On The Balance Index Sets of Graphs

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Abstract Let G be a graph with vertex set V(G) and edge set E(G), and let A ={0, 1}. A labeling f : V(G) → A induces a partial edge labeling f^* : E(G) → A defined by $f^*(xy)$ = f(x), if and only if f(x) = f(y), for each edge xy ∈ E(G). For i ∈ A, let $v_f(i)$ = card{v ∈ V(G) : f(v) = i} and $e_{f^*}(i)$ = card{e ∈ E(G) : $f^*(e)$ = i}. A labeling f of a graph G is said to be friendly if | $v_f(0) - v_f(1)$ | ≤ 1. If, | $e_f(0) - e_f(1)$ | ≤ 1 then G is said to be ***balanced***. The ***balance index set*** of the graph G, BI(G), is defined as {|$e_f(0) - e_f(1)$| : the vertex labeling f is friendly}. Results parallel to the concept of friendly index sets are presented.

1. Introduction

In [2], A. Liu, S.K. Tan and the second author considered a new labeling problem of graph theory. Let G be a graph with vertex set V(G) and edge set E(G). A vertex labeling of G is a mapping f from V(G) into the set {0, 1}. For each vertex labeling f of G, we can define a partial edge labeling f* of G in the following way. For each edge (u, v) ∈ E(G), where u, v ∈ V(G), we have f*(u, v) = 1 if f(u) = f(v) = 1, and = 0 if f(u) = f(v) = 0. Note that if f(u) ? f(v), the edge (u, v) is not labeled by f*. Thus f* is a partial function from E(G) into the set {0, 1}, and we shall refer to f* as the induced partial edge labeling.

For i = 0, 1, let $v_f(i)$ = |{v ∈ V(G) | f(v) = i}| and $e_f(i)$ = |{e ∈ E(G) | f*(e) = i}|. The mapping f is said to be **friendly** if | $v_f(0)$- $v_f(1)$ | = 1

With these notations, we now introduce the notion of a balanced graph.

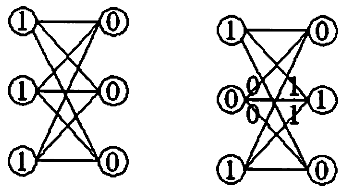
Definition 1. A graph G is said to be a ***balanced*** graph or G is ***balanced*** if there is a vertex labeling f of G satisfying |$v_f(0) - v_f(1)$| = 1 and |$e_f(0) - e_f(1)$ | = 1.

JCMCC **66** (2008), pp. 135-150

We will use v(0), v(1), e(0), e(1) instead of the more complicated $v_f(0)$, $v_f(1)$, $e_f(0)$, $e_f(1)$, when the context is clear. A graph G is said to be ***strongly vertex-balanced*** if G is balanced and v(0) = v(1). Similarly, a graph G is ***strongly edge-balanced*** if it is balanced and e(0) = e(1). If G is a strongly vertex-balanced and strongly edge-balanced graph, then we say that G is a ***strongly balanced*** graph.

Definition 2. The ***balance index set*** of a graph G, BI(G), is defined as $\{|e_f(0) - e_f(1)| :$ the vertex labeling f is friendly$\}$.

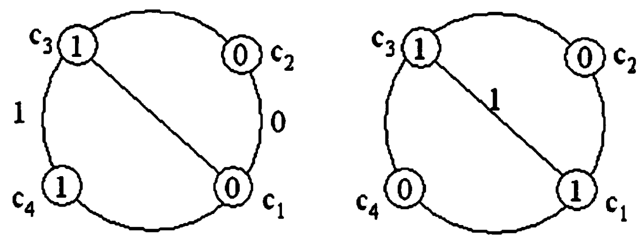
Example 1. $BI(K_{3,3}) = \{0\}$.



|e(1)-e(0)| = 0

Figure 1

Example 2. BI(G) = {0,1}.



|e(1)-e(0)| = 0 |e(1)-e(0)| = 1

Figure 2.

Theorem 1.1. Let G be a graph with p vertices. Then max{BI(G)} = k(k – 1)/2 if p = 2k or 2k – 1, depending on whether p is even or odd.
Proof. By friendliness, v(1) = k if p = 2k is even, and k or k – 1 if p = 2k – 1 is odd. Thus the maximum value of e(1) = k(k – 1)/2. The minimum value of e(1) is of course 0. The same argument gives the same maximum and minimum values for e(0). The result follows.

Theorem 1.2. Let T be a tree with p vertices. Then max{BI(T)} = (p/2) – 1 if p is even, and (p – 1)/2 if p is odd.

Proof. By friendliness, $v(1) = p/2$ if p is even, and $(p - 1)/2$ or $(p + 1)/2$ if p is odd.

For a friendly vertex labeling, consider the subgraph of T containing all the edges labeled 1 and their vertices. Each connected component of this subgraph is a tree, and so the number of edges (all labeled 1) is 1 less than the number of vertices (all labeled 1). Thus $e(1) \leq v(1)$ – the number of connected components $\leq v(1) - 1 = (p/2) - 1$ if p is even, and $(p - 1)/2$ if p is odd. The minimum value of $e(1)$ is of course 0. The same argument gives the same maximum and minimum values for $e(0)$. The result follows.

Definition 3. A subset X of Z is said to be ***BI-representable*** if there exists a graph G such that $BI(G) = X$.

We investigate sets of integers that are BI-representable. Some balance graphs were considered in [6]. The notion of friendly index sets of graphs which is similar to balance index sets were considered in [3,4,5].

2. On balance index sets of some trees.

In this section we first consider balance index sets of the star St(n), which is the one-point union of n copies of K_2. We note that when computing the balance index set of a graph, we may fix an arbitrary vertex in the graph and label it 0. If another vertex labeling gives it the label 1, simply replace each vertex label by its complement. Then $v(0)$ and $v(1)$ are interchanged, and $e(0)$ and $e(1)$ are interchanged. Since we are only concerned with absolute values, interchanging $v(0)$ and $v(1)$, $e(0)$ and $e(1)$ would not make any difference.

