ORTHOGONAL DOUBLE COVERS OF COMPLETE GRAPHS BY LOBSTERS OF DIAMETER 5

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ABSTRACT. An orthogonal double cover (ODC) of the complete graph K_n by a graph G is a collection $\mathcal{G} = \{G_i | i = 1, 2, \dots, n\}$ of spanning subgraphs of K_n, all isomorphic to G, with the property that every edge of K_n belongs to exactly two members of $\mathcal{G}$ and any two distinct members of $\mathcal{G}$ share exactly one edge.

A lobster of diameter five is a tree arising from a double star by attaching any number of pendant vertices to each of its vertices of degree one. We show that for any double star $R(p, q)$ there exists an ODC of K_n by all lobsters of diameter five (with finitely many possible exceptions) arising from $R(p, q)$.

1. Introduction

An *orthogonal double cover (ODC)* of the complete graph K_n is a collection $\mathcal{G} = \{G_i | i = 1, 2, \dots, n\}$ of spanning subgraphs of K_n, called *pages*, satisfying the following two properties:

(1) *Double cover property*: Every edge of K_n belongs to exactly two pages of $\mathcal{G}$.
(2) *Orthogonality property*: Any two distinct pages of $\mathcal{G}$ share exactly one edge.

The definition immediately implies that every page of $\mathcal{G}$ must have exactly $n - 1$ edges. If all pages of $\mathcal{G}$ are isomorphic to the same graph G, then $\mathcal{G}$ is called an ODC of K_n by G.

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ODCs have been investigated for more than 25 years, and there is an extensive literature on the subject. For motivation and an overview of results and problems in the area we refer to the survey paper [2].

As the condition on the number of edges of G is necessary for the existence of an ODC by G, it is natural to ask whether there exist ODCs for some classes of trees. It can be easily verified that there is no ODC of K_4 by P_4, the path of length three. For all other non-trivial trees on at most 14 vertices ODCs of the corresponding complete graphs exist [2], which supports the following conjecture by Gronau, Mullin, and Rosa [3].

Conjecture. (Gronau, Mullin, and Rosa) *If $T \neq P_4$ is a tree on $n \geq 2$ vertices, then there is an ODC of K_n by T.*

The conjecture trivially holds for stars, and it was shown to be true for all trees of diameter three [3] (see also [4]). On the other hand, even for paths a complete solution is not known. Therefore, we do not expect the conjecture to be completely solved soon.

It was also proved for all caterpillars of diameter four [5]. A *caterpillar* is a tree that is obtained by attaching pendant vertices of degree one to a path. By $R(p_1, p_2, \ldots, p_t)$ we denote a caterpillar with t spine vertices and p_i pendant vertices attached to the i-th spine vertex (in natural order). We of course assume that $p_1, p_t \geq 1$.

The only other class of trees of diameter four and five is the class of lobsters. In general, a *lobster* is a tree which is not a caterpillar but deleting its pendant vertices (and edges incident with them) results in a caterpillar. Leck and Leck [5] proved that for a fixed r, *almost all* lobsters of diameter four with the central vertex (i.e., the only vertex of eccentricity 2) of degree r admit an ODC of the appropriate K_n. The author strengthened their result by proving that for a fixed r, *all but possibly finitely many* lobsters of diameter four with the central vertex of degree r admit an ODC [1]. In this paper, we prove analogous result for lobsters of diameter five. A lobster L of diameter five arises from a double star $R(p, q)$ by attaching pendant vertices to the leaves of the double star. We say that $R(p, q)$ is the *base caterpillar* or just *base* of the lobster L.

2. Another generalization of adding construction

In this section we describe a recursive method for constructing ODCs. The method is a modification of the method originally developed by Leck and Leck in [5], who call it the *adding construction.* The author generalized the adding construction in [1]. A simple version appeared already as a part of a construction of ODCs by double stars in [3]. To find an ODC by a graph G, we will need two subgraphs of G, say G^*, G', that both admit certain type of orthogonal double cover and arise from G by deleting some vertices of degree one.

