Ramsey $(K_{1,2}, C_4)$−minimal Graphs

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Abstract. For any given graphs G and H, we write $F \to (G, H)$ to mean that any red-blue coloring of the edges of F contains a red copy of G or a blue copy of H. Graph F is (G, H)-minimal (Ramsey-minimal) if $F \to (G, H)$ but $F^* \nrightarrow (G, H)$ for any proper subgraph $F^* \subset F$. The class of all (G, H)-minimal graphs is denoted by $\mathcal{R}(G, H)$. In this paper we will determine the graphs in $\mathcal{R}(K_{1,2}, C_4)$.

1 Introduction

We consider simple graphs, namely finite undirected graphs without loops and multiple edges. Let $G = (V, E)$. We say that G *contains* H if G contains a subgraph isomorphic to H. The subgraph of G isomorphic to C_4 is defined as a *basic cycle*.

Let G and H be graphs. We say that $F \to (G, H)$ if any red-blue coloring of the edges of F contains a red copy of G or a blue copy of H. Graph F is *(G, H)-minimal (Ramsey-minimal)* if $F \to (G, H)$ but $F^* \nrightarrow (G, H)$ for any proper subgraph $F^* \subset F$. The class of all (G, H)-minimal graphs is denoted by $\mathcal{R}(G, H)$. In general we follow the terminology in [7]. Recent results on Ramsey numbers can be found in [11].

Here are some previous results dealing with the problem of finding graphs in $\mathcal{R}(G, H)$. Burr, Erdös and Lovász [6] proved that $\mathcal{R}(2K_2, 2K_2) = \{3K_2, C_5\}$ and $\mathcal{R}(K_{1,2}, K_{1,2}) = \{K_{1,3}, C_{2n+1}\}$ for $n \geq 1$. Furthermore, Burr et.al [5] determined all graphs in $\mathcal{R}(2K_2, K_3)$. Later, Burr et.al [4] showed that $\mathcal{R}(K_{1,m}, K_{1,n}) = \{K_{1,m+n-1}\}$ for odd m and n.

In [10] Mengersen and Oeckermann characterized all graphs in $\mathcal{R}(2K_2, K_{1,n})$ for $n \geq 3$. Another result is the characterization of all graphs in $\mathcal{R}(K_{1,2}, K_{1,m})$ for $m \geq 3$ by Borowiecki et.al in [2]. Then Borowiecki et.al [3] determined all graphs in $\mathcal{R}(K_{1,2}, K_3)$. Recently, Baskoro et.al [1] showed that $W_{3t+1} \in \mathcal{R}(P_3, C_3^t)$, where W_{3t+1} is a wheel with $6t+2$ edges and C_3^t is a windmill graph, i.e. a graph obtained by connecting a vertex c (called a hub) to all vertices of tK_2.

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The problem of characterizing pairs of graphs (G, H) for which the set $\mathcal{R}(G, H)$ is finite or infinite has also been investigated in numerous papers. In particular, all pairs of two forests for which the set $\mathcal{R}(G, H)$ is finite are specified in a theorem of Faudree [8]. Then Luczak [9] stated that if G is a forest other than a matching and H is a graph containing at least one cycle then $\mathcal{R}(G, H)$ is infinite. It follows that the set $\mathcal{R}(K_{1,2}, C_4)$ is infinite. In this paper we will determine the graphs in $\mathcal{R}(K_{1,2}, C_4)$.

2 Some classes of graphs

We define some classes of graphs needed to prove our main results.

Let k be positive integer, $k \geq 2$. A graph G with

$$\begin{aligned} V(G) &= \{w_i \mid 1 \leq i \leq k\} \cup \{u_i \mid 1 \leq i \leq k\} \cup \{v_j \mid 1 \leq j \leq k+1\}, \\ E(G) &= \{v_i w_i \mid 1 \leq i \leq k\} \cup \{v_i u_i \mid 1 \leq i \leq k\} \cup \{u_i v_{i+1} \mid 1 \leq i \leq k\} \\ &\quad \cup \{w_i v_{i+1} \mid 1 \leq i \leq k\} \end{aligned}$$

is called the *C_4-path*. We define vertices v_1 and v_{k+1} as the *end vertices* of the C_4-path.

A *C_4-cycle* is constructed by identifying two end vertices of C_4-path. The *length* of a C_4-path (a C_4-cycle) is the number of basic cycles in the C_4-path (the C_4-cycle).

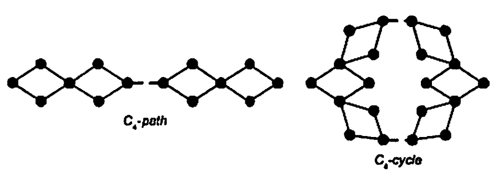

Fig. 1. C_4-path and C_4-cycle

We use C_4-path and C_4-cycle (Figure 1), graphs L_1 and L_2 (Figure 2) to define some classes of graphs below.

