On the connected partition dimension of unicyclic graphs

Imran Javaid

School of Mathematical Sciences,
Government College University,
68-B, New Muslim Town, Lahore, Pakistan
imranjavaid45@gmail.com

Abstract. Let G be a connected graph. For a vertex $v \in V(G)$ and an ordered k-partition $\Pi = \{S_1, S_2, ..., S_k\}$ of $V(G)$, the representation of v with respect to Π is the k-vector $r(v|\Pi) = (d(v, S_1), d(v, S_2), ..., d(v, S_k))$ where $d(v, S_i) = \min_{w \in S_i}(x, w)(1 \leq i \leq k)$. The k-partition Π is said to be resolving if the k-vectors $r(v|\Pi)$, $v \in V(G)$, are distinct. The minimum k for which there is a resolving k-partition of $V(G)$ is called the partition dimension of G, denoted by $pd(G)$. A resolving k-partition $\Pi = \{S_1, S_2, ..., S_k\}$ of $V(G)$ is said to be connected if each subgraph $< S_i >$ induced by S_i $(1 \leq i \leq k)$ is connected in G. The minimum k for which there is a connected resolving k-partition of $V(G)$ is called the connected partition dimension of G, denoted by $cpd(G)$. In this paper, the connected partition dimension of the unicyclic graphs is calculated and bounds are proposed.

Keywords: unicyclic graph, resolving partition, partition dimension, connected partition dimension.

1 Introduction

If G is a connected graph, the distance $d(u, v)$ between two vertices u and v in G is the length of a shortest path between them. The diameter of G is the largest distance between two vertices in $V(G)$. For a vertex v of a graph G and a subset S of $V(G)$, the distance between v and S is $d(v, S) = \min\{d(v, x)|x \in S\}$. Let $\Pi = \{S_1, S_2, \ldots, S_k\}$ be an ordered k-partition of vertices of G and let v be a vertex of G. The representation $r(v|\Pi)$ of v with respect to Π is the k-tuple $(d(v, S_1), d(v, S_2), \ldots, d(v, S_k))$. If distinct vertices of G have distinct representations with respect to Π, then Π is called a resolving partition for G. The cardinality of a minimal resolving partition is called the partition dimension of G, denoted by $pd(G)$ ([1],[2]). A resolving partition $\Pi = \{S_1, S_2, ..., S_k\}$ of $V(G)$ is called connected if each subgraph $< S_i >$ induced by S_i $(1 \leq i \leq k)$

is connected in G. The minimum k for which there is a connected resolving k-partition of $V(G)$ is called the connected partition dimension of G, denoted by $cpd(G)$ [9].

The concepts of resolvability have previously appeared in the literature (see [1]–[4], [6]–[9]). These concepts have some applications in chemistry for representing chemical compounds [3] or to problems of pattern recognition and image processing, some of which involve the use of hierarchical data structures [5].

If $d(x, S) \neq d(y, S)$ we shall say that the class S separates vertices x and y. If a class S of Π separates vertices x and y we shall also say that Π separates x and y. From these definitions it can be observed that the property of a given partition Π of the vertices of a graph G to be a resolving partition of G can be verified by investigating the pairs of vertices in the same class. Indeed, every vertex $x \in S_i$ $(1 \leq i \leq k)$ is at distance 0 from S_i, but is at a distance different from zero from any other class S_j with $j \neq i$. It follows that $x \in S_i$ and $y \in S_j$ are separated either by S_i or by S_j for every $i \neq j$.

A connected graph with exactly one cycle is called unicyclic graph. Every unicyclic graph that is not a cycle is decomposable into a cycle and one or more trees, the connected partition dimension of whose are known[9]. In this paper, we divide the unicyclic graphs that are not cycles into two types and prove that the connected partition dimension of unicyclic graphs of type 1 is 3 and propose bounds for the connected partition dimension of unicyclic graphs of type 2.