Theorem 2.1. The balance index set of the star St(n) is
(1) $\{k\}$, if $n = 2k + 1$ is odd,
(2) $\{k - 1, k\}$, if $n = 2k$ is even.
Proof. (1) Let $n = 2k + 1$. Without loss of generality, let the center be labeled 0. Then k of the other vertices are labeled 0, while the remaining $(k + 1)$ vertices are labeled 1. Thus $BI(St(2k + 1)) = \{k\}$.
(2) Let $n = 2k$. Without loss of generality, let the center be labeled 0. Then either k of the other vertices are labeled 0 while the remaining k vertices are labeled 1, or $(k - 1)$ of the other vertices are labeled 0 while the remaining $(k + 1)$ vertices are labeled 1. Thus $BI(St(2k)) = \{k - 1, k\}$.□

Note. Theorem 2.1 shows that the maximum in Theorem 1.2 is attainable.

Corollary 2.2. For any $k > 0$, the sets $\{k\}$ and $\{k - 1, k\}$ are BI-representable.

The double star D(m, n) is a tree of diameter three such that there are m appended edges on one end of P_2 and n appended edges on the other end (Figure 3). Without loss of generality, we assume $m \leq n$.

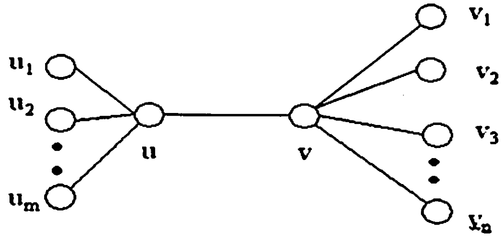


Figure 3.

Theorem 2.3. The balance index set of the double star D(m, n), where $m \le n$, is
(1) $\{(n-m)/2, (n+m)/2\}$ if m + n is even, and
(2) $\{(n-m-1)/2, (n-m+1)/2, (n+m-1)/2, (n+m+1)/2\}$ if m + n is odd.

Proof.
(1) Let m + n = 2k.

There are 2k + 2 vertices. Without loss of generality, assume that the vertex u has label 0.

First label the vertex v by 0. Assume j of the vertices $u_1, \dots, u_m$ are labeled by 0, and the other (m − j) vertices are labeled by 1. By friendliness, (k − j − 1) of the vertices $v_1, \dots, v_n$ are labeled by 0, and the other (n − k + j + 1) vertices are labeled by 1. Then e(0) = k and e(1) = 0, making e(0) − e(1) = k = (n + m)/2.

Then label the vertex v by 1. Assume j of the vertices $u_1, \dots, u_m$ are labeled by 0, and the other (m − j) vertices are labeled by 1. By friendliness, (k − j) of the vertices $v_1, \dots, v_n$ are labeled by 0, and the other (n − k + j) vertices are labeled by 1. Then e(0) = j and e(1) = n − k + j, making e(0) − e(1) = k − n = (m − n)/2, with absolute value (n − m)/2.

Thus BI(D(m, n)) = $\{(n-m)/2, (n+m)/2\}$.

(2) Let m + n = 2k + 1.

There are 2k + 3 vertices. Without loss of generality, assume that the vertex u has label 0.

First label the vertex v by 0. Assume j of the vertices $u_1, \dots, u_m$ are labeled by 0, and the other (m − j) vertices are labeled by 1. By friendliness, either (k − j − 1) or (k − j) of the vertices $v_1, \dots, v_n$ are labeled by 0, and the other (n − k + j + 1) or (n − k + j) vertices are labeled by 1 respectively. Then e(0) = k or k + 1 and e(1) = 0, making e(0) − e(1) = k = (n + m − 1)/2, or k + 1 = (n + m + 1)/2 respectively.

Then label the vertex v by 1. Assume j of the vertices $u_1, \dots, u_m$ are labeled by 0, and the other (m − j) vertices are labeled by 1. By friendliness, either (k − j) or (k − j + 1) of the vertices $v_1, \dots, v_n$ are labeled by 0, and the other (n − k + j) or (n − k + j − 1) vertices are labeled by 1 respectively. Then e(0) = j and e(1) = n − k + j or n − k + j − 1, making e(0) − e(1) = k − n = (m − n − 1)/2, with absolute value (n − m + 1)/2, or k − n + 1 = (m − n + 1)/2, with absolute value (n − m − 1)/2.

Thus BI(D(m, n)) = $\{(n-m-1)/2, (n-m+1)/2, (n+m-1)/2, (n+m+1)/2\}$.□

Corollary 2.4. For any non-negative integers a and b, with $a < b$, $\{a, b\}$ is BI-representable.
Proof. Let $m = b - a$ and $n = b + a$. Use (1) in Theorem 2.3.□

Corollary 2.5. For any non-negative integers a and b, with $a + 1 < b$, $\{a, a + 1, b, b + 1\}$ is BI-representable.
Proof. Let $m = b - a$ and $n = b + a + 1$. Use (2) in Theorem 2.3.□

Corollary 2.6. For any non-negative integer a, $\{a, a + 1, a + 2\}$ is BI-representable.
Proof. Let $m = 1$ and $n = 2a + 2$. Use (2) in Theorem 2.3.□

Example 3. Figure 4 shows the balance index set of the double star D(m, m), m = 2 and 3.