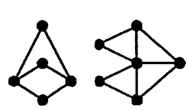

Fig. 2. L_1 with root x and L_2 with root y

Let $\mathcal{A}$ be a family of graphs which contains

(1) A_1, A_2 and A_3 (Figure 3),

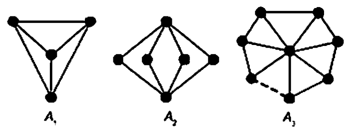

Fig. 3. A_1, A_2 and A_3

(2) $A_4(k)$, $k \geq 0$. This graph consists of two copies of L_1 with a C_4-path of length k joining two roots of L_1 (if $k = 0$ then we have two copies of L_1 with a common root),
(3) $A_5(k)$, $k \geq 0$. This graph consists of a L_1 and a L_2 with a C_4-path of length k joining the root of L_1 and the root of L_2,
(4) $A_6(k)$, $k \geq 0$. This graph consists of two copies of L_2 with a C_4-path of length k joining two roots of L_2 (for $k = 0$ we have two copies of L_2 with a common root).

Let $\mathcal{B}$ be a family of graphs which contains

(1) $B_1(k,t)$, $k \geq 0$, $t \geq 2$. This graph consists of a C_4-cycle of length t and a L_1 with a C_4-path of length k joining the root of L_1 and an arbitrary vertex of degree 4 of C_4-cycle,
(2) $B_2(k,t)$, $k \geq 0$, $t \geq 2$. This graph consists of a C_4-cycle of length t and a L_2 with a C_4-path of length k joining the root of L_2 and an arbitrary vertex of degree 4 of C_4-cycle,
(3) $B_3(k,t)$, $k \geq 0$, $t \geq 2$. This graph consists of a C_4-cycle of length t and a L_1 with a C_4-path of length k joining the root of L_1 and an arbitrary vertex of degree 2 of C_4-cycle,
(4) $B_4(k,t)$, $k \geq 0$, $t \geq 2$. This graph consists of a C_4-cycle of length t and a L_2 with a C_4-path of length k joining the root of L_2 and an arbitrary vertex of degree 2 of C_4-cycle.

Let $\mathcal{D}$ be a family of graphs which contains

(1) $D_1(k,t_1,t_2)$, $k \geq 0$, $t_1 \geq 2$, $t_2 \geq 2$. This graph is constructed by joining two copies of C_4-cycles of length t_1 and t_2 and a C_4-path of length k. One of the end vertices of C_4-path is identified with an arbitrary vertex of degree 4 of the first C_4-cycle and the other end vertex is identified with an arbitrary vertex of degree 4 of the second C_4-cycle,
(2) $D_2(k,t_1,t_2)$, $k \geq 0$, $t_1 \geq 2$, $t_2 \geq 2$. This graph is constructed by joining two copies of C_4-cycles of length t_1 and t_2 and a C_4-path of length k. One of the end vertices of C_4-path is identified with an arbitrary vertex of degree 4 of the first C_4-cycle and the other end vertex is identified with an arbitrary vertex of degree 2 of the second C_4-cycle,

(3) $D_3(k, t_1, t_2)$, $k \geq 0$, $t_1 \geq 2$, $t_2 \geq 2$. This graph is constructed by joining two copies of C_4-cycles of length t_1 and t_2 and a C_4-path of length k. One of the end vertices of C_4-path is identified with an arbitrary vertex of degree 2 of the first C_4-cycle and the other end vertex is identified with an arbitrary vertex of degree 2 of the second C_4-cycle.

Let $\mathcal{I} = \{I(t, k_1, k_2),\ t \geq 2,\ k_1 \geq 0,\ k_2 \geq 0\}$ where $I(t, k_1, k_2)$ is constructed by joining a C_4-cycle of length t with two copies of C_4-paths of length k_1 and k_2. Two of the end vertices of the C_4-paths are identified with two arbitrary vertices of C_4-cycle (the C_4-paths are attached to different vertices of C_4-cycle).

3 Main Results

The *distance* $d(u, v)$ between two (not necessary distinct) vertices u and v in a graph G is the length of a shortest path between them. When u and v are identical, their distance is 0. When u and v are unreachable from each other, their distance is defined to be ∞. The *diameter of G*, *diam*(G), is the greatest distance between any two vertices in that graph. In Theorems 1, 2 and 3 we present a collection of graphs that belongs to $\mathcal{R}(K_{1,2}, C_4)$.

Theorem 1. *If* $\mathcal{R}_1 = \{F \in \mathcal{R}(K_{1,2}, C_4) \mid diam(F) = 1\}$ *then* $\mathcal{R}_1 = \{A_1\}$.

Proof. It can be easily seen that $A_1 \simeq K_4$. Consider any red-blue coloring of A_1 that implies an edge-decomposition $A_1 = A_{11} \oplus A_{12}$. Let $A_{11} \not\supseteq K_{1,2}$. Then A_{11} contains at most one edge. Therefore $A_{12} \supseteq C_4$. Consequently $A_1 \rightarrow (K_{1,2}, C_4)$.

To prove the minimality of A_1, let $V(A_1) = \{v_i \mid 1 \leq i \leq 4\}$. Consider $A_1^* \simeq A_1 \backslash \{e\}$ for any fixed edge $e \in E(A_1)$. Without loss of generality, we assume that $e = v_1v_2$. Then the edges of A_1^* can be partitioned into two classes, namely E_1 and E_2, with $E_1 = \{v_2v_4, v_1v_3\}$ and $E_2 = \{v_1v_4, v_2v_3, v_3v_4\}$ such that $A_1^*[E_1] \not\supseteq K_{1,2}$ and $A_1^*[E_2] \not\supseteq C_4$. Thus $A_1^* \nrightarrow (K_{1,2}, C_4)$. Therefore $A_1 \in \mathcal{R}(K_{1,2}, C_4)$.