2 The connected partition dimension of the Unicyclic graphs

In [6], metric dimension of unicyclic graphs was given. An identification graph was defined and was given as $G = G[G_1, G_2, u, v]$ which is obtained from G_1 and G_2 by identifying u and v where G_1 and G_2 are non-trivial connected graphs with $u \in V(G_1)$ and $v \in V(G_2)$. Therefore $u = v$ in G. We name the vertex $u = v$ in G, junction vertex. The identification is said to be of type 1 if an end vertex of a path is identified with a vertex of degree two of a cycle in a graph or an end vertex of a path is identified with a vertex of degree 1 of a graph otherwise identification is said to be of type 2. We use this terminology given in [6] to find the connected partition dimension of unicyclic graphs. A unicyclic graph can be obtained by the addition of a single edge between two vertices of a tree. Also a unicyclic graph that is not a cycle can be obtained from a cycle and one or more tree by identifying some specified vertices on the cycle and on the trees.
We calculate the connected partition dimension of unicyclic graph by establishing the relationship between the connected partition dimension of a unicyclic graph and those of its cycles and rooted trees. Here we present two lemmas which will be used in finding the connected partition dimension of unicyclic graphs.

Lemma 1. *Let G_1 and G_2 are nontrivial connected graphs with $u \in V(G_1)$ and $v \in V(G_2)$ and let $G = G[G_1, G_2, u, v]$. If identification is of type 1, then $cpd(G) = cpd(G_1)$*

Proof. Let G_1 be any non-trivial connected graph and G_2 be a path on $n(\geq 2)$ vertices. We suppose that $cpd(G_1) = k$ and in [9], it was shown that $cpd(G) = 2$ if and only if $G = P_n$ for $n \geq 2$ so $cpd(G_2) = 2$. Then there exists resolving partitions $\Pi_1 = \{S_1, S_2, \ldots, S_k\}$ where $S_i \subset V(G_1)$ for $i = 1, 2, \ldots, k$ and $\Pi_2 = \{T_1, T_2\}$ where $T_i \subset V(G_2)$ for $i = 1, 2$.
Let v be the junction vertex and let $v \in S_k$ in Π_1 and $v \in T_1$ in Π_2. We define a new partition $\Pi_v = \{S_1, S_2, \ldots, S_k \cup T_1 \cup T_2\}$. Now, we shall prove that this is a resolving partition of $V(G)$. We shall discuss three cases.
(a). Suppose $v_1, v_2 \in V(G_1)$ where v_1 and v_2 are distinct. Then $r(v_1|\Pi_v) \neq r(v_2|\Pi_v)$ because v_1 and v_2 will be at same distances in G as in G_1 from $S_1, S_2, \ldots$, S_{k-1} and also $d(v_i, S_k) = d(v_i, S_k \cup T_1 \cup T_2)$ for $i = 1, 2$. Since Π_1 is resolving partition, this means Π_v is also resolving partition for all vertices in G_1.
(b). Suppose $v_1, v_2 \in V(G_2)$ where v_1 and v_2 are distinct then $d(v_1, v) \neq d(v_2, v)$ where v is the end vertex. This means that $r(v_i|\Pi_v) = (d(v_i, v) + d(v, S_1), d(v_i, v) + d(v, S_2), \ldots, 0)$ for $i = 1, 2$. This means Π_v resolves all vertices of G_2.
(c). Suppose $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$ and belong to $S_k \cup T_1 \cup T_2$ then there are two cases to be discussed.
Case 1: If the identification is such that an end-vertex of a path is identified with a vertex of degree 1 of a graph then $d(v_1, S_i) < d(v_2, S_i)(1 \leq i \leq k-1)$ which yields $r(v_1|\Pi_v) \neq r(v_2|\Pi_v)$.
Case 2: If the identification is such that an end-vertex of a path is identified with a vertex of degree 2 in a cycle of a graph then as the the vertices of cycle are divided into at least three classes it is easy to see that v_1 and v_2 are different distance from a class containing the vertices of the cycle which implies $r(v_1|\Pi_v) \neq r(v_2|\Pi_v)$.
This means Π_v is a resolving partition for all the vertices of G which implies