Let $\mathcal{G} = \{G_1, G_2, \dots, G_n\}$ be an ODC of K_n by G defined by isomorphisms $\phi_i : G \to G_i$ for $i = 1, 2, \dots, n$. A vertex v of G is *surjective* if $\{\phi_i(v) | i = 1, 2, \dots, n\} = V(K_n)$. Notice that in [5] surjective vertices are called rotational.

The following lemma is an essential part of the adding construction. A proof can be found in [1] or in [5] as a part of the proof of Lemma 6.

Lemma 1. *Let $K_{r,s}$ be a complete bipartite graph with bipartition $X = \{x_1, x_2, \dots, x_r\}, Y = \{y_1, y_2, \dots, y_s\}$. Let G be a subgraph of $K_{r,s}$ such that $\deg_G(y_j) = 1$ for every $j = 1, 2, \dots, s$ and every y_j is adjacent to one of $x_1, x_2, \dots, x_m$ with $1 \le m \le r$. Define G_i for $i = 1, 2, \dots, r$ by an isomorphism $\phi_i : G \to G_i$ with the property that $\phi_i(x_p) \ne \phi_j(x_p)$ for $p = 1, 2, \dots, m$ if $i \ne j$ and $\phi_i(y_q) = y_q$ for $q = 1, 2, \dots, s$.*

Let F be the star induced by $X \cup \{y_s\}$. Let F_i for $i = 1, 2, \dots, s$ be defined by an isomorphism $\psi_i : F \to F_i$ with the property that $\psi_i(y_s) = y_i$ for $i = 1, 2, \dots, s$ and $\psi_i(x_q) = x_q$ for $q = 1, 2, \dots, r$. Then

(1) every edge of $K_{r,s}$ appears in exactly one of the graphs $G_1, G_2, \dots, G_r$,

(2) every edge of $K_{r,s}$ appears in exactly one of the graphs $F_1, F_2, \dots, F_s$, and

(3) for any $i \in \{1, 2, \dots, r\}, j \in \{1, 2, \dots, s\}$ the graphs G_i and F_j share exactly one edge.

The following lemma will be also useful. The proof is an easy application of Lemma 1 and can be found in [1].

Lemma 2. *Let $H = K_r + \overline{K}_s$ (i.e., H is the complete graph K_{r+s} with all edges of some K_s removed). Let $V(H) = X \cup Y, X \cap Y = \emptyset$, where $\langle X \rangle = K_r$ and $\langle Y \rangle = \overline{K}_s$. Let $\mathcal{G}' = \{G'_1, G'_2, \dots, G'_r\}$ be an ODC of K_r by G' with $V(G') = X$ defined by $G'_i = \phi'_i(G')$ and $S \subseteq X$ be the set of surjective vertices of G'. Let G be a graph with the vertex set $X \cup Y$ that arises from G' by joining each vertex of Y to exactly one surjective vertex of G'. Let G_i for $i = 1, 2, \dots, r$ be defined by $\phi_i(G)$, where $\phi_i(x) = \phi'_i(x)$ for every $x \in X$ and $\phi_i(y) = y$ for every $y \in Y$. Then every two pages G_i, G_j share exactly one edge $x_p x_q$, every edge $x_p x_q$ appears in exactly two distinct pages, and every edge $x_p y_t$ appears in exactly one page.*

Next we present another generalization of the adding construction.

Lemma 3. *Let G be a graph with the vertex set $V, |V| = n = \ell k + m, 0 \le m \le \min\{\ell - 1, k - 1\}$ and $U \subset V$ be a set of vertices of degree one with the following properties:*

(1) $W = V \setminus U$, $|W| = k, |U| = (\ell - 1)k + m$, the graph $G^* = \langle W \rangle$ allows an ODC $\mathcal{G}^*$ of K_k and $S \subseteq W$ is the set of surjective vertices of G^* with respect to $\mathcal{G}^*$,

(2) in the bipartite graph with bipartition W, U there are at least b edge-disjoint stars $K_{1,k+1}$ with central vertices in S, where $b = \min\{m, \lfloor \frac{\ell}{2} \rfloor\}$, and if $b < \lfloor \frac{\ell}{2} \rfloor$, then $\lfloor \frac{\ell}{2} \rfloor - m$ more edge-disjoint stars $K_{1,k}$.

