Global Alliance Partition in Trees

Linda Eroh
Department of Mathematics
University of Wisconsin Oshkosh,
Oshkosh, WI, 54901
eroh@uwosh.edu
and
Ralucca Gera *
Department of Applied Mathematics
Naval Postgraduate School,
Monterey, CA, 93943
rgera@nps.edu

Abstract

Let G be a connected graph with vertex set $V(G)$ and edge set $E(G)$. A (defensive) alliance in G is a subset S of $V(G)$ such that for every vertex $v \in S$, $|N[v] \cap S| \geq |N(v) \cap (V(G) - S)|$. The alliance partition number, $\psi_a(G)$, was defined (and further studied in [11]) to be the maximum number of sets in a partition of $V(G)$ such that each set is a (defensive) alliance. Similarly, $\psi_g(G)$ is the maximum number of sets in a partition of $V(G)$ such that each set is a global alliance, i.e. each set is an alliance and a dominating set. In this paper, we give bounds for the global alliance partition number in terms of the minimum degree, which gives exactly two values for $\psi_g(G)$ in trees. We concentrate on conditions that classify trees to have $\psi_g(G) = i$ $(i = 1, 2)$, presenting a characterization for binary trees.

Key Words: alliance, global alliance, domination, partition, alliance partition.

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1 Introduction and motivation

A *dominating set* S is a subset of the vertices of a graph such that every vertex either belongs to S or has a neighbor in S. The topic has long been studied by researchers [9, 10]. This definition also leads naturally to the associated optimization problem: find a dominating set of minimum cardinality. Numerous variants of this problem have been studied [2, 9, 10, 15].

Global and defensive alliances were introduced by Kristiansen, Hedetniemi, and Hedetniemi in [13] and [14]. The definitions were motivated by the study of alliances between different people, between different countries, and between species of plants in botany. In a graph G, a nonempty set of vertices S is a *(defensive) alliance* if for every vertex $v \in S$, $|N[v] \cap S| \geq |N(v) \cap (V(G) - S)|$. Then for every v in S, every neighbor of v in S is an ally of v, and every neighbor not in S is an enemy of v. So v must be adjacent to at least as many allies as enemies, where v itself is counted as an ally. Note that it is possible for v not to have any enemies. A (defensive) alliance is *global* if it is also a dominating set for the graph. The minimum order of a (defensive) alliance in a graph G is the *(defensive) alliance number* written $a(G)$, and the minimum order of a global alliance is the *global (defensive) alliance number* $\gamma_a(G)$. In this paper we will abbreviate global defensive alliance to global alliance. Algorithmic complexity of alliances in graphs was first studied in [3] with more studies of complexity of different variants of alliances. For other graph theory terminology the reader should refer to [4].

The *alliance partition number of* G, $\psi_a(G)$, was defined in [13] to be the maximum number of sets in a partition of $V(G)$ such that each set is an alliance. Similarly, *the global alliance partition number*, $\psi_g(G)$, is defined to be the maximum number of sets in a partition of $V(G)$ such that each set is a global alliance, i.e. each set is an alliance and a dominating set. Also the *offensive global alliance partition number* is the maximum number of sets in a partition of $V(G)$ such that each set is a global offensive alliance.

Similar concepts have been studied in which the vertex set has been partitioned into exactly two sets, each of which is some type of alliance. In [5], [6], and [7], R. D. Dutton and H. S. Khurram defined an *alliance-free partition* to be a partition of the vertex set into two nonempty sets if neither one of the two sets contains a strong defensive alliance as a subset. Also, they defined an *alliance cover set* to be a subset of the vertices of a graph that contains at least one vertex from every alliance of the graph. It turns out that the complement of an alliance cover set is an alliance free set, that is, a set that does not contain any alliance as a subset. They characterize the graphs that can be partitioned into alliance free and alliance cover sets. Gerber and Kobler in [8] introduced the *satisfactory partition problem*,

which, restated in our notation, involves determining whether a particular graph has a partition into two strong defensive alliances. In [12], this idea was generalized to the *k- Satisfactory Graph Partitioning problem (k-SGP)*, which consists in determining if a graph is k-satisfiable or not, i.e., whether a given graph can be partitioned into two k-defensive alliances. An alliance A is k-defensive, if for each vertex $v \in A$, we have that $\deg_A(v) \geq \deg_{V-A} v + k$, where k is an integer. Note that if $k = 0$, then the problem reduces to finding which graphs have a partition of the vertex set into exactly two alliances. A graph G is 0-satisfiable if and only if $\psi_a(G) \geq 2$, so the alliance partition number could be viewed as a generalization of 0-satisfiability. In a similar fashion the *unfriendly graph partition problem* was introduced by Aharoni et al. [1], where the vertex set is partitioned into two sets such that each vertex has most of its neighbors in the complement of the set it belongs to.