Since F has diameter 1, F must be a complete graph. Furthermore for $n \geq 5$, K_n always contain A_1. Thus K_n cannot be a $(K_{1,2}, C_4)$-minimal for $n \geq 5$. □

Theorem 2. *If* $\mathcal{R}_2 = \{F \in \mathcal{R}(K_{1,2}, C_4) \mid diam(F) = 2\}$ *then* $\{A_2, A_3\} \subseteq \mathcal{R}_2$.

Proof. **Case 2.1** Graph A_2.
To show that $A_2 \in \mathcal{R}_2$, we consider any red-blue coloring of A_2 that implies an edge-decomposition $A_2 = A_{21} \oplus A_{22}$. Let $A_{21} \not\supseteq K_{1,2}$. Thus A_{21} consists of at most two disjoint edges. Therefore, $A_{22} \supseteq C_4$. It follows that $A_2 \rightarrow (K_{1,2}, C_4)$.

To prove the minimality of A_2, let $V(A_2) = \{v_i \mid 1 \leq i \leq 4\} \cup \{u_j \mid 1 \leq j \leq 2\}$ such that v_i is the vertex of degree 2 and u_j is the vertex of degree 4. Consider $A_2^* \simeq A_2 \backslash \{e\}$ for any fixed edge $e \in E(A_2)$. W.l.o.g we assume that $e = v_1u_1$. Then the edges of A_2^* can be partitioned into two classes, namely E_1 and E_2, with $E_1 = \{v_2u_1, v_4u_2\}$ and $E_2 = \{v_1u_2, v_2u_2, v_3u_2, v_3u_1, v_4u_1\}$ such that $A_2^*[E_1] \not\supseteq K_{1,2}$ and $A_2^*[E_2] \not\supseteq C_4$. Thus $A_2^* \nrightarrow (K_{1,2}, C_4)$. Therefore $A_2 \in \mathcal{R}_2(K_{1,2}, C_4)$.

Case 2.2 Graph A_3.
A_3 is a wheel W_{2n+1} with odd number of spokes. We define

$$\begin{aligned} V(W_{2n+1}) &= \{c\} \cup \{v_i \mid 1 \le i \le 2n+1\}, \\ E(W_{2n+1}) &= E_1 \cup E_2, \text{ where} \\ E_1 &= \{cv_i \mid 1 \le i \le 2n+1\} \text{ and} \\ E_2 &= \{v_j v_{j+1} \mid 1 \le j \le 2n\} \cup \{v_{2n+1}v_1\}. \end{aligned}$$

To show that $A_3 \in \mathcal{R}_2$, we consider any red-blue coloring of A_3 that implies an edge-decomposition $A_3 = A_{31} \oplus A_{32}$. Let $A_{31} \not\supseteq K_{1,2}$. Thus A_{31} contains at most one spoke. If A_{31} contains one spoke, say cv_i for some $i, 1 \le i \le 2n+1$, then $\{c, v_{i-1}, v_i, v_{i+1}\}$ forms a blue C_4. In case that A_{31} contains no spoke, the parity of $|C_{2n+1}|$ in W_{2n+1} forces two incident edges in C_{2n+1} to be in A_{32}. Therefore, $A_{32} \supseteq C_4$. It follows that $A_3 \rightarrow (K_{1,2}, C_4)$.

Let us consider the graph $A_3^* \simeq W_{2n+1} \backslash \{e\}$ for any fixed edge $e \in E(W_{2n+1})$. Consider the following two cases.

Case 2.2.1 $e \in E_1$.
W.l.o.g let $e = cv_1$. If $n = 1$ then color cv_2 by red and other edges by blue. Therefore, $W_3 \backslash \{cv_1\}$ has no red $K_{1,2}$ neither blue C_4. If $n \ge 2$ then the edges of $A_3^*(k)$ can be partitioned into two classes, namely E_3 and E_4, as follows

$$\begin{aligned} E_3 &= \{cv_2\} \cup \{v_{2s}v_{2s+1} \mid 2 \le s \le n\}, \\ E_4 &= (E_1 \backslash \{cv_2\}) \cup \{v_{2n+1}v_1, v_1v_2, v_2v_3, v_3v_4\} \cup \{v_{2t-1}v_{2t} \mid 3 \le t \le n\}. \end{aligned}$$

Color the edges of E_3 by red and E_4 by blue. Under this coloring $A_3^*[E_3] \not\supseteq K_{1,2}$ and $A_3^*[E_4] \not\supseteq C_4$.

Case 2.2.2 $e \in E_2$.
W.l.o.g let $e = v_1v_2$. The edges of $A_3^*(k)$ can be partitioned into two classes, namely E_5 and E_5, as follows

$$\begin{aligned} E_5 &= \{cv_1\} \cup \{v_{2p}v_{2p+1} \mid 1 \le p \le n\}, \\ E_6 &= (E_1 \backslash \{cv_1\}) \cup \{v_{2n+1}v_1\} \cup \{v_{2q-1}v_{2q} \mid 2 \le q \le n\}. \end{aligned}$$

Color the edges of E_5 by red and E_6 by blue. Consequently, $A_3^*[E_5] \not\supseteq K_{1,2}$ and $A_3^*[E_6] \not\supseteq C_4$. Thus we have that $A_3^* \not\rightarrow (K_{1,2}, C_4)$. Therefore $A_3 \in \mathcal{R}_2(K_{1,2}, C_4)$. □