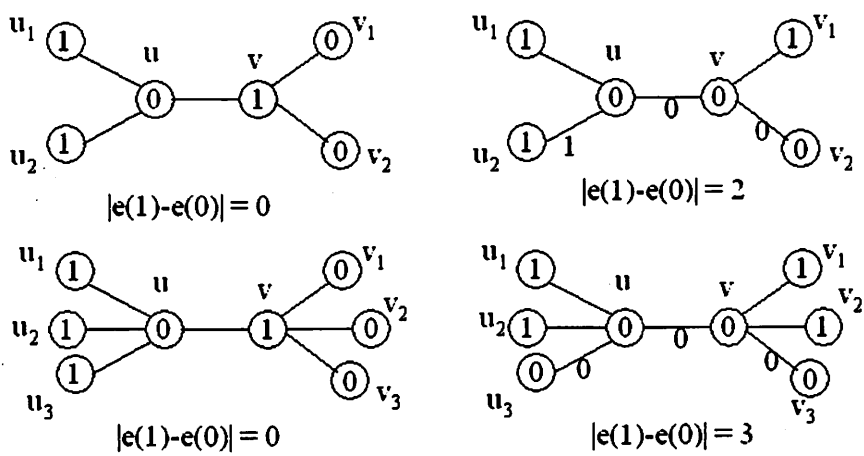


Figure 4.

Example 4. Figure 5 shows BI(D(3, 4)) = {0, 1, 3, 4}.

By Theorem 1.2, $\max\{BI(B(2, d))\} = (p - 1)/2 = (2^{d+1} - 2)/2 = 2^d - 1$. This finishes the proof. □

<u>Example 5.</u> $BI(B(2, 2)) = \{0, 1, 2, 3\}$.

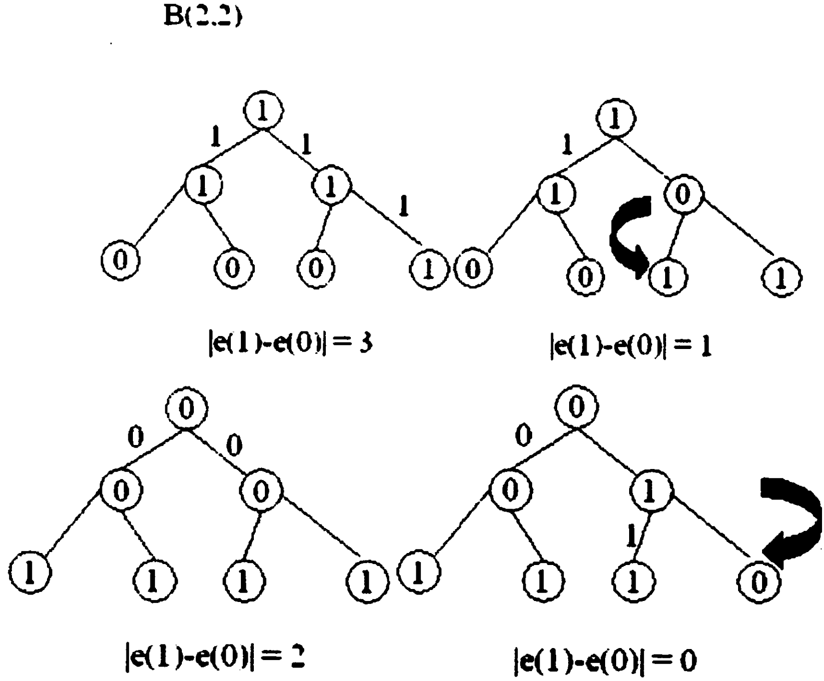


Figure 6.

<u>Example 6.</u> $BI(B(2, 3)) = \{0, 1, 2, 3, 4, 5, 6, 7\}$.

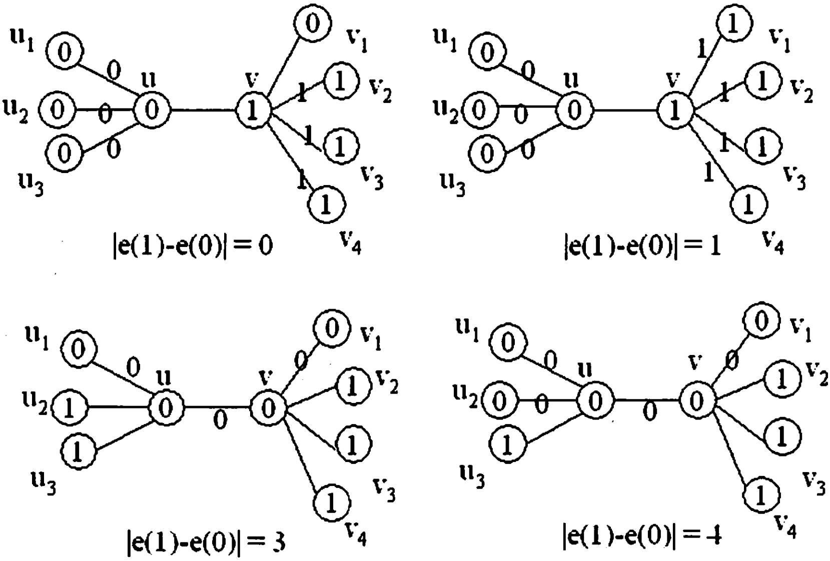


Figure 5.

Let $B(2, d)$ denote the full binary tree of depth d. We denote its vertices at depth t from left to right by $v_{1,t}, v_{2,t}, \dots, v_{2^t,t}$.

Theorem 2.7. For any $d \geq 1$, the balance index set of the full binary tree $B(2, d)$ of depth d is the set $\{0, 1, 2, \dots, 2^d - 1\}$.

Proof. The tree $B(2, d)$ has $1 + 2 + 4 + \dots + 2^d = 2^{d+1} - 1$ vertices, and $2 + 4 + \dots + 2^d = 2^{d+1} - 2$ edges.

First we label all the vertices at depth less than d by 1. Then we label all the vertices at depth d except the rightmost one by 0, and the rightmost vertex by 1. We see that $e(1) = 2^d - 1$, $e(0) = 0$, and so $2^d - 1 \in BI(B(2, d))$.