If $m \geq 1$, then let $U' \subset V$ be a set of vertices of degree one (not necessarily disjoint from U) with the following properties:

(3) $W' = V \setminus U'$, $|W'| = k+1, |U'| = (\ell-1)k+m-1$, the graph $G' = \langle W' \rangle$ allows an ODC $\mathcal{G}'$ of K_{k+1} and $S' \subseteq W'$ is the set of surjective vertices of G' with respect to $\mathcal{G}'$,

(4) in the bipartite graph with bipartition W', U' there are at least b' edge-disjoint stars $K_{1,k+1}$ with central vertices in S', where $b' = \min\{m - 1, \lfloor \frac{\ell}{2} \rfloor\}$, and if $b' < \lfloor \frac{\ell}{2} \rfloor$, then $\lfloor \frac{\ell}{2} \rfloor - b'$ more edge-disjoint stars $K_{1,k}$.

Then G allows an ODC $\mathcal{G}$ of K_n.

Proof. We split V into ℓ disjoint subsets, $V_1, V_2, \ldots, V_\ell$, with $|V_i| = k+1$ for $i = 1, 2, \ldots, m$ and $|V_i| = k$ for $i = m+1, m+2, \ldots, \ell$ if $m < \ell$. The conditions (2) and (4) guarantee that whenever we place a copy of G' into one of $V_1, V_2, \ldots, V_m$, or G^* into one of $V_{m+1}, V_{m+2}, \ldots, V_\ell$, we have enough stars $K_{1,k+1}$ and/or $K_{1,k}$ whose vertices of degree one can be placed into at least $\lfloor \frac{\ell}{2} \rfloor$ other sets V_j such that each star "fills" with its leaves the whole set V_j. More precisely, for every $i = 1, 2, \ldots, \ell$, we can fill with the leaves of a star $K_{1,k+1}$ or $K_{1,k}$ (as needed) each of the sets $V_{i+1}, V_{i+2}, \ldots, V_{i+\lfloor \frac{\ell}{2} \rfloor}$, where the addition is taken modulo ℓ.

Now let $G_{i,p}$ and $G_{i,q}$ be pages of $\mathcal{G}$ isomorphic to G that are placed such that their subgraphs $G'_{i,p}$ and $G'_{i,q}$ isomorphic to G' (or $G^*_{i,p}$ and $G^*_{i,q}$ isomorphic to G^*) both belong to the set V_i. We apply Lemma 2 with $V_i = X, V \setminus V_i = Y$ and observe that $G_{i,p}$ and $G_{i,q}$ share exactly one edge with both endvertices in V_i and each such an edge is contained in exactly two pages $G_{i,p'}$ and $G_{i,q'}$ for some p', q'. It also follows that $G_{i,p}$ and $G_{i,q}$ do not share any edge $v_i x$ with $v_i \in V_i, x \notin V_i$ or an edge xy with $x, y \notin V_i$.

We now denote by $\tilde{G}_{i,p}$ a page of $\mathcal{G}^*$ placed in V_i if $m+1 \leq i \leq \ell$ or a page of $\mathcal{G}'$ placed in V_i if $1 \leq i \leq m$. If we then look at $G_{i,p}$ with $\tilde{G}_{i,p}$ placed in the set V_i and $G_{j,q}$ with $\tilde{G}_{j,q}$ placed in the set V_j, we observe that they can intersect only in some edge $v_i v_j$, where $v_i \in V_i, v_j \in V_j$. We can assume WLOG that $j \leq i + \lfloor \frac{\ell}{2} \rfloor$. Therefore, there is the star $K_{1,k}$ (if $m+1 \leq j \leq \ell$) or $K_{1,k+1}$ (if $1 \leq j \leq m$) with the central vertex v_i in $G_{i,p}$, while in $G_{j,q}$ all vertices of V_i are of degree one. Thus, we can apply Lemma 1 with $V_i = Y, V_j = X$ and conclude that $G_{i,p}$ and $G_{j,q}$ share exactly one edge. Again, if we let $\tilde{G}_{i,p}$ run through V_i and $\tilde{G}_{j,q}$ through V_j, from Lemma 1 we can see that each edge of the complete bipartite graph with the partite sets V_i, V_j is contained in exactly two pages $G_{i,p'}$ and $G_{j,q'}$ for some p', q'. □