Theorem 3. *If $\mathcal{R}_3 = \{F \in \mathcal{R}(K_{1,2}, C_4) \mid diam(F) \ge 4\}$ then*

$$\mathcal{R}_3 \supseteq \{A_4(k), A_5(k), A_6(k), B_1(k,t), B_2(k,t), D_1(k, t_1, t_2)\}.$$

Proof. **Case 3.1** Graph $A_4(k)$.
Consider any red-blue coloring of $A_4(k)$ that implies an edge-decomposition $A_4(k) = A_{41}(k) \oplus A_{42}(k)$. Let $A_{41}(k) \not\supseteq K_{1,2}$. Observe that if there exists a blue C_4 in the C_4-path that belongs to $A_{42}(k)$ then the proof is complete. So now

we assume that in each basic cycle in C_4-path there is an edge belongs to $A_{41}(k)$. We call such an edge as a *red edge.*

Define the *end cycles* as the basic cycles that are attached to L_1. Thus the end vertices of the C_4-path belong to the end cycles. We claim that one of the end vertices, say v_1, is incident to a red edge. The claim is easy to justify by noting that if the end vertex is not incident to that red edge then the condition $A_{41}(k) \not\supseteq K_{1,2}$ forces the other end vertex, say v_{k+1}, to be incident to a red edge (See Figure 4). Then there is a blue C_4 in one of the L_1s. It follows that $A_{42}(k) \supseteq C_4$. Thus $A_4(k) \rightarrow (K_{1,2}, C_4)$.

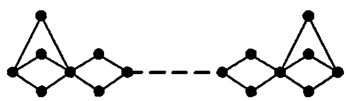

Fig. 4. $A_4(k)$

We denote the set of vertices of the L_1s by $\{x_i \mid 1 \le i \le 4\} \cup \{y_j \mid 1 \le j \le 6\}$ where x_i is the vertex of degree 3 and y_j is the vertex of degree 2. The root of the first L_1, x_2, is identified with v_1 and the root of the second L_1, x_3, is identified with v_{k+1}.

To prove the minimality of $A_4(k)$, let us consider the graph $A_4^* \simeq A_4\backslash\{e\}$ for any fixed edge $e \in E(A_4)$. Consider the following two cases.

Case 3.1.1 e is an edge of one of the L_1s.
W.l.o.g assume that e is in the first L_1s, say $e = x_1y_1$. The edges of $A_4^*(k)$ can be partitioned into two classes, namely E_1 and E_2, as follows

$$\begin{aligned}
E_1 &= \{x_1y_2, x_3y_5, x_4y_4\} \cup \{v_iw_i \mid 1 \le i \le k\},\\
E_2 &= \{x_1y_3, x_2y_1, x_2y_2, x_2y_3, x_3y_4, x_3y_6, x_4y_5, x_4y_6\}\\
&\quad \cup \{v_iu_i \mid 1 \le i \le k\} \cup \{u_iv_{i+1} \mid 1 \le i \le k\} \cup \{w_iv_{i+1} \mid 1 \le i \le k\}.
\end{aligned}$$

Color the edges of E_1 by red and E_2 by blue. Under this coloring $A_4^*[E_1] \not\supseteq K_{1,2}$ and $A_4^*[E_2] \not\supseteq C_4$.

Case 3.1.2 e is an edge of the C_4-path.
W.l.o.g assume that $e = v_1w_1$. Then the edges of $A_4^*(k)$ can be partitioned into two classes, namely E_3 and E_4, as follows

$$\begin{aligned}
E_3 &= \{x_1y_2, x_2y_1, x_3y_5, x_4y_4\} \cup \{v_jw_j \mid 2 \le j \le k\},\\
E_4 &= \{x_1y_1, x_1y_3, x_2y_2, x_2y_3, x_3y_4, x_3y_6, x_4y_5, x_4y_6\}\\
&\quad \cup \{v_iu_i \mid 1 \le i \le k\} \cup \{u_iv_{i+1} \mid 1 \le i \le k\} \cup \{w_iv_{i+1} \mid 1 \le i \le k\}.
\end{aligned}$$

Color the edges of E_3 by red and E_4 by blue. Under this coloring $A_4^*[E_3] \not\supseteq K_{1,2}$ and $A_4^*[E_4] \not\supseteq C_4$. Thus $A_4^* \nrightarrow (K_{1,2}, C_4)$. Therefore $A_4 \in \mathcal{R}_2(K_{1,2}, C_4)$.

Case 3.2 Graph $A_5(k)$.
The proof is similar with the above case. Consider any red-blue coloring of $A_5(k)$ that implies an edge-decomposition $A_5(k) = A_{51}(k) \oplus A_{52}(k)$. Let $A_{51}(k) \not\supseteq K_{1,2}$. If there is no blue C_4 in the C_4-path that belongs to $A_{52}(k)$ then we claim that a red edge is incident to one of the end vertices of the C_4-path (in this case we define the end cycles as the basic cycles that are attached to L_1 and L_2). Then there is a blue C_4 in L_1 or L_2 (depends on whether the red edge is incident to the root of L_1 or the root of L_2). It follows that $A_{52}(k) \supseteq C_4$. Thus $A_5(k) \to (K_{1,2}, C_4)$.