In this paper, we give bounds for the global alliance partition number in terms of the minimum degree, which gives exactly two values for $\psi_g(G)$ in trees. Section 3 particularly concentrates on trees, where the global alliance number can have only two values, 1 or 2, respectively. We provide sufficient conditions for the global alliance partition number to be of each value, and we present a characterization for binary trees. We recently learned that T. W. Haynes and J. A. Lachniet have independently worked on this topic, and we refer to [11] and any of their future papers for the study of the global alliance partition number in classes of graphs and general bounds in terms of the order of the graph.

2 Preliminary Results and Observations

In this section we make quick observations about the global alliance partition number in general, concentrating on trees in Section 3. We will start by showing an example of the global alliance partition number on the Petersen graph. Let P be the Petersen graph of order $n = 10$. We will be using labels on the vertices to denote which global alliance each vertex belongs to.

Recall that each global alliance must be a dominating set. Observe that a dominating set must have at least 3 vertices. Moreover, each global alliance must be an alliance, and so each vertex must have at least one ally in its global alliance. Since there is no dominating set of order 3 that is an alliance (i.e. at least one vertex of any dominating set of order 3 has no allies in the dominating set), it follows that at least four vertices must be in each global alliance. Thus $\psi_g(P) \leq \lfloor \frac{10}{4} \rfloor = 2$. Also, the labeling below shows that there is a global alliance partition with 2 global alliances, and so $\psi_g(P) = 2$.

We consider disconnected graphs, so that we can concentrate on con-

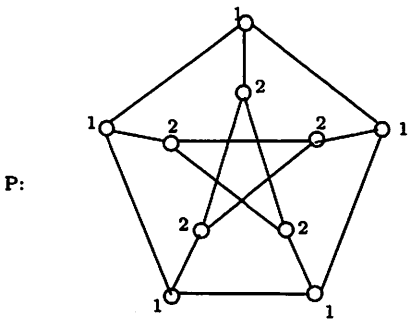

Figure 1: $\psi_g(P) = 2$

nected graphs for the rest of the paper. We present the relationship between the global alliance partition number of a disconnected graph and its components. First, observe that each global alliance must contain at least one vertex in each component of G, in particular, it must be a global alliance of that component. Thus we have the following result.

Proposition 2.1 *Let G be a disconnected graph whose components are G_1, $G_2, \ldots, G_p$. Then*

$$\psi_g(G) = \min_{1 \le i \le p} \psi_g(G_i).$$

The first bound is in terms of the minimum degree in the graph, which will be very useful in studying the global alliance partition number in trees.

Theorem 2.2 *Let G be a connected graph with minimum degree δ. Then*

$$1 \le \psi_g(G) \le 1 + \left\lceil \frac{\delta}{2} \right\rceil.$$

The bound is sharp.

Proof. Let $v \in V(G)$ such that $\deg v = \delta$. Then v can have at most $\lceil \frac{\delta}{2} \rceil$ enemies. Since at least one vertex in each global alliance must dominate v, it follows that there are at most $\lceil \frac{\delta}{2} \rceil$ possible global alliances containing the enemies of v, plus one global alliance that contains v and its allies. Thus $\psi_g(G) \le 1 + \lceil \frac{\delta}{2} \rceil$. To see the sharpness, let $G = K_{t+1} \times K_t$. □

For trees, this bound is sharp for P_{2k} $k \ge 2$. We now restrict our attention to trees only.

3 Global Alliance Partition in Trees

We begin the section with two definitions. The *corona of* G, $Cor(G)$, is the graph obtained from G by adding a pendant to each vertex of G. Similarly, we define the *double-corona of* G, $DblCor(G)$, to be the graph obtained from G by adding two pendants to each vertex of G. For the purposes of this paper, we use *end-vertex* to be a vertex of degree 1, and a *pendant* to be the vertex of degree 1 together with the edge with which it is incident. Also, we call *stem* a vertex that is adjacent to some end-vertex.