At the $(d - 1)$st level, interchange the vertex labels of $v_{2^{d-1},d-1}$ and its left child $v_{2^d-1,d}$. Then $v_{2^{d-1},d-1}$ and $v_{2^d-1,d}$ have labels 0 and 1 respectively, and the tree has two fewer 1-edges. This decreases the value of $e(1) - e(0)$ for the tree to $(2^d - 3)$. Now interchange the vertex labels of $v_{2^{d-1}-1,d-1}$ and its left child $v_{2^d-3,d}$. This decreases the value of $e(1)$ by 1 and increases the value of $e(0)$ by 1, making $e(1) - e(0) = 2^d - 5$. Repeating this procedure till the vertex $v_{2,d-1}$ produces the values $1, 3, 5, \dots, 2^d - 1$.

Now we start over, and label all the vertices at depth less than d by 1, and all the vertices at depth d by 0. We see that $e(1) = 2^d - 2$, $e(0) = 0$, and so $2^d - 2 \in BI(B(2, d))$.

At the $(d - 1)$st level, interchange the vertex labels of $v_{2^{d-1},d-1}$ and its right child $v_{2^d,d}$. Then $v_{2^{d-1},d-1}$ and $v_{2^d,d}$ have labels 0 and 1 respectively. This decreases the value of $e(1)$ by 1 and increases the value of $e(0)$ by 1, changing the value $e(1) - e(0)$ to $2^d - 4$. Repeating this procedure till the vertex $v_{2,d-1}$ produces the values $0, 2, 4, \dots, 2^d - 2$.

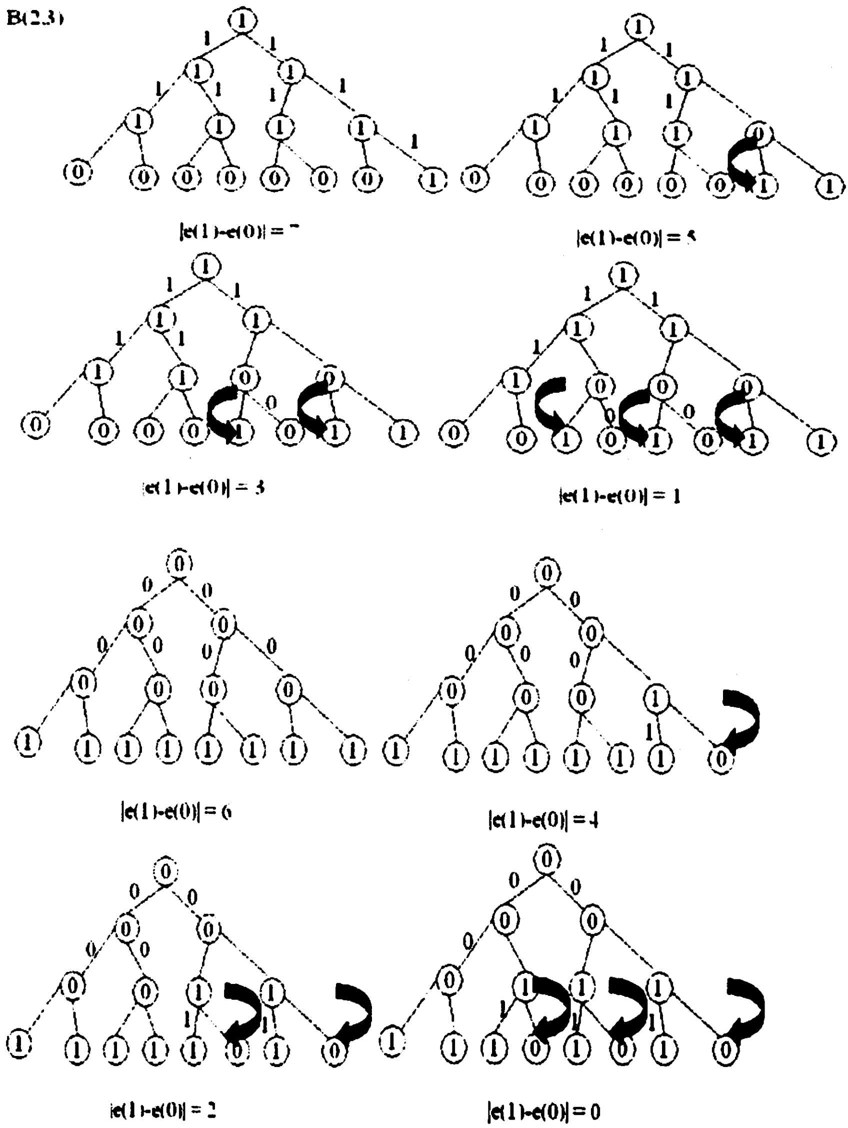


Figure 7.

3. Balance index sets of complete graphs, complete bipartite graphs and $K_n \cup N_n$.

Theorem 3.1. For any $n \geq 3$, $BI(K_n)$ is
(1) $\{0\}$ if n is even, and
(2) $\{k\}$ if $n = 2k + 1$ is odd.

Proof. It is clear that the balance index set of the complete graph K_{2t} is {0} for all $t \geq 1$.

Now consider K_{2k+1}. Without loss of generality, assume that $v(1) - v(0) = 1$. Let $f(v_i) = 1$, for $i = 1, 2, \dots, k+1$ and $f(v_j) = 0$, for $j = k+2, \dots, 2k+1$. Then we have $e(1) = C(k+1, 2)$ and $e(0) = C(k, 2)$. Thus $|e(0) - e(1)| = k$.□

For any $m, n \geq 1$, we denote the complete bipartite graph by $K_{m,n}$. Without loss of generality, we assume that $m \leq n$.

Theorem 3.2. The balance index set of complete bipartite graph $BI(K_{m,n})$ is
(1) $\{|(n-m)(i-\frac{1}{2}m)| : i = 0, 1, 2, \dots, m\}$ if $(m+n)$ is even, and
(2) $\{|(n-m)(i-\frac{1}{2}m)-\frac{1}{2}m|, |(n-m)(i-\frac{1}{2}m)+\frac{1}{2}m| : i = 0, 1, 2, \dots, m\}$ if $(m+n)$ is odd.