Let $V(L_1) = \{x_i \mid 1 \leq i \leq 2\} \cup \{y_j \mid 1 \leq j \leq 3\}$, where x_i is the vertex of degree 3 and y_j is the vertex of degree 2. The root of L_1, x_2, is identified with v_1. Let $V(L_2) = \{p_j \mid 1 \leq j \leq 3\} \cup \{q_i \mid 1 \leq i \leq 2\} \cup \{r_1\}$ where p_j is the vertex of degree 3, q_j is the vertex of degree 2 and r_1 is the vertex of degree 4. The root of L_2, p_1, is identified with v_{k+1}.

It can be proved similarly that $A_5(k) \backslash \{e\} \nrightarrow (K_{1,2}, C_4)$ for any fixed edge $e \in E(A_5(k))$. Therefore $A_5(k) \in \mathcal{R}_2(K_{1,2}, C_4)$.

Case 3.3 Graph $A_6(k)$.
The proof is similar with two cases above. Consider any red-blue coloring of $A_6(k)$ that implies an edge-decomposition $A_6(k) = A_{61}(k) \oplus A_{62}(k)$. Let $A_{61}(k) \not\supseteq K_{1,2}$. If there is no blue C_4 in the C_4-path that belongs to $A_{62}(k)$ then we claim that a red edge is incident to one of the end vertices of the C_4-path (in this case we define the end cycles as the basic cycles that are attached to L_2s). Then there is a blue C_4 in one of the L_2s. It follows that $A_{62}(k)[E_2] \supseteq C_4$. Thus $A_6(k) \to (K_{1,2}, C_4)$.

We denote the set of vertices of the L_2s by $\{p_j \mid 1 \leq j \leq 6\} \cup \{q_i \mid 1 \leq i \leq 4\} \cup \{r_t \mid 1 \leq t \leq 2\}$ where p_j is the vertex of degree 3, q_i is the vertex of degree 2 and r_t is the vertex of degree 4. The root of the first L_2, p_1, is identified with v_1 and the root of the second L_2, p_4, is identified with v_{k+1}.

It can be proved similarly that $A_6(k) \backslash \{e\} \nrightarrow (K_{1,2}, C_4)$ for any fixed edge $e \in E(A_6(k))$. Therefore $A_6(k) \in \mathcal{R}_2(K_{1,2}, C_4)$.

Case 3.4 Graph $B_1(k, t)$.
Consider any edge-decomposition $B_1(k,t) = B_{11}(k,t) \oplus B_{12}(k,t)$ by a red-blue coloring. Let $B_{11}(k,t) \not\supseteq K_{1,2}$. If there is no blue C_4 in the C_4-path that belongs to $B_{12}(k,t)$ then we claim that a red edge is incident to one of the end vertices of the C_4-path. If the red edge is incident to v_1 then we have a blue C_4 in one of the basic cycles of C_4-cycle. If the red edge is incident to v_{k+1} then there is a blue C_4 in L_1. It follows that $B_{12}(k,t) \supseteq C_4$. Thus $B_1(k,t) \to (K_{1,2}, C_4)$.

The vertices of C_4-cycle of length t are denoted by $\{z_s \mid 1 \leq s \leq t\} \cup \{a_s \mid 1 \leq s \leq t\} \cup \{b_s \mid 1 \leq s \leq t\}$, where z_s is the vertex of degree 4, a_s and b_s are the vertices of degrees 2. For L_1, the notation of their vertices is similar with the L_1 in $A_4(k)$. W.l.o.g, we assume that the first end vertex of C_4-path, v_1, is identified with z_s for an arbitrary s, $1 \leq s \leq t$ and the second end vertex, v_{k+1}, is identified with x_2, the root of L_1.

To prove the minimality of $B_1(k,t)$, we consider $B_1^*(k,t) = B_1(k,t) \backslash \{e\}$ for any fixed edge $e \in E(B_1(k,t))$. Consider the following cases.

Case 3.4.1 e is an edge in C_4-cycle.
W.l.o.g we assume that $e = z_t a_t$. Then the edges of $B_1^*(k,t)$ can be partitioned into two classes, namely E_1 and E_2, as follows

$$\begin{aligned} E_1 &= \{a_p z_{p+1} \mid 1 \le p \le t-1\} \cup \{v_i w_i \mid 1 \le i \le k\} \cup \{x_1 y_2, x_2 y_1\}, \\ E_2 &= E(B_1^*(k,t)) \setminus E_1. \end{aligned}$$

Color the edges of E_1 by red and E_2 by blue. Under this coloring $B_1^*[E_1] \not\supseteq K_{1,2}$ and $B_1^*[E_2] \not\supseteq C_4$.

Case 3.4.2 e is an edge in C_4-path.
W.l.o.g we assume that $e = v_1 w_1$. Then the edges of $B_1^*(k,t)$ can be partitioned into two classes, namely E_3 and E_4, as follows

$$\begin{aligned} E_3 &= \{z_s a_s \mid 1 \le s \le t\} \cup \{v_j w_j \mid 2 \le j \le k\} \cup \{x_1 y_1, x_2 y_2\}, \\ E_4 &= E(B_1^*(k,t)) \setminus E_3. \end{aligned}$$

Color the edges of E_3 by red and E_4 by blue. Under this coloring $B_1^*[E_3] \not\supseteq K_{1,2}$ and $B_1^*[E_4] \not\supseteq C_4$.