$$cpd(G) \leq cpd(G_1) + cpd(G_2) - 2.$$

Now, we show that $cpd(G) \geq cpd(G_1) + cpd(G_2) - 2$. Suppose that this is not the case, then $cpd(G) \leq cpd(G_1) + cpd(G_2) - 3$ which means that $cpd(G) \leq cpd(G_1) - 1$. This suggests that vertices of G_1 can be resolved with fewer connected classes than k which is a contradiction. So $cpd(G) \geq cpd(G_1) + cpd(G_2) - 2$.
Hence

$$cpd(G) = cpd(G_1) + cpd(G_2) - 2.$$

Now, we present a lemma for the connected partition dimension of a graph G when the identification is of type 2.

Lemma 2. *Let G_1 and G_2 are nontrivial connected graphs with $u \in V(G_1)$ and $v \in V(G_2)$ and let $G = G[G_1, G_2, u, v]$. If identification is of type 2, then $cpd(G) \leq cpd(G_1) + cpd(G_2) - 1$.*

Proof. Let G_1 and G_2 be non-trivial connected graph with $cpd(G_1) = k$ and $cpd(G_2) = l$. Then there exist resolving partitions $\Pi_1 = \{S_1, S_2, \ldots, S_k\}$ where $S_i \subset V(G_1)$ for $i = 1, 2, \ldots, k$ and $\Pi_2 = \{T_1, T_2, \ldots, T_l\}$ where $T_j \subset V(G_2)$ for $j = 1, 2, \ldots, l$.
Let v be the junction vertex and let $v \in S_k$ in Π_1 and $v \in T_1$ in Π_2. We define a new partition $\Pi_v = \{S_1, S_2, \ldots, S_k \cup T_1, T_2, \ldots, T_l\}$. Now, we shall prove that

this is a resolving partition of $V(G)$. We shall discuss three cases.
(a). Suppose $v_1, v_2 \in V(G_1)$ where v_1 and v_2 are distinct. Then $r(v_1|\Pi_v) \neq r(v_2|\Pi_v)$ because v_1 and v_2 will be at same distances in G as in G_1 from $S_1, S_2, \ldots, S_{k-1}$ and also $d(v_i, S_k) = d(v_i, S_k \cup T_1)$ for $i = 1, 2$. And $d(v_i, T_j) = d(v_i, v) + d(v, T_j)$ where $i = 1, 2$ and $j = 2, 3, \ldots, l$. Since Π_1 is a resolving partition for $V(G_1)$. This means Π_v is also a resolving partition for all vertices in G_1.
(b). Suppose $v_1, v_2 \in V(G_2)$ where v_1 and v_2 are distinct then similar situations for vertices of G_2 follows as does in the above case for vertices of G_1. So $r(v_1|\Pi_v) \neq r(v_2|\Pi_v)$ for all $v_1, v_2 \in V(G_2)$ where v_1 and v_2 are distinct.
(c). Suppose $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$ and belong to $S_k \cup T_1$ then $d(v_1, S_i) \neq d(v_2, S_i)(1 \leq i \leq k-1)$ which yields $r(v_1|\Pi_v) \neq r(v_2|\Pi_v)$.
This means Π_v is a resolving partition for all the vertices of G. This implies $cpd(G) \leq cpd(G_1) + cpd(G_2) - 1$.

But the inequality $cpd(G) \geq cpd(G_1) + cpd(G_2) - 1$ is not true in general. For example, consider the graph shown in figure below with connected partition dimension 3. If we identify a path at v_2, v_3 or v_4 then again connected partition dimension of the identified graph is 3 which is not consistent with this inequality.

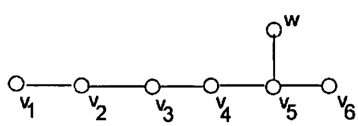

Fig. 1. A tree with connected partition dimension 3

Now, we state the terminology as given in [6]. Let G be the unicyclic graph and let C_n be the unique cycle of G. Let $u_1, u_2, \ldots, u_k$ be the distinct vertices of C with $deg(u_i) \geq 3$ where $1 \leq i \leq k$, and let T_i be the subtree of G rooted at u_i. A unicyclic graph G is said to be of type 1 if and only if every tree T_i of G is a path, one of whose end-vertex is $u_i \in V(C)$ otherwise unicyclic graph is of type 2.