Proof. The set of vertices of $K_{m,n}$ can be partitioned into two subsets, one with m vertices, and the other with n vertices. Two vertices are adjacent if and only if they come from different subsets.

(1) Assume that i vertices from the first subset are labeled 0. Then the remaining $(m-i)$ vertices in this subset must be labeled 1. By friendliness, in the second subset, $(\frac{1}{2}(n+m)-i)$ vertices must be labeled 0, and the remaining $(\frac{1}{2}(n-m)+i)$ vertices must be labeled 1. It follows that $e(0) = i(\frac{1}{2}(n+m)-i)$ and $e(1) = (m-i)(\frac{1}{2}(n-m)+i)$. Simplification gives $e(0) - e(1) = (n-m)(i-\frac{1}{2}m)$. Since $i = 0, 1, 2, \dots, m$, by exhausting all the values of i, we obtain the balance index set stated in the theorem.

(2) We consider two subcases:

Case 2.1: $v(0) = \frac{1}{2}(m+n-1)$ and $v(1) = \frac{1}{2}(m+n+1)$.
If the first subset has i vertices labeled 0, then the remaining $(m-i)$ vertices will have label 1. Then the second subset must have $(\frac{1}{2}(m+n-1)-i)$ vertices labeled 0, and the remaining $(\frac{1}{2}(n-m+1)+i)$ vertices labeled 1. Then $e(0) = i(\frac{1}{2}(m+n-1)-i)$ and $e(1) = (m-i)(\frac{1}{2}(n-m+1)+i)$, giving $e(0) - e(1) = (n-m)(i-\frac{1}{2}m)-\frac{1}{2}m$. Since $i = 0, 1, 2, \dots, m$, by exhausting all the values of i, we obtain the first half of the set in (2).

Case 2.2: $v(0) = \frac{1}{2}(m+n+1)$ and $v(1) = \frac{1}{2}(m+n-1)$.
Again if the first subset has i vertices labeled 0, then the remaining $(m-i)$ vertices will be labeled 1. Then the other subset must have $(\frac{1}{2}(m+n+1)-i)$ vertices labeled 0, and the remaining $(\frac{1}{2}(n-m-1)+i)$ vertices labeled 1. Then $e(0) = i(\frac{1}{2}(m+n+1)-i)$ and $e(1) = (m-i)(\frac{1}{2}(n-m-1)+i)$, giving $e(1) - e(0) = (n-m)(i-\frac{1}{2}m)+\frac{1}{2}m$. By exhausting all the values of $i = 0, 1, 2, \dots, m$, we obtain the second half of the set in (2).□

In particular, we have

Corollary 3.3. The balance index set of the complete bipartite graph $BI(K_{m,m})$ is {0}.

Example 7. Figure 8 illustrates $BI(K_{2,5}) = \{1,2,4\}$.

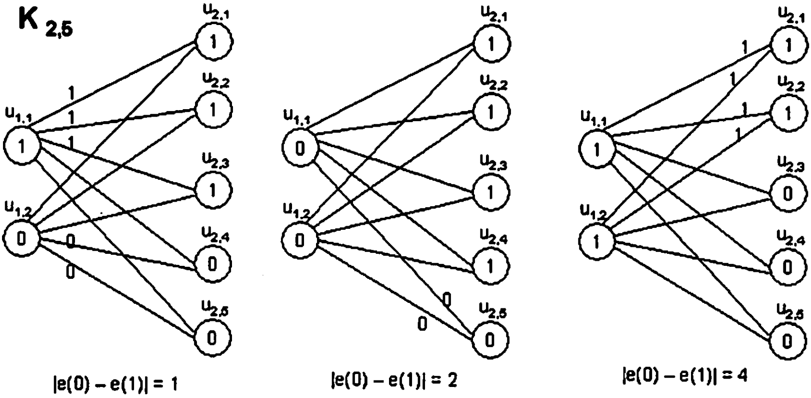


Figure 8.

The following result shows that we can generate arbitrarily large BI-representable sets.

__Corollary 3.4.__ The balance index set of the complete bipartite graph BI($K_{m,m+1}$) is {0, 1, 2, … , m}.

__Example 8.__ Figure 9 shows the balance index sets of the complete bipartite graphs $K_{2,3}$ and $K_{3,4}$.

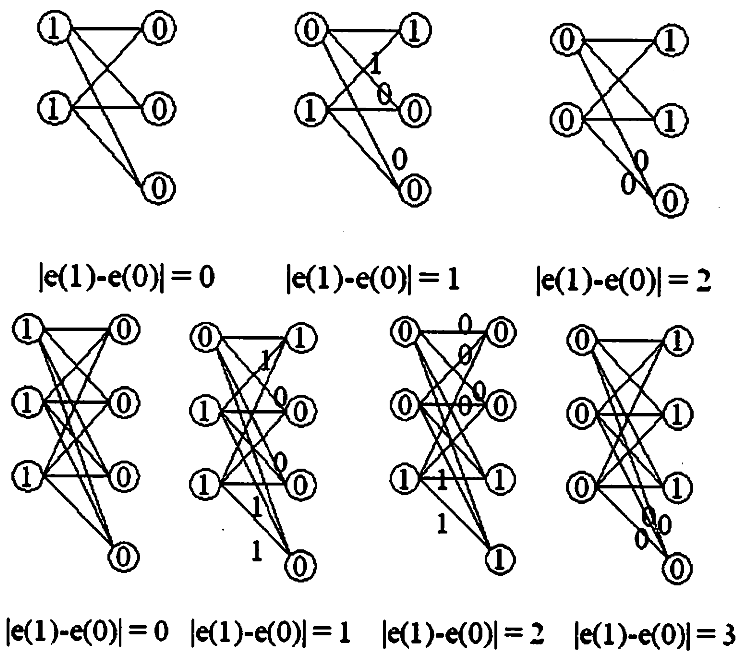


Figure 9.