Case 3.4.3 $e \in L_1$.
W.l.o.g we assume that $e = x_1 y_1$. Then the edges of $B_1^*(k,t)$ can be partitioned into two classes, namely E_5 and E_6, as follows

$$\begin{aligned} E_5 &= \{z_s a_s \mid 1 \le s \le t\} \cup \{w_i v_{i+1} \mid 1 \le i \le k\} \cup \{x_1 y_2\}, \\ E_6 &= E(B_1^*(k,t)) \setminus E_5. \end{aligned}$$

Color the edges of E_5 by red and E_6 by blue. Under this coloring $B_1^*[E_5] \not\supseteq K_{1,2}$ and $B_1^*[E_6] \not\supseteq C_4$. Thus $B_1^*(k,t) \nrightarrow (K_{1,2}, C_4)$. Therefore $B_1(k,t) \in \mathcal{R}(K_{1,2}, C_4)$.

Case 3.5 Graph $B_2(k,t)$.
The notation for C_4-cycle is similar with the above case, while the notation for L_2 is similar with the notation of L_2 in $A_5(k)$. W.l.o.g we assume that the first end vertex of C_4-path, v_1, is identified with z_s for an arbitrary s, $1 \le s \le t$ and the second end vertex, v_{k+1}, is identified with p_1, the root of L_2.

The proof is similar with Case 3.4 above. We consider any edge-decompo sition $B_2(k,t) = B_{21}(k,t) \oplus B_{22}(k,t)$ by a red-blue coloring. Let $B_{21}(k,t) \not\supseteq K_{1,2}$. If there is no blue C_4 in the C_4-path that belongs to $B_{22}(k,t)$ then we claim that a red edge is incident to one of the end vertices of the C_4-path. If the red edge is incident to v_1 then we have a blue C_4 in one of the basic cycles of C_4-cycle. If the red edge is incident to v_{k+1} then there is a blue C_4 in L_2. It follows that $B_{22}(k,t) \supseteq C_4$. Thus $B_2(k,t) \rightarrow (K_{1,2}, C_4)$.

It can be proved similarly that $B_2(k,t) \setminus \{e\} \nrightarrow (K_{1,2}, C_4)$ for any fixed edge $e \in E(B_2(k,t))$. Therefore $B_2(k,t) \in \mathcal{R}(K_{1,2}, C_4)$.

Case 3.6 Graph $D_1(k, t_1, t_2)$.
We denote the set of vertices of C_4-cycle of length t_1 by

$$\{z_{mt_1} \mid 1 \le m \le t_1\} \cup \{a_{mt_1} \mid 1 \le m \le t_1\} \cup \{b_{mt_1} \mid 1 \le m \le t_1\}$$

and the set of vertices of the C_4-cycle of length t_2 by

$$\{z_{nt_2} \mid 1 \le n \le t_2\} \cup \{a_{nt_2} \mid 1 \le n \le t_2\} \cup \{b_{nt_2} \mid 1 \le n \le t_2\}.$$

The degrees of z_{mt_1} and z_{nt_2} are 4, while the vertices a_{mt_1}, b_{mt_1} and a_{nt_2}, b_{nt_2} are of degrees 2. W.l.o.g we identify one of the end vertex of C_4-path, v_1, with z_{1t_1} and the other end vertex, v_{k+1}, is identified with z_{1t_2}.

Consider any edge-decomposition $D_1(k,t_1,t_2) = D_{11}(k,t_1,t_2) \oplus D_{12}(k,t_1,t_2)$ by a red-blue coloring. Let $D_{11}(k,t_1,t_2) \not\supseteq K_{1,2}$. If there is no blue C_4 in the C_4-path that belongs to $D_{12}(k,t_1,t_2)$ then we claim that a red edge is incident to one of the end vertices of the C_4-path. If the red edge is incident to v_1 then we have a blue C_4 in one of the basic cycles of the first C_4-cycle. If the red edge is incident to v_{k+1} then there is a blue C_4 in the second C_4-cycle. It follows that $D_{12}(k,t_1,t_2)) \supseteq C_4$. Thus $D_3(k,t_1,t_2) \to (K_{1,2}, C_4)$.

To prove the minimality of $D_1(k,t_1,t_2)$, we consider $D_1^*(k,t_1,t_2) = D_1(k,t_1,t_2)\backslash\{e\}$ for any fixed edge $e \in E(D_1(k,t_1,t_2))$. Consider the following cases.

Case 3.6.1 e is an edge of one of the C_4-cycles.
W.l.o.g we assume that $e = z_{1t_1}a_{1t_1}$. Then the edges of $D_1^*(k,t_1,t_2)$ can be partitioned into two classes, namely E_1 and E_2, as follows

$$\begin{aligned} E_1 &= \{z_{pt_1}a_{pt_1} \mid 2 \le p \le t_1\} \cup \{v_iw_i \mid 1 \le i \le k\} \\ &\quad \cup \{a_{st_2}z_{(s+1)t_2} \mid 1 \le s \le t_2 - 1\} \cup \{a_{t_2t_2}z_{1t_2}\}, \\ E_2 &= E(D_1^*(k,t_1,t_2)) \setminus E_1. \end{aligned}$$

Color the edges of E_1 by red and E_2 by blue. Under this coloring $D_1^*(k,t_1,t_2)^*$ $[E_1] \not\supseteq K_{1,2}$ and $D_1^*(k,t_1,t_2)[E_2] \not\supseteq C_4$.

