$,

$\beta_2(x_i^j x_{i,k}^j) = \beta_1(x_i^j x_{i,k}^j)$ for all i, k and j without the case when $i = 2$, $k = t$ and $\frac{m}{2} + 1 \le j \le m$.

If $i = 2$, $k = t$ and $\frac{m}{2} + 1 \le j \le m$ then we put $\beta_2(x_i^j x_{i,k}^j) = \frac{9m(t+1)}{2} + 1 - j$.

By direct computation we obtain that the set of edge-weights consists of the arithmetic sequence $\left\{\frac{(9t+13)m}{2} + 2, \frac{(9t+13)m}{2} + 3, \dots, \frac{(15t+17)m}{2} + 1\right\}$ and therefore the total labeling β_2 is super $\left(\frac{(9t+13)m}{2} + 2, 1\right)$-edge-antimagic total.

Case 2. $n = 5$.

Let us consider a total labeling $\beta_3 : V(mS_{t_1,t_2,\dots,t_n}) \cup E(mS_{t_1,t_2,\dots,t_n}) \to \{1, 2, \dots, 2mn(t+1) - m\}$, for $t_1 = t_2 = \dots = t_n = t$, where

$$\beta_3(x_i^j) = \begin{cases} j, & \text{if } i = 1 \text{ and } 1 \le j \le \frac{m}{2} \\ (n - i + 1)m + j, & \text{if } 2 \le i \le n \text{ and } 1 \le j \le \frac{m}{2} \\ \left(\frac{n-1}{2} - i\right) m + j, & \text{if } 1 \le i \le \frac{n-1}{2} \text{ and } \frac{m+2}{2} \le j \le m \\ \left(\frac{3n-1}{2} - i\right) m + j, & \text{if } \frac{n+1}{2} \le i \le n \text{ and } \frac{m+2}{2} \le j \le m \end{cases}$$

$$\beta_3(x_i^j x_{i+1}^j) = \begin{cases} \frac{mn}{2}(4t+3) + \frac{m}{2} + 1 - j, & \text{if } i = 1 \text{ and } 1 \le j \le \frac{m}{2} \\ mn(2t+1) + \frac{m}{2}(2i-1) + 1 - j, & \text{if } 2 \le i \le n-1 \text{ and } 1 \le j \le \frac{m}{2} \\ 2mn(t+1) + (2i - n + 2)\frac{m}{2} + 1 - j, & \text{if } 1 \le i \le \frac{n-3}{2} \\ & \text{and } \frac{m+2}{2} \le j \le m \\ \frac{mn}{2}(3t+2) + m + 1 - j, & \text{if } i = \frac{n-1}{2} \text{ and } \frac{m+2}{2} \le j \le m \\ \frac{mn}{2}(4t+1) + m(i+1) + 1 - j, & \text{if } \frac{n+1}{2} \le i \le n-1 \\ & \text{and } \frac{m+2}{2} \le j \le m \end{cases}$$

and for $1 \le k \le t$ in the following way

$$\beta_3(x_{i,k}^j) = \begin{cases} (nk + n - i)m + j, & \text{if } 1 \le i \le n \text{ and } 1 \le j \le \frac{m}{2} \\ \left(\frac{n-3}{2} + nk - i\right) m + j, & \text{if } 1 \le i \le \frac{n-3}{2} \text{ and } \frac{m+2}{2} \le j \le m \\ \left(\frac{3n-3}{2} + nk - i\right) m + j, & \text{if } \frac{n-1}{2} \le i \le n \text{ and } \frac{m+2}{2} \le j \le m \end{cases}$$

$$\beta_3(x_i^j x_{i,k}^j) = \begin{cases} \frac{mn}{2}(3t - k + 2) + m + 1 - j, & \text{if } i = 1 \text{ and } 1 \le j \le \frac{m}{2} \\ \frac{mn}{2}(4t + 2 - k) + \frac{m}{2}(2i-1) + 1 - j, & \text{if } 2 \le i \le \frac{n+1}{2} \text{ and } 1 \le j \le \frac{m}{2} \\ \frac{mn}{2}(3t + 1 - k) + mi + 1 - j, & \text{if } \frac{n+3}{2} \le i \le n \text{ and } 1 \le j \le \frac{m}{2} \\ \frac{mn}{2}(3t - k + 3) + 1 - j, & \text{if } 1 \le i \le \frac{n-3}{2} \\ & \text{and } \frac{m+2}{2} \le j \le m \\ \frac{mn}{2}(3t - k + 2) + m + 1 - j, & \text{if } i = \frac{n-1}{2} \text{ and } \frac{m+2}{2} \le j \le m \\ \frac{mn}{2}(4t + 1 - k) + m(i+1) + 1 - j, & \text{if } \frac{n+1}{2} \le i \le n \\ & \text{and } \frac{m+2}{2} \le j \le m. \end{cases}$$