The graphs we will use in our constructions as G^*, G' have a *cyclic* orthogonal double cover, or CODC for short. We say that K_n has a CODC $\mathcal{G} = \{G_1, G_2, \ldots, G_n\}$ by a graph G if $V(K_n) = \{x_1, x_2, \ldots, x_n\}$ and the isomorphisms $\phi_1, \phi_2, \ldots, \phi_n; \phi_i : G \to G_i$ are defined as $\phi_i(x_j) = x_{j+i-1}$ for every $i, j = 1, 2, \ldots, n$, where the addition is taken modulo n with 0 replaced by n. Notice that if $\mathcal{G}$ is cyclic, then *all* vertices of G are surjective.

3. The result

We noticed above that a lobster of diameter five is a tree arising from a double star by attaching any number of pendant vertices to each of its vertices of degree one. We will call the only two vertices of eccentricity 3 the *primary* vertices, and their neighbors (of eccentricity 4) the *secondary vertices*. The vertices of degree one and eccentricity 5 are called *leaves*. Recall that $R(p_1, p_2, p_3)$ is the caterpillar of diameter 5 with p_1 and p_3 leaves adjacent to the endvertices of the path P_3, respectively, and p_2 leaves attached to the central vertex of P_3.

We will use the following result by Leck and Leck [5].

Theorem 4. (Leck and Leck [5]) *The caterpillar $R(p_1, p_2, p_3)$ admits a CODC if $p_2 \leq |p_1 - p_3|$.*

Now we can prove our main result.

Theorem 5. *Let $L = L(p, q; 5)$ be a lobster of diameter 5 with the base double star $R(p, q)$ and $n \geq 4(p+q)(p+q+1)$ vertices. Then there is an ODC of K_n by L.*

Proof. First we show that at least one secondary vertex has $p+q$ or more neighboring leaves. Suppose it is not the case. Then the number of leaves is at most $(p+q)(p+q-1) = (p+q)^2 - (p+q)$ and the number of vertices of L is at most $(p+q)^2+2$, which is a contradiction. Therefore, L contains either $R(p+q, p-1, q)$ or $R(p, q-1, p+q)$. Each of them satisfies assumptions of Theorem 4. Therefore, we can suppose WLOG it is the former. By Theorem 4 it admits a CODC of $K_{2(p+q)+2}$, all of its vertices are surjective, and we can choose it for G' of Lemma 3. Similarly, $R(p+q-1, p-1, q)$ satisfies assumptions of Theorem 4. Hence it admits a CODC of $K_{2(p+q)+1}$, and we can choose it for G^* of Lemma 3. Note that then $k = 2(p+q)+1$.

Now we need to show that L has enough leaves to satisfy assumptions (2) and (4) of Lemma 3. We again proceed by contradiction. The number of secondary vertices of the base double star is $p+q$ and the maximum number of leaves that do not induce a star $K_{1,k+1}$ is $2p+2q+1$ at each secondary vertex. Therefore, we have at most $(p+q)(2p+2q+1)$ leaves not included in the stars. These leaves fill *at most* $p+q$ of the sets V_i. It remains to show that the number of stars $K_{1,k+1}$ and/or $K_{1,k}$ that fill all remaining sets V_i is *at least* $p+q$. If this is so, then the number of sets V_i filled with stars satisfies assumptions of Lemma 3. Suppose it is not the case. Then we have at most $p+q-1$ stars, and can suppose that they are all the bigger ones, $K_{1,2p+2q+2}$. This gives at most $(p+q-1)(2p+2q+2)$ vertices in the stars. The total number of vertices is then at most

$$(2p+2q+2)+(p+q)(2p+2q+1)+(p+q-1)(2p+2q+2)=(p+q)(4p+4q+3).$$

This contradicts our assumption that the number of vertices is at least $4(p+q)(p+q+1)$. Therefore, the number of stars is sufficient. Now we have verified all assumptions of Lemma 3 and the proof is complete. □

The corollary now follows immediately.

Corollary 6. *Let* $p,q\geq 1$. *Then all lobsters of diameter* 5 *with the base double star* $R(p,q)$ *with at most finitely many possible exceptions admit an orthogonal double cover of the complete graph* K_n.

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