Example 9. $BI(K_{3,6}) = \{0, 3, 6\}$ and $BI(K_{3,7}) = \{2, 6\}$.

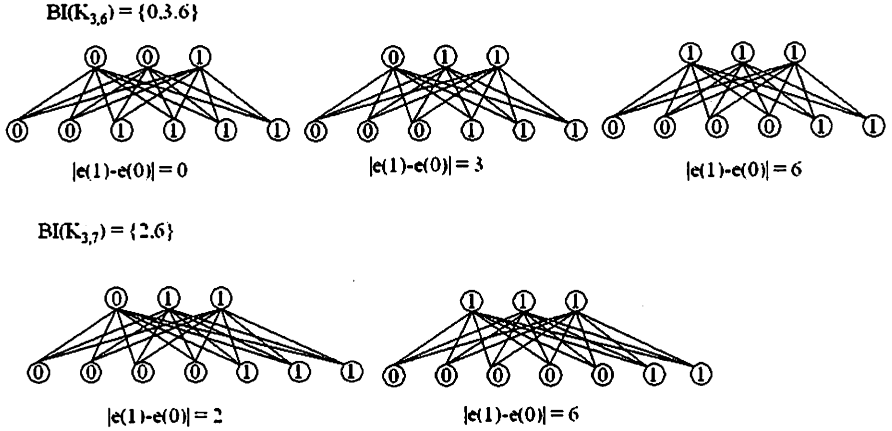


Figure 10.

Corollary 3.5. For any $n \geq 1$, $BI(K_{n,n+2}) = \{n, n-2, \ldots, 0\}$ if n is even, and $\{n, n-2, \ldots, 1\}$ if n is odd.

Corollary 3.6. For any $m \geq 0$, even $n \geq 2$, $BI(K_{n,n+2m}) = \{mn, m(n-2), \ldots, 0\}$.

Corollary 3.7. For any odd $n \geq 1$, $\{n, n-2, \ldots, 1\}$ is BI-representable.

Corollary 3.8. For any $m \geq 0$, even $n \geq 2$, $\{mn, m(n-2), \ldots, 0\}$ is BI-representable.

Now we consider the disjoint union of two graphs $K_n \cup N_n$.

Theorem 3.9. For any $n \geq 2$, $BI(K_n \cup N_n) = \{\frac{1}{2}(n-1)(n-2j) : 0 \leq j \leq n/2\}$. This is $\{0, 2k-1, 2(2k-1), \ldots, k(2k-1)\}$ if $n = 2k$, and $\{k, 3k, 5k, \ldots, (2k+1)k\}$ if $n = 2k+1$.

Proof. By changing all vertex labels to their complements if necessary, we may assume that K_n has no more 0-vertices than 1-vertices. Thus let K_n have j vertices labeled 0, where $0 \leq j \leq n/2$. Then the other $(n-j)$ vertices of K_n are labeled 1. We have $e(0) = j(j-1)/2$ and $e(1) = (n-j)(n-j-1)/2$. Thus $|e(0) - e(1)| = \frac{1}{2}(n-1)(n-2j)$. Now let j exhaust all possible values.□

Note. Theorem 3.9 shows that the maximum in Theorem 1.1 is attainable.

Corollary 3.10. For any $k \geq 1$, $\{0, 2k-1, 2(2k-1), \ldots, k(2k-1)\}$ and $\{k, 3k, 5k, \ldots, (2k+1)k\}$ are BI-representable.

Example 10. $BI(K_5 \cup N_5) = \{2, 6, 10\}$ and $BI(K_6 \cup N_6) = \{0, 5, 10, 15\}$.

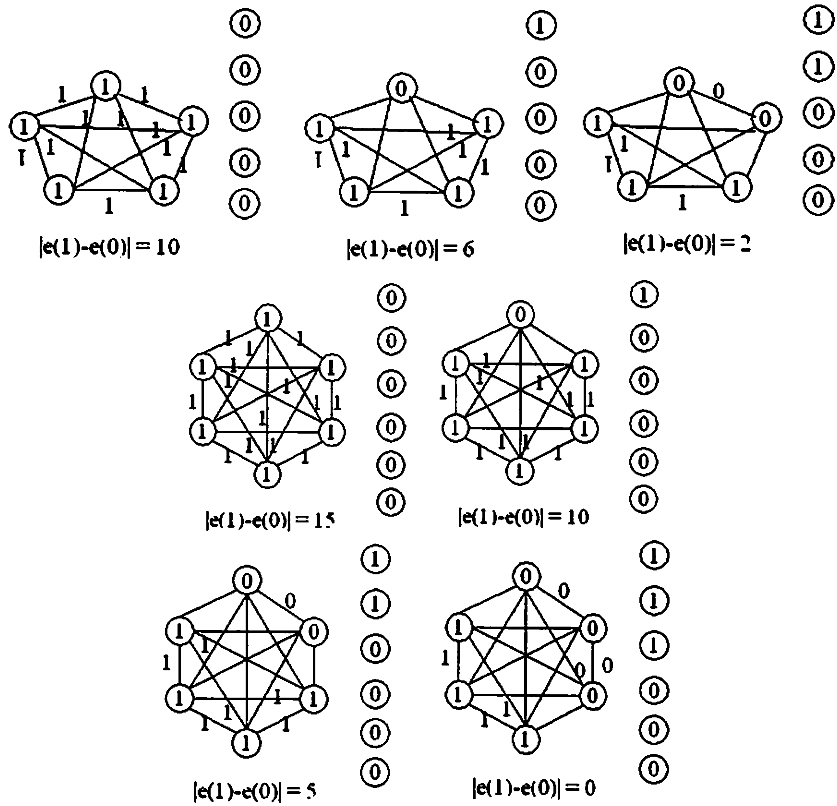


Figure 11.