The following is a corollary to Theorem 2.2.

Corollary 3.1 *Let T be a tree of order $n \geq 3$. Then $1 \leq \psi_g(T) \leq 2$.*

We define a tree T to be of *Class* 1 if $\psi_g(T) = 1$ and of *Class* 2 if $\psi_g(T) = 2$. Each class contains several families of trees:
Class 1 : paths on an odd number of vertices, spiders, stars, double stars with maximum degree at least 4, k-subdivided stars (where each edge of the star is subdivided k times for a fixed integer k).
Class 2 : paths on even number of vertices, Cor(T), DblCor(T) (as we will show in this section).

We will be using colors to denote the global alliance 1 or the global alliance 2 that each vertex belongs to, with the coloring $c : V(T) \rightarrow \{1, 2\}$, where $c(v) = i$ if vertex v belongs to global alliance i, for $i = 1, 2$.

Characterizing exactly which trees are of Class 1 and which are of Class 2 appears to be a difficult problem. However, we begin with a sufficient condition for a tree to be Class 2.

Theorem 3.2 *Let T be a tree. If T has a perfect matching, then T is of Class 2.*

Proof. Define the two global alliances by labeling the vertices of the graph either with 1 or 2 as follows: pick the root of T to be an end vertex, say x, label x with 1, and label the rest of the vertices traveling on all possible paths away from x by changing the label every time an edge of the matching is crossed, and keeping the same label as the label of the previous vertex if no edge of the matching was used. □

Next, we will show that every tree is the induced subgraph of some tree of Class 2.

Proposition 3.3 *Let T be a tree. Then there is a tree T' of Class 2 that contains T as an induced subgraph.*

Proof. Let T be a tree. If $\psi_g(T) = 2$, then $T' = T$. Otherwise, if $\psi_g(T) = 1$, let $T' = Cor(T)$. Then the set $V(T)$ forms a global alliance

in T' since (1) each of its vertices has exactly one enemy and at least two allies, (2) and each vertex not in T is dominated by its stem in T. Also, the set $V(T') - V(T)$ is a global alliance since (1) each of its vertices has exactly one enemy and exactly one ally (itself), and (2) each vertex of $V(T)$ is dominated by its pendant in $V(T') - V(T)$. Thus $\psi_g(T') = 2$. □

As noted in the proof above, the corona of any tree has the global alliance partition number of 2, with the original tree as one global alliance, and a maximum independent set forming the other one.

The previous result works for any graph G, not only for trees. Similarly, if instead of just a pendant one would add two pendants or a P_3 to each vertex of the original graph, the same result follows.

Corollary 3.4 *Let T be a tree. Then*

$$\psi_g\ (Cor(T)) = 2 \quad \textit{and} \quad \psi_g\ (DblCor(T)) \ = \ 2.$$

We next present two sufficient conditions for a tree to be of Class 1. For $v \in V(G)$, we define $\delta_P(v)$ to be the number of end vertices that v is adjacent to. Also, we denote by $\mathcal{E}$ the set of all end vertices of G.

Proposition 3.5 *Let T be a tree of order $n \geq 4$. If there is a vertex v such that $\delta_P(v) \geq \lceil \frac{\deg v}{2} \rceil + 1$, then T is of Class 1.*

Proof. Suppose T is a tree of order $n \geq 4$ and v is a vertex of T with $\delta_P(v) \geq \left\lceil \frac{\deg v}{2} \right\rceil + 1$. Suppose, to the contrary, that there is a Class 2 coloring of T. If u is an end-vertex adjacent to v, then u must be a different color than v. Without loss of generality, then, we may assume that v is color 2, and so every end-vertex adjacent to v is color 1. Thus, v has at least $\left\lceil \frac{\deg v}{2} \right\rceil + 1$ enemies and at most $\left\lfloor \frac{\deg v}{2} \right\rfloor$ allies, which is a contradiction. There is no Class 2 coloring of T. □

The next sufficient condition involves the distance between end vertices in the tree.