The reader can easily verify that the labeling β_3 uses each integer $1, 2, \dots, 2mn(t+1) - m$ exactly once and the edge-weights of all the edges constitute the set $\{\frac{mn}{2}(3t+4)+\frac{m}{2}+2, \frac{mn}{2}(3t+4)+\frac{m}{2}+3, \dots, \frac{mn}{2}(5t+6)-\frac{m}{2}+1\}$. So the labeling β_3 is super $(\frac{mn}{2}(3t+4)+\frac{m}{2}+2, 1)$-edge-antimagic total labeling.

Case 3. $n \geq 7$.

Let us construct a total labeling β_4 from $V(mS_{t_1,t_2,\dots,t_n}) \cup E(mS_{t_1,t_2,\dots,t_n})$, for $t_1 = t_2 = \dots = t_n = t$, to the set of consecutive integers $\{1, 2, \dots, 2mn(t+1)-m\}$ as follows

$\beta_4(x_i^j) = \beta_3(x_i^j)$ for $1 \leq i \leq n$ and $1 \leq j \leq m$,

$$\beta_4(x_i^j x_{i+1}^j) = \begin{cases} \frac{mn}{2}(4t+3)+\frac{m}{2}+1-j, & \text{if } i=1 \text{ and } 1 \leq j \leq \frac{m}{2} \\ mn(2t+1)+\frac{m}{2}(2i-1)+1-j, & \text{if } 2 \leq i \leq n-1 \text{ and } 1 \leq j \leq \frac{m}{2} \\ 2mn(t+1)+(2i-n+2)\frac{m}{2}+1-j, & \text{if } 1 \leq i \leq \frac{n-3}{2} \\ & \text{and } \frac{m+2}{2} \leq j \leq m \\ 2mnt+m+1-j, & \text{if } i=\frac{n-1}{2} \text{ and } \frac{m+2}{2} \leq j \leq m \\ \frac{mn}{2}(4t+1)+m(i+1)+1-j, & \text{if } \frac{n+1}{2} \leq i \leq n-1 \\ & \text{and } \frac{m+2}{2} \leq j \leq m \end{cases}$$

and for $1 \leq k \leq t$ in the following way

$\beta_4(x_{i,k}^j) = \beta_3(x_{i,k}^j)$,

$$\beta_4(x_i^j x_{i,k}^j) = \begin{cases} \frac{mn}{2}(4t-k)+m+1-j, & \text{if } i=1 \text{ and } 1 \leq j \leq \frac{m}{2} \\ \frac{mn}{2}(4t+2-k)+\frac{m}{2}(2i-1)+1-j, & \text{if } 2 \leq i \leq \frac{n+1}{2} \text{ and } 1 \leq j \leq \frac{m}{2} \\ \frac{mn}{2}(4t-1-k)+mi+1-j, & \text{if } \frac{n+3}{2} \leq i \leq n \text{ and } 1 \leq j \leq \frac{m}{2} \\ \frac{mn}{2}(4t-k)+\frac{m}{2}(2i+3)+1-j, & \text{if } 1 \leq i \leq \frac{n-3}{2} \\ & \text{and } \frac{m+2}{2} \leq j \leq m \\ \frac{mn}{2}(4t-k)+m+1-j, & \text{if } i=\frac{n-1}{2} \text{ and } \frac{m+2}{2} \leq j \leq m \\ \frac{mn}{2}(4t+1-k)+m(i+1)+1-j, & \text{if } \frac{n+1}{2} \leq i \leq n \\ & \text{and } \frac{m+2}{2} \leq j \leq m. \end{cases}$$

There is no problem in seeing that the labeling β_4 uses each integer $1, 2, \dots, 2mn(t+1)-m$ exactly once and this implies that the labeling β_4 is a bijection. We can observe that under the labeling β_4 the edge-weights of all the edges constitute the sets

$W_{\beta_4}^1 = \{w_{\beta_4}^1(x_i^j x_{i+1}^j) = \frac{mn}{2}(4t+5)-\frac{m}{2}+1+j : \text{ if } i=1 \text{ and } 1 \leq j \leq \frac{m}{2}\}$,