Now, we present theorem that states that the connected partition dimension of unicyclic graphs of type 1 is 3.

Theorem 21 *Let G be a unicyclic graph of type 1 with unique cycle C_n of order n then $cpd(G) = 3$.*

Proof. Let G be a unicyclic graph of type 1 and let $C_n : v_1, v_2, \ldots, v_n, v_1$ be the unique cycle on n vertices. Let $u_1, u_2, \ldots, u_k$ be the distinct vertices of C_n at which k paths are rooted. We know that $cpd(C_n) = 3$ where $n \geq 3$ and in [9], it was shown that $cpd(P_m) = 2(m \geq 2)$. We first assume that $k = 1$. Let G_1 be the unicyclic graph obtained from the cycle C_n and a path P_1 by identifying a vertex in C_n and a vertex of degree one in P_1 and labeling it u_1. By lemma 2.1 $cpd(G_1) = cpd(C_n)$. Now let G_2 be the unicyclic graph obtained from G_1 and a path P_2 by identifying a vertex in C_n in G_1 and a vertex of degree one in P_2 and labeling it u_2. By lemma 2.1, $cpd(G_2) = cpd(C_n)$. Repeating this procedure for $k \geq 3$, we let $G = G_k$ be obtained from G_{k-1} and a path P_k by identifying a vertex in C_n in G_{k-1} and a vertex of degree one in P_k and labeling it u_k. By lemma 2.1, $cpd(G) = cpd(C_n)$. Hence if G is a unicyclic graph of type 1, then $cpd(G) = 3$.

Now, we present the theorem that gives the connected partition dimension of unicyclic graph of type 2.

Theorem 22 *Let G be a unicyclic graph of type 2 with unique cycle C_n of order n then $4 \leq cpd(G) \leq 3 + \sum_{i=1}^{k} cpd(T_i) - k$.*

Proof. Let G be a unicyclic graph of type 2. Since G is a unicyclic graph of type 2 then it has at least one identification of type 2. This means by lemma 2.2 that $cpd(G) \geq 4$. Hence lower bound is verified.
We now prove that $cpd(G) \leq 3 + \sum_{i=1}^{k} cpd(T_i) - k$. Let $C_n : v_1, v_2, \ldots, v_n, v_1$ be the unique cycle of G on n vertices. Let $u_1, u_2, \ldots, u_k$ be the distinct vertices of C_n at which k trees are rooted. We first assume that $k = 1$. Let G_1 be the unicyclic graph obtained from the cycle C_n and a tree T_1 by identifying a vertex in C_n and a vertex of degree one in T_1 and labeling it u_1. By lemma 2.1 and 2.2, $cpd(G_1) \leq 3 + cpd(T_1) - 1$. Now let G_2 be the unicyclic graph obtained from G_1 and a tree T_2 by identifying a vertex in C_n of G_1 and a vertex of degree one in T_2 and labeling it u_2. By lemma 2.1 and 2.2, $cpd(G_2) \leq 3 + cpd(T_1) - 1 + cpd(T_2) - 1$. Repeating this procedure for $k \geq 3$, we let $G = G_k$ be obtained from G_{k-1} and a tree T_k by identifying a vertex in C_n in G_{k-1} and a vertex of degree one in T_k and labeling it u_k. By lemma 2.1 and 2.2, $cpd(G) = cpd(G_k) \leq 3 + cpd(T_1) - 1 + cpd(T_2) - 1 + \ldots + cpd(T_k) - 1$. This means