4. Windmill graphs

. Shee and Ho [7] considered one-point union of graphs which are cordial.

Notation. Let $K(m_1, m_2, \ldots, m_n)$ be the one-point amalgamation of the complete graphs $m_1, m_2, \ldots, m_n$ vertices. Call the point at which the complete graphs are amalgamated the center of $K(m_1, \ldots m_n)$. If k of the m values are equal to the same value a, and if no confusion could arise, we use a^k to denote these values.

Theorem 4.1. For any $m \geq 0$, $n \geq 2$, $BI(K(n + 2m + 1, n)) = \{|(2m + 1)j - mn - 2m^2| : m \leq j \leq n + m - 1\}$.

Proof. In $K(n + 2m + 1, n)$, there are $(2n + 2m)$ vertices. Thus a friendly vertex labeling must have $v(0) = v(1) = n + m$. Without loss of generality, let the center be labeled 0. If K_{n+2m+1} has j non-center vertices labeled 0, then the other $(n + 2m - j)$ vertices must be labeled 1. By friendliness, K_n has $(n + m - j - 1)$ non-center vertices labeled 0, and the other $(j - m)$ vertices labeled 1. For these numbers to make sense, we must have $m \leq j \leq n + m - 1$. Then in K_{n+2m+1}, $e(0) = (j + 1)j/2$ and $e(1) = (n + 2m - j)(n + 2m - j - 1)/2$, and in K_n, $e(0) = (n + m -$

j)(n + m − j − 1)/2 and e(1) = (j − m)(j − m − 1)/2. Algebraic calculations show that for the whole graph K(n + 2m + 1, n), e(0) − e(1) = (2m + 1)j − mn − 2m^2.□

Example 11. BI(K(5, 2)) = {1, 2}.

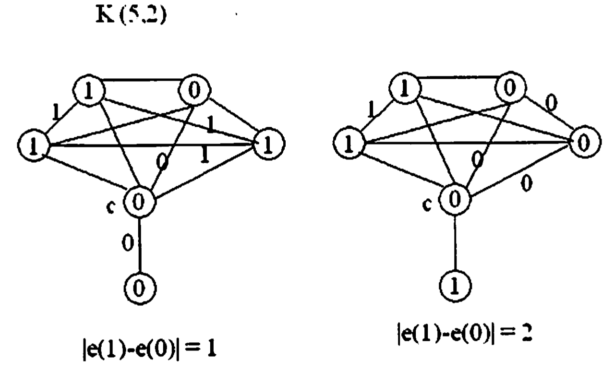


Figure 12.

Example 12. BI(K(6, 3)) = {1, 2, 4}.

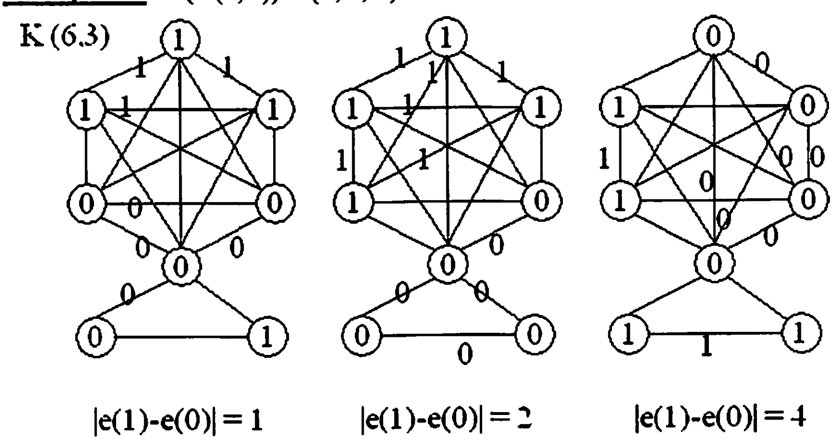


Figure 13.

Corollary 4.2. BI(K(n + 1, n)) = {0, 1, … , n − 1}.

Corollary 4.3. For any n ≥ 1, the set {0, 1, … , n − 1} is BI-representable.

Theorem 4.4. For any n ≥ 3, the balanced index set of the graph K(n, n) is {0, n − 1}.

Proof. Call the vertex at which the two copies of K_n are amalgamated the center of K(n, n). Without loss of generality, let the center be labeled 0. There remain (2n − 2) vertices to be labeled.

Case 1: In one copy of K_n, m non-center vertices are labeled 0, while the other (n − m − 1) vertices are labeled 1, where 0 ≤ m ≤ (n − 1). In the other copy of

K_n, (n – m – 1) non-center vertices are labeled 0, while the other m vertices are labeled 1. In the first K_n, there are (m + 1) vertices labeled 0, and (n – m – 1) vertices labeled 1. Thus e(0) = (m + 1)m/2, and e(1) = (n – m – 1)(n – m – 2)/2. In the second K_n, there are (n – m) vertices labeled 0, and m vertices labeled 1. Thus e(0) = (n – m)(n – m – 1)/2, and e(1) = m(m – 1)/2. Then for the entire $K_n(2)$, e(0) = (m + 1)m/2 + (n – m)(n – m – 1)/2, and e(1) = (n – m – 1)(n – m – 2)/2 + m(m – 1)/2, making e(0) – e(1) = m + (n – m – 1) = n – 1.
Case 2: In one copy of K_n, m non-center vertices are labeled 0, while the other (n – m – 1) vertices are labeled 1, where $0 \le m \le (n-2)$. In the other copy of K_n, (n – m – 2) non-center vertices are labeled 0, while the other (m + 1) vertices are labeled 1. In the first K_n, there are (m + 1) vertices labeled 0, and (n – m – 1) vertices labeled 1. Thus e(0) = (m + 1)m/2, and e(1) = (n – m – 1)(n – m – 2)/2. In the second K_n, there are (n – m – 1) vertices labeled 0, and (m + 1) vertices labeled 1. Thus e(0) = (n – m – 1)(n – m – 2)/2, and e(1) = (m + 1)m/2. Then for the entire K(n,n), e(0) = (m + 1)m/2 + (n – m – 1)(n – m – 2)/2, and e(1) = (n – m – 1)(n – m – 2)/2 + (m + 1)m/2, making e(0) – e(1) = 0.□