$W_{\beta_4}^2 = \{w_{\beta_4}^2(x_i^j x_{i+1}^j) = mn(2t+3)+\frac{m}{2}(1-2i)+1+j : \text{ if } 2 \leq i \leq \frac{n+1}{2} \text{ and } 1 \leq j \leq \frac{m}{2}\}$,

$W_{\beta_4}^3 = \{w_{\beta_4}^3(x_i^j x_{i+1}^j) = mn(2t+3)+\frac{m}{2}(1-2i)+1+j : \text{ if } \frac{n+3}{2} \leq i \leq n-1 \text{ and } 1 \leq j \leq \frac{m}{2}\}$,

$W_{\beta_4}^4 = \{w_{\beta_4}^4(x_i^j x_{i+1}^j) = \frac{mn}{2}(4t+5)-m(i+1)+1+j : \text{ if } 1 \leq i \leq \frac{n-3}{2} \text{ and } \frac{m+2}{2} \leq j \leq m\}$,

$W_{\beta_4}^5 = \{w_{\beta_4}^5(x_i^j x_{i+1}^j) = mn(2t+1)+1+j : \text{ if } i=\frac{n-1}{2} \text{ and } \frac{m+2}{2} \leq j \leq m\}$,

$W^6_{\beta_4} = \{w^6_{\beta_4}(x^j_i x^j_{i+1}) = \frac{mn}{2}(4t+7) - m(i+1) + 1 + j :$ if $\frac{n+1}{2} \le i \le n-1$ and $\frac{m+2}{2} \le j \le m\}$,

$W^7_{\beta_4} = \{w^7_{\beta_4}(x^j_i x^j_{i,k}) = \frac{mn}{2}(4t+k+2) + 1 + j :$ if $i = 1$ and $1 \le j \le \frac{m}{2}\}$,

$W^8_{\beta_4} = \{w^8_{\beta_4}(x^j_i x^j_{i,k}) = \frac{mn}{2}(4t+k+6) + \frac{m}{2}(1-2i) + 1 + j :$ if $2 \le i \le \frac{n+1}{2}$ and $1 \le j \le \frac{m}{2}\}$.

$W^9_{\beta_4} = \{w^9_{\beta_4}(x^j_i x^j_{i,k}) = \frac{mn}{2}(4t+k+3) + m(1-i) + 1 + j :$ if $\frac{n+3}{2} \le i \le n$ and $1 \le j \le \frac{m}{2}\}$,

$W^{10}_{\beta_4} = \{w^{10}_{\beta_4}(x^j_i x^j_{i,k}) = \frac{mn}{2}(4t+k+2) - \frac{m}{2}(2i+1) + 1 + j :$ if $1 \le i \le \frac{n-3}{2}$ and $\frac{m+2}{2} \le j \le m\}$,

$W^{11}_{\beta_4} = \{w^{11}_{\beta_4}(x^j_i x^j_{i,k}) = \frac{mn}{2}(4t+k+2) + 1 + j :$ if $i = \frac{n-1}{2}$ and $\frac{m+2}{2} \le j \le m\}$,

$W^{12}_{\beta_4} = \{w^{12}_{\beta_4}(x^j_i x^j_{i,k}) = \frac{mn}{2}(4t+k+7) - m(i+1) + 1 + j :$ if $\frac{n+1}{2} \le i \le n$ and $\frac{m+2}{2} \le j \le m\}$,

The set $\bigcup_{r=1}^{12} W^r_{\beta_4} = \{mn(2t+1) + \frac{m}{2} + 2, mn(2t+1) + \frac{m}{2} + 3, \ldots, mn(3t+2) - \frac{m}{2} + 1\}$ consists of consecutive integers, which implies that β_4 is a super $\left(mn(2t+1) + \frac{m}{2} + 2, 1\right)$-edge-antimagic total labeling. □

The next theorem describes an $(a, 2)$-edge-antimagic vertex labeling for the disjoint union of m copies of a regular caterpillar when $t = 2$.