$$4 \leq cpd(G) \leq 3 + \sum_{i=1}^{k} cpd(T_i) - k.$$

It is frequent question in graph theory that how the value of a graphical parameter is affected when a small change is made in a graph. In this context, we answer the question in the case of connected partition dimension when a single edge is added to a tree. Now we present a theorem that gives the bounds for the connected partition dimension of unicyclic graphs when a single edge is added to a tree. We show that the connected partition dimension can increase by 1 or decrease by 2 analogous to metric dimension[3]. For this purpose, we follow the terminology given in [3] and [9]. A vertex of degree at least 3 in a graph G will be called a major vertex of G. Any end-vertex u of G is said to be a terminal vertex of a major vertex v of G if $d(u, v) < d(u, w)$ for every other major vertex w of G. The terminal degree $ter(v)$ of a major vertex v is the number of terminal vertices of v. A major vertex v of G is an exterior major vertex of G if it has a positive terminal degree. Let $\sigma(G)$ denote the sum of terminal degrees of the major vertices of G, let $ex(G)$ denote the number of exterior major vertices of

G. In [9], connected partition dimension of a tree of order $n \geq 4$ was given as $cpd(T) = \sigma(T) - ex(T) + 1$ and it was also shown that $cpd(G) \geq \sigma(G) - ex(G) + 1$.

Theorem 23 *If T is a tree of order at least 3 and e is an edge of $\overline{T}$, then* $cpd(T) - 2 \leq cpd(T + e) \leq cpd(T) + 1$.

Proof. It was shown in [9] that $cpd(G) \geq \sigma(G) - ex(G) + 1$ and $cpd(T) = \sigma(T) - ex(T) + 1$. It is easy to see that $\sigma(T + e) \geq \sigma(T) - 2$ and $ex(T + e) \leq ex(T)$ from where it follows that $\sigma(T+e) - ex(T+e) + 1 \geq \sigma(T) - ex(T) - 2 + 1 = cpd(T) - 2$. This means $cpd(T + e) \geq cpd(T) - 2$.
It remains to show that $cpd(T + e) \leq cpd(T) + 1$. Suppose that T contains p exterior major vertices $v_1, v_2, \ldots, v_p$. For each i with $1 \leq i \leq p$, let $u_{i1}, u_{i2}, \ldots, u_{ik_i}$ be the terminal vertices of v_i. For each i with $1 \leq i \leq p$, let P_{ij} be the $v_i - u_{ij}$ path in T for all $1 \leq j \leq k_i$ and let x_{ij} be a vertex in P_{ij} that is adjacent to v_i. Then let Q_{ij} be the $x_{ij} - u_{ij}$ subpath of P_{ij} for all $1 \leq i \leq p$ and $1 \leq j \leq k_i$.
Let $U = \{v_1, u_{11}, u_{21}, \ldots, u_{p1}\}$ and let T_1 be the subtree of T of smallest size such that T_1 contains U. Let $S_0 = V(T_1)$ and $S_{ij} = V(Q_{ij})$ for all $1 \leq i \leq p$ and $2 \leq j \leq k_i$. Define a k-partition Π of $V(T)$ by $\Pi = \{S_0, S_{12}, S_{13}, \ldots, S_{1k_1}, S_{22}, S_{23}, \ldots, S_{2k_2}, \ldots, S_{p2}, S_{p3}, \ldots, S_{pk_p}\}$. Then Π is connected and resolving as was shown in Theorem 3.3 of [9]. It is noted that the vertices in one class are separated by more than one class. Let C denote the unique cycle of $T + e$. We consider two cases.
Case 1. If C contains at least two major vertices v and w then resolving partition for T is also a resolving partition for $T + e$. So $cpd(T + e) \leq |\Pi| \leq cpd(T)$.
Case 2. If C contains only one major vertex v then there are two subcases to be discussed.
Subcase 2a. If the edge is between two paths incident at v then resolving partition for T is also a resolving partition for $T + e$. So $cpd(T + e) \leq |\Pi| \leq cpd(T)$.
Subcase 2b. If the edge is between two vertices of a path with more than 3 vertices incident at v then we define a new partition by putting any vertex of the path other than the major vertex in a new class. This will be a resolving partitions for $T + e$. So $cpd(T + e) \leq |\Pi| + 1 \leq cpd(T) + 1$.
Hence

$$cpd(T) - 2 \leq cpd(T + e) \leq cpd(T) + 1.$$

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