__Corollary 4.5.__ For any $n \ge 3$, the set {0, n – 1} is BI-representable.

__Example 13.__ Figure 14 shows BI(K(6,6)) = {0, 5}.

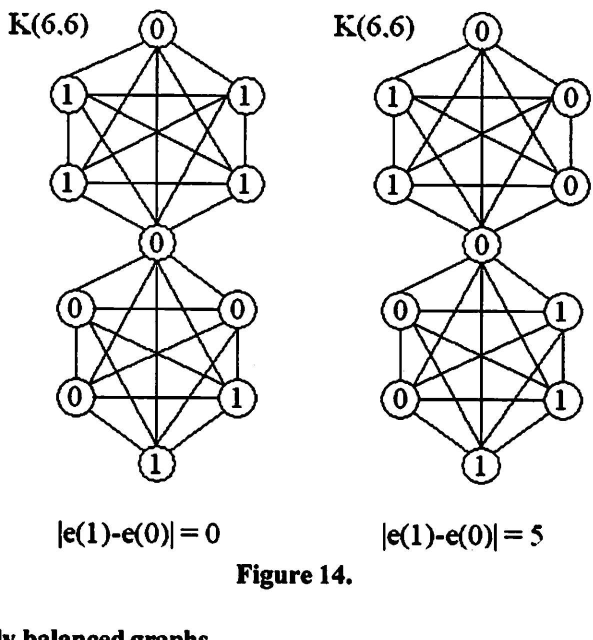


Figure 14.

5. Uniformly balanced graphs.

In [1], Chartrand, Zhang and the second author characterized graphs whose friendly index sets are subset of {0,1}.

Recall that a graph G is balanced if there exists a binary labeling f such that $|v_f(0) - v_f(1)| \le 1$ and $|e_f(0) - e_f(1)| \le 1$, or equivalently, min BI(G) ≤ 1.

Definition 4. A graph G is said to be ***uniformly balanced*** if it is balanced for any friendly binary labeling f.

Remark. A graph G is uniformly balanced if and only if $BI(G) \subseteq \{0, 1\}$.

In Section 3, we showed that the balance index set of the complete bipartite graph $BI(K_{m,m})$ is $\{0\}$. Thus the complete bipartite graph $K_{m,m}$ is uniformly balanced. We now show that the cycle C_n is uniformly balanced for all $n \geq 3$.

Lemma 5.1. For any (not necessarily friendly) vertex labeling of C_n, the differences $e(0) - e(1)$ and $v(0) - v(1)$ are the same.
Proof. Obviously it is impossible for all vertices to be labeled 0 or for all vertices to be labeled 1. In other words, there must be two adjacent vertices with complementary labels. Call them $v_{1,1}$ and $v_{1,0}$, with labels 1 and 0 respectively. Start from $v_{1,0}$, and go in the direction opposite to $v_{1,1}$. Consider the value of $c = (v(0) - v(1)) - (e(0) - e(1))$ as we traverse the cycle. When we start from $v_{1,0}$, $c = 1$. Let the next vertex labeled 1 be $v_{2,1}$. Before we reach $v_{2,1}$, $v(0)$ and $e(0)$ increase by the same amount, while $v(1)$ and $e(1)$ are unchanged. The edge leading to $v_{2,1}$ has no label. Thus at $v_{2,1}$, the value of c becomes 0. Continue to traverse the cycle in the same direction. Let the next vertex labeled 0 be $v_{2,0}$. Before we reach $v_{2,0}$, $v(1)$ and $e(1)$ increase by the same amount, while $v(0)$ and $e(0)$ are unchanged. The edge leading to $v_{2,0}$ has no label. Thus at $v_{2,0}$, the value of c becomes 1 again. Let the next vertex labeled 1 be $v_{3,1}$. The same argument shows that the value of c at $v_{3,1}$ is 0. Let the next vertex labeled 0 be $v_{3,0}$. The same argument shows that the value of c at $v_{3,0}$ is 1. Continue in this fashion. Eventually we will return to $v_{1,1}$ and $v_{1,0}$. At $v_{1,1}$, the value of c is 0. The edge leading from $v_{1,1}$ to $v_{1,0}$ has no label. Since the vertex label of $v_{1,0}$ has been counted when we start the process, we conclude that the value of $c = 0$ when we finish traversing the cycle. □

Theorem 5.2. $BI(C_n) = \{0\}$ if n is even, and $= \{1\}$ if n is odd.
Proof. As the vertex labeling is friendly, we have $|v(0) - v(1)| = 0$ when n is even, and $= 1$ when n is odd. The fact from Lemma 5.1 that $e(0) - e(1) = v(0) - v(1)$ finishes the proof.□

Theorem 5.3. Let G be a 2-regular graph, i.e., G is the disjoint union of cycles. Then $BI(G) = \{0\}$ or $\{1\}$, if the number of vertices is even or odd respectively.
Proof. Assume that G is the disjoint union of k cycles. Let $c_i = (v(0) - v(1)) - (e(0) - e(1))$ for the ith cycle, where $i = 1, 2, \ldots, k$, and v and e are the counts corresponding to that cycle. By Lemma 5.1, $c_i = 0$ for each i. Summing $c_1, c_2, \ldots, c_k$, we have $(v(0) - v(1)) - (e(0) - e(1)) = 0$ for the whole graph G. Using the fact that the vertex labeling is friendly, we establish the result.□

In [1], uniformly cordial graphs are completely characterized. However, at present the following problem is still unsolved.

Problem. Characterize uniformly balanced graphs.

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