Theorem 9 *There is $(m+2, 2)$-edge-antimagic vertex labeling for $mS_{t_1,t_2,\ldots,t_n}$, for $t_1 = t_2 = \cdots = t_n = 2$ and every $m, n \ge 2$.*

Proof. For $t_1 = t_2 = \cdots = t_n = 2$ and $m, n \ge 2$, define the bijection $\alpha_2 : V(mS_{t_1,t_2,\ldots,t_n}) \to \{1, 2, \ldots, 3mn\}$ as follows

$\alpha_2(x^j_i) = (3i-2)m + j$, if $1 \le i \le n$ and $1 \le j \le m$

$$\alpha_2(x^j_{i,k}) = \begin{cases} (3i-3)m + j, & \text{if } 1 \le i \le n,\ 1 \le j \le m \text{ and } k = 1 \\ (3i-1)m + j, & \text{if } 1 \le i \le n,\ 1 \le j \le m \text{ and } k = 2. \end{cases}$$

Then for the edge-weights we have

$W^1_{\alpha_2} = \{w^1_{\alpha_2}(x^j_i x^j_{i+1}) = (6i-1)m + 2j :$ if $1 \le i \le n-1$ and $1 \le j \le m\}$,

$W^2_{\alpha_2} = \{w^2_{\alpha_2}(x^j_i x^j_{i,k}) = (6i-5)m + 2j :$ if $1 \le i \le n$, $1 \le j \le m$ and $k = 1\}$,

$W^3_{\alpha_2} = \{w^3_{\alpha_2}(x^j_i x^j_{i,k}) = (6i-3)m + 2j :$ if $1 \le i \le n$, $1 \le j \le m$ and $k = 2\}$ and the set $\bigcup_{r=1}^{3} W^r_{\alpha_2} = \{m+2, m+4, \ldots, 6mn-m\}$ contains an arithmetic sequence with the first term $m+2$ and common difference 2. Thus α_2 is a $(m+2, 2)$-edge-antimagic vertex labeling. □

With respect to Theorem 5, the $(m+2, 2)$-edge-antimagic vertex labeling α_2 can be extended to a super edge-antimagic total labeling. Thus for $p = 3mn$ and $q = 3mn - m$ the following theorem holds.

Theorem 10 *The graph $mS_{t_1,t_2,\ldots,t_n}$, for $t_1 = t_2 = \cdots = t_n = 2$, has a super $(6mn+2, 1)$-edge-antimagic total labeling and a super $(3mn+m+3, 3)$-edge-antimagic total labeling, for every $m \ge 2$ and $n \ge 2$.*

3 Conclusion

We summarize that the graph $mS_{t_1,t_2,\ldots,t_n}$, for $t_1 = t_2 = \cdots = t_n = t$, has a super (a,d)-edge-antimagic total labeling for

(i) $d \in \{0,2\}$, if mn is odd and $t \geq 1$,

(ii) $d = 1$, if either mnt is odd, or m is even and n is odd, $t \geq 1$, or $t = 2$ and $m,n \geq 2$,

(iii) $d = 3$, if $m,n \geq 2$ and $t = 2$.

Constructions that will produce a super (a,d)-edge-antimagic total labelings of $mS_{t_1,t_2,\ldots,t_n}$, for $t_1 = t_2 = \cdots = t_n = t$, $d \in \{0,1,2\}$ and every $m,n \geq 2$, have not been found yet. Nevertheless, we suggest the following conjecture.

Conjecture 2 *There is a super (a,d)-edge-antimagic total labeling of the graph $mS_{t_1,t_2,\ldots,t_n}$, for $t_1 = t_2 = \cdots = t_n = t \geq 1$, $d \in \{0,1,2\}$ and for every $m \geq 2$ and $n \geq 2$.*

For the graph $mS_{t_1,t_2,\ldots,t_n}$, for $t_1 = t_2 = \cdots = t_n = t \neq 2$, so far we have not found any super $(a,3)$-edge-antimagic total labeling. So, we propose the following open problem.

Open Problem 2 *For the graph $mS_{t_1,t_2,\ldots,t_n}$, for $t_1 = t_2 = \cdots = t_n = t$, determine if there is a super $(a,3)$-edge-antimagic total labeling, for every $m \geq 2$, $n \geq 2$ and $t \neq 2$.*

In the case when graph $mS_{t_1,t_2,\ldots,t_n}$ does not have any restriction for the values of $t_1, t_2, \ldots, t_n$, the problem to find a super (a,d)-edge-antimagic total labeling seems to be difficult. For further investigation it leads us to suggest the following

Open Problem 3 *Find, if possible, some structural characteristics of a graph $mS_{t_1,t_2,\ldots,t_n}$ which make a super (a,d)-edge-antimagic total labeling impossible.*

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