Case 3.6.2 e is an edge in C_4-path.
W.l.o.g we assume that $e = v_1w_1$. Then the edges of $D_1^*(k,t_1,t_2)$ can be partitioned into two classes, namely E_3 and E_4, as follows

$$\begin{aligned} E_3 &= \{a_{rt_1}z_{(r+1)t_1} \mid 1 \le r \le t_1 - 1\} \cup \{v_jw_j \mid 2 \le j \le k\} \\ &\quad \cup \{a_{st_2}z_{(s+1)t_2} \mid 1 \le s \le t_2 - 1\} \cup \{a_{t_2t_2}z_{1t_2}\}, \\ E_4 &= E(D_1^*(k,t_1,t_2)) \setminus E_3. \end{aligned}$$

Color the edges of E_3 by red and E_4 by blue. Under this coloring $D_1^*(k,t_1,t_2)$ $[E_3] \not\supseteq K_{1,2}$ and $D_1^*(k,t_1,t_2)[E_4] \not\supseteq C_4$. Therefore $D_1(k,t_1,t_2) \in \mathcal{R}(K_{1,2}, C_4)$. □

Theorem 4. $B_3(k,t), B_4(k,t) \notin \mathcal{R}(K_{1,2}, C_4)$.

Proof. **Case 4.1** Graph $B_3(k,t)$.
The notation of $V(B_3(k,t))$ is similar with notation of $V(B_1(k,t))$. W.l.o.g we assume that the first end vertex of C_4-path, v_1, is identified with a_1 and the second end vertex, v_{k+1}, is identified with x_2, the root of L_1.

The edges of $B_3(k,t)$ can be partitioned into two classes, namely E_1 and E_2, as follows

$$\begin{aligned} E_1 &= \{z_s b_s \mid 1 \le s \le t\} \cup \{v_i w_i \mid 1 \le i \le k\} \cup \{x_1 y_1, x_2 y_2\}, \\ E_2 &= E(B_3(k,t)) \setminus E_1. \end{aligned}$$

Under this coloring $B_3(k,t)[E_1] \not\supseteq K_{1,2}$ and $B_3(k,t)[E_2] \not\supseteq C_4$.
Thus
$B_3(k,t) \nrightarrow (K_{1,2}, C_4)$. Therefore $B_3(k,t) \notin \mathcal{R}(K_{1,2}, C_4)$.

Case 4.2 Graph $B_4(k,t)$.
The notation of $V(B_4(k,t))$ is similar with notation of $V(B_2(k,t))$. W.l.o.g we assume that the first end vertex of C_4-path, v_1, is identified with a_1 and the second end vertex, v_{k+1}, is identified with p_1, the root of L_2.

Similar with Case 4.1 above, the edges of $B_4(k,t)$ can be partitioned into two classes, namely E_1 and E_2, as follows

$$\begin{aligned} E_1 &= \{z_s b_s \mid 1 \le s \le t\} \cup \{v_i w_i \mid 1 \le i \le k\} \cup \{p_1 p_3, p_2 q_1, q_2 r_1\}, \\ E_2 &= E(B_4(k,t)) \setminus E_1. \end{aligned}$$

Under this coloring $B_4(k,t)[E_1] \not\supseteq K_{1,2}$ and $B_4(k,t)[E_2] \not\supseteq C_4$. Thus $B_4(k,t) \nrightarrow (K_{1,2}, C_4)$. Therefore $B_4(k,t) \notin \mathcal{R}(K_{1,2}, C_4)$. □

Theorem 5. $D_2(k,t_1,t_2), D_3(k,t_1,t_2) \notin \mathcal{R}(K_{1,2}, C_4)$.

Proof. **Case 5.1** Graph $D_2(k,t_1,t_2)$.
The notation of $V(D_2(k,t_1,t_2))$ is similar with $V(D_1(k,t_1,t_2))$. W.l.o.g we assume that v_1 is identified with z_{1t_1} and v_{k+1} is identified with a_{1t_2}.

The edges of $D_2(k,t_1,t_2)$ can be partitioned into two classes, namely E_1 and E_2, as follows

$$\begin{aligned} E_1 &= \{a_{rt_1} z_{(r+1)t_1} \mid 1 \le r \le t_1 - 1\} \cup \{a_{t_1 t_1} z_{1t_1}\} \\ &\quad \cup \{w_i v_{i+1} \mid 1 \le i \le k\} \cup \{z_{nt_2} b_{nt_2} \mid 1 \le n \le t_2\}, \\ E_2 &= E(D_2(k,t_1,t_2)) \setminus E_1. \end{aligned}$$

Under this coloring $D_2(k,t_1,t_2)[E_1] \not\supseteq K_{1,2}$ and $D_2(k,t_1,t_2)[E_2] \not\supseteq C_4$. Thus $D_2(k,t_1,t_2) \nrightarrow (K_{1,2}, C_4)$. Therefore $D_2(k,t_1,t_2) \notin \mathcal{R}(K_{1,2}, C_4)$.

Case 5.2 Graph $D_3(k,t_1,t_2)$.
The notation of $V(D_3(k,t_1,t_2))$ is similar with $V(D_1(k,t_1,t_2))$. W.l.o.g we assume that v_1 is identified with a_{1t_1} and v_{k+1} is identified with a_{1t_2}.