Proposition 3.6 *Let T be a tree of order $n \geq 4$. If there is an end vertex $v \in \mathcal{E}$, such that for $\forall u \in \mathcal{E}$, $d(v, u)$ is even, then T is of Class 1.*

Proof. Let T be a tree of order $n \geq 4$ and let v be an end-vertex such that for all $u \in \mathcal{E}$, $d(v, u)$ is even. Suppose to the contrary that T is Class 2 and consider a Class 2 coloring of T. Without loss of generality, v is color 1. It follows that v must be adjacent to a vertex v_1 of color 2. If v_1 has another neighbor besides v, then at least one of the neighbors of v_1, say v_2, must be color 2. Since v_2 is dominated by color 1, v_2 must have a neighbor v_3 which is color 1. We continue building the path in this fashion until

we reach an end-vertex v_k. Thus, we have a path $v = v_0, v_1, v_2, \ldots, v_k$, in which v_i is color 1 if $i \equiv 0$ or $3 (\mathrm{mod}\, 4)$ and v_i is color 2 if $i \equiv 1$ or $2 (\mathrm{mod}\, 4)$. Since $d(v, v_k)$ is even, k is congruent to either 0 or $2 (\mathrm{mod}\, 4)$. Thus, v_{k-1} is the same color as v_k. Since v_k is an end-vertex, it does not get dominated by the opposite color, which is a contradiction. T must be Class 1. □

Recall that a *binary tree* is a tree of maximum degree 3. A *complete binary tree* is a binary tree of height n, with 2^i vertices at distance i from the root ($1 \leq i \leq n$). And so, in a complete binary tree, each vertex has two children, except for the 2^n end-vertices at distance n from the root. As a quick consequence of the above proposition, any complete binary tree is of Class 1.

For (general) binary trees, we show that the sufficient condition for Class 1 in Proposition 3.6 is also necessary, obtaining a characterization for general binary trees.

Theorem 3.7 *Let T be a binary tree of order $n \geq 3$. T is Class 2 if and only if there exist a pair of end-vertices in T that are an odd distance from one another.*

Proof. First, we assume to the contrary, that T does not have a pair of end-vertices that are an odd distance from one another. Then the distance between any two end-vertices is even, and by Proposition 3.6 then T is of Class 1.

For the converse, let T be a binary tree of order n and let u and v be two end-vertices in T such that $d(u, v)$ is odd. We will show that T is Class 2 using induction on the number of end-vertices in T.

If T only has the two end-vertices u and v, then T is a path of odd length, that is, a path with an even number of vertices. We have already seen that a path with an even number of vertices is Class 2.

Suppose any binary tree with $k - 1$ or fewer end-vertices and with two end-vertices an odd distance from each other is Class 2. Let T be a binary tree with $k \geq 3$ end-vertices, such that two end-vertices u and v are an odd distance from each other. Let w be an end-vertex of T other than u and v. Notice that T must contain at least one vertex of degree 3. Let x be the vertex of degree 3 closest to w, and let $w = w_0, w_1, \ldots, w_j = x$ be the path from w to x, where $j \geq 1$. Notice that each vertex w_i with $1 \leq i \leq j - 1$ must have degree 2. Define $T' = T - \{w_0, w_1, \ldots, w_{j-1}\}$. By the inductive hypothesis, T' is Class 2.

Consider a Class 2 coloring of T'. We may assume without loss of generality that $x = w_j$ is color 1. Notice that x has degree 2 in T', so x must be adjacent to one vertex that is color 1 and one vertex that is color 2. Thus, if we use the same colors in T, x is still defended in color 1 and dominated by color 2 no matter what color we use for w_{j-1}. If $j \equiv 0$ or

$3 \bmod 4$, we can color w_i with color 1 for $i \equiv 0$ or $3 \bmod 4$ and with color 2 otherwise. If $j \equiv 1$ or $2 \bmod 4$, we can color w_i with color 2 for $i \equiv 0$ or $3 \bmod 4$ and with color 1 otherwise. In either case, $w_j = x$ is assigned Color 1, the color it already has in T'. For each i, $1 \leq i \leq j-1$, w_i has one neighbor that is color 1 and one neighbor that is color 2, so w_i is both defended in its own color and dominated by the opposite color. Notice also that w_0 is adjacent to a vertex in the opposite color, so it is dominated by that color. An end-vertex can always defend itself. Thus, we have a Class 2 coloring of T. □

Notice that this result does not hold for trees in general. For instance, the tree T with $V(T) = \{u, v, w, x, y, z\}$ and $E(T) = \{uv, vw, wx, wy, wz\}$ is Class 1 by Proposition 3.5, even though $d(u, x)$ is odd. And so we post the following open problem.

Problem 3.8 *Is there a characterization for trees in general?*

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