The edges of $D_3(k,t_1,t_2)$ can be partitioned into two classes, namely E_1 and E_2, as follows

$$\begin{aligned} E_1 &= \{z_{mt_1} a_{mt_1} \mid 1 \le m \le t_1\} \cup \{w_i v_{i+1} \mid 1 \le i \le k\} \\ &\quad \cup \{b_{st_2} z_{(s+1)t_2} \mid 1 \le s \le t_2 - 1\} \cup \{b_{t_2 t_2} z_{1t_2}\}, \\ E_2 &= E(D_3(k,t_1,t_2)) \setminus E_1. \end{aligned}$$

Under this coloring $D_3(k,t_1,t_2)[E_1] \not\supseteq K_{1,2}$ and $D_3(k,t_1,t_2)[E_2] \not\supseteq C_4$. Thus $D_3(k,t_1,t_2) \nrightarrow (K_{1,2}, C_4)$. Therefore $D_3(k,t_1,t_2) \notin \mathcal{R}(K_{1,2}, C_4)$. □

Theorem 6. $I(t,k_1,k_2) \notin \mathcal{R}(K_{1,2},C_4)$.

Proof. We denote the set of vertices of C_4-path of length k_1 by

$$\{v_{pk_1} \mid 1 \leq p \leq k_1+1\} \cup \{w_{qk_1} \mid 1 \leq q \leq k_1\} \cup \{u_{qk_1} \mid 1 \leq q \leq k_1\}$$

and the set of vertices of C_4-path of length k_2 by

$$\{v_{rk_2} \mid 1 \leq r \leq k_2+1\} \cup \{w_{sk_2} \mid 1 \leq s \leq k_2\} \cup \{u_{sk_2} \mid 1 \leq s \leq k_2\}.$$

The notation of the vertices of C_4-cycle of length t are similar with its notation in Case 3.4. Consider these following cases.

Case 6.1 $v_{(k_1+1)k_1}$ and v_{1k_2} are identified with two vertices of degrees 4 of the C_4-cycle.
W.l.o.g we assume that $v_{(k_1+1)k_1}$ is identified with z_1 and v_{1k_2} is identified with z_t. Then the edges of $I(t,k_1,k_2)$ can be partitioned into two classes, namely E_1 and E_2, as follows

$$\begin{aligned} E_1 &= \{v_{ik_1}w_{ik_1} \mid 1 \leq i \leq k_1\} \cup \{w_{jk_2}v_{(j+1)k_2} \mid 1 \leq j \leq k_2\} \\ &\cup \{a_s z_{s+1} \mid 1 \leq s \leq t-1\} \cup \{a_t z_1\}, \\ E_2 &= E(I(t,k_1,k_2)) \setminus E_1. \end{aligned}$$

Under this coloring $I(t,k_1,k_2)[E_1] \not\supseteq K_{1,2}$ and $I(t,k_1,k_2)[E_2] \not\supseteq C_4$.

Case 6.2 $v_{(k_1+1)k_1}$ is identified with a vertex of degrees 4 and v_{1k_2} is identified with a vertex of degree 2 of the C_4-cycle.
W.l.o.g we assume that $v_{(k_1+1)k_1}$ is identified with z_1 and v_{1k_2} is identified with a_t. Then the edges of $I(t,k_1,k_2)$ can be partitioned into two classes, namely E_3 and E_4, as follows

$$\begin{aligned} E_3 &= \{v_{ik_1}w_{ik_1} \mid 1 \leq i \leq k_1\} \cup \{w_{jk_2}v_{(j+1)k_2} \mid 1 \leq j \leq k_2\} \\ &\cup \{a_s z_{s+1} \mid 1 \leq s \leq t-1\} \cup \{a_t z_1\}, \\ E_4 &= E(I(t,k_1,k_2)) \setminus E_3. \end{aligned}$$

Under this coloring $I(t,k_1,k_2)[E_3] \not\supseteq K_{1,2}$ and $I(t,k_1,k_2)[E_4] \not\supseteq C_4$.

Case 6.3 $v_{(k_1+1)k_1}$ and v_{1k_2} are identified with two vertices of degrees 2 of the C_4-cycle.
W.l.o.g we assume that $v_{(k_1+1)k_1}$ is identified with a_1 and v_{1k_2} is identified with a_t. Then the edges of $I(t,k_1,k_2)$ can be partitioned into two classes, namely E_5 and E_6, as follows

$$\begin{aligned} E_5 &= \{v_{ik_1}w_{ik_1} \mid 1 \leq i \leq k_1\} \cup \{w_{jk_2}v_{(j+1)k_2} \mid 1 \leq j \leq k_2\} \\ &\cup \{a_s z_{s+1} \mid 1 \leq s \leq t-1\} \cup \{a_t z_1\}, \\ E_6 &= E(I(t,k_1,k_2)) \setminus E_5. \end{aligned}$$

Under this coloring $I(t,k_1,k_2)[E_5] \not\supseteq K_{1,2}$ and $I(t,k_1,k_2)[E_6] \not\supseteq C_4$. Thus $I(t,k_1,k_2) \nrightarrow (K_{1,2},C_4)$. Therefore $I(t,k_1,k_2) \notin \mathcal{R}(K_{1,2},C_4)$. □

As a final remark, we present some problems that are raised from this paper.

(1) Characterize all graphs $F \in \mathcal{R}(K_{1,2}, C_4)$ with $diam(F) = 2$,
(2) Does there exist a graph F such that $F \in \mathcal{R}(K_{1,2}, C_4)$ with $diam(F) = 3$?
(3) Characterize all graphs $F \in \mathcal{R}(K_{1,2}, C_4)$ with $diam(F) \geq 4$.

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