# The Edge-graceful Spectra of Connected Bicyclic Graphs Without Pendant*

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**Abstract**

Let $G$ be a connected simple $(p, q)$-graph and $k$ a non-negative integer. The graph $G$ is said to be $k$-edge-graceful if the edges can be labeled with $k, k+1, \ldots, k+q-1$ so that the vertex sums are distinct modulo $p$. The set of all $k$ where $G$ is $k$-edge-graceful is called the edge-graceful spectrum of $G$. In 2004, Lee, Cheng and Wang analyzed the edge-graceful spectra of certain connected bicyclic graphs, leaving some cases as open problems. Here, we determine the edge-graceful spectra of all connected bicyclic graphs without pendant.

# 1 Introduction

Let $G = (V, E)$ be a connected simple graph having $p$ vertices and $q$ edges. Given an integer $k \geq 0$, a bijection $f : E(G) \to \{k, k+1, \ldots, k+q-1\}$ is called an *edge-labeling* of $G$. Any such edge-labeling induces a map $f^+ : V(G) \to \mathbb{Z}_p$, defined by $f^+(v) = \Sigma f(uv)$ (mod $p$), where the sum is over all $uv \in E(G)$. If there exists an edge-labeling $f$ whose induced map $f^+$ is a bijection, we say that $f$ is a *$k$-edge-graceful labeling* of $G$ and that $G$ is *$k$-edge-graceful*. The set

$$\text{Egsp}(G) = \{k \in \mathbb{N} \cup \{0\} \mid G \text{ is } k\text{-edge-graceful}\}$$

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is called the *edge-graceful spectrum* of $G$.

In 1985, Lo [18] introduced the concept of an edge-graceful (ie, 1-edge-graceful) graph. Since then, a large body of research has emerged in the area of edge-graceful labelings of graphs [2,4–23,26]. Within this rich area of study, several open problems remain unresolved to this day. Examples of these include Lee's [8] conjecture that all trees of odd order are edge-graceful, as well as Lo's condition being a necessary and sufficient condition for a graph to be edge-graceful. The interested reader is directed to Gallian's [3] excellent survey of general labeling problems, as well as to Wallis' [25] monograph on magic labelings.

The concept of $k$-edge-graceful labelings was first introduced in 2004 by Lee, Chen and Wang [9]. In their paper, the edge-graceful spectra of two classes of bicyclic graphs, namely the dumbbell graphs and cycles with one chord, were analyzed. For these types of graphs, some open cases remained unresolved. Eventually, the edge-graceful spectra of cycles with one chord was completely determined by Shiu, Ling and Low [24]. In this paper, we completely determine the edge-graceful spectra for the class of connected bicyclic graphs without pendant, of which dumbbell graphs and cycles with one chord belong to.

## 2 A necessary condition

In [18], Lo gave a necessary condition for a graph $G$ to be 1-edge-graceful. This can be naturally extended to give the following result.

**Theorem 2.1.** *If $(p, q)$-graph $G$ is $k$-edge-graceful, then*

$$q(q + 2k - 1) \equiv \frac{p(p+1)}{2} \pmod{p}. \qquad (2.1)$$

We observe the following for a $k$-edge-graceful $(p, q)$-graph:

- If $p$ is odd, (2.1) is equivalent to $q(q + 2k - 1) \equiv 0 \pmod{p}$.
- If $p$ is even, (2.1) is equivalent to $q(q+2k-1) \equiv \frac{p^2-p}{2} \equiv \frac{-p}{2} \equiv \frac{p}{2}$ $\pmod{p}$.

**Corollary 2.2.** *If $(p,q)$-graph $G$ has a $k$-edge-graceful labeling, then* $p \equiv 0, 1,$ *or* $3 \pmod 4$.

*Proof.* Let $G$ be a $k$-edge-graceful graph and assume that $p \equiv 2 \pmod 4$. Now, set $p = 4l + 2$. Thus, we have that $q(q + 2k - 1) \equiv 2l + 1 \pmod p$. This implies that $q(q+2k-1)-(2l+1) \equiv 0 \pmod p$. Since $q(q+2k-1)$ is even and $2l+1$ is odd, $q(q+2k-1)-(2l+1)$ is odd. Thus, we have an odd number which is congruent to $0 \pmod p$, where $p$ is even. This is impossible and we reach a desired contradiction. □

# 3 Bicyclic graphs

A connected $(p, p+1)$-graph $G$ is called a *bicyclic* graph. For a bicyclic $(p, p+1)$-graph $G$, condition (2.1) and Corollary 2.2 imply the following:

- If $p$ is odd, then $\text{Egsp}(G) \subseteq \{k \in \mathbb{N} \cup \{0\} \mid k \equiv 0 \pmod p\} = \{sp \mid s = 0, 1, 2, \dots\}$.
- If $p = 4n$, then $\text{Egsp}(G) \subseteq \{k \in \mathbb{N} \cup \{0\} \mid k \equiv \frac{p}{4} \pmod{\frac{p}{2}}\} = \{sn \mid s = 1, 3, 5, \dots\}$.
- If $p \equiv 2 \pmod 4$, then $\text{Egsp}(G) = \varnothing$.

Furthermore, we note that $k \in \text{Egsp}(G)$ if and only if $k + p \in \text{Egsp}(G)$. Thus to find $\text{Egsp}(G)$, we only need to consider all the values of $k$ between $0$ and $p-1$ which satisfy condition (2.1). Hence for a $(p, p+1)$-graph $G$ where $p$ is odd, we only need to determine if $G$ is 0-edge-graceful. Similarly for $p = 4n$, we only need to determine if $G$ is $k$-edge-graceful for $k = n$, $3n$.

**Definition 1.** A vertex of degree $t$ is called a *$t$-vertex*. A vertex of degree greater than $t$ is called a *$t^+$-vertex*.

**Lemma 3.1.** *Let $G$ be a $(p, p+1)$-bicyclic graph without pendant. Then, the number of $2^+$-vertices in $G$ is at most two.*

*Proof.* It is known that $\sum_{v \in V(G)} d(v) = 2(p+1)$, where $d(v)$ denotes the degree of $v$. Let $x$ be the number of $2^+$-vertices in $G$. Then, $2(p-x) + 3x \leq 2(p+1)$. Hence, $x \leq 2$. □

**Definition 2.** A *one-point union of two cycles* is a simple graph obtained from two cycles, say $C_m$ and $C_n$ where $m, n \geq 3$, by identifying one vertex from each cycle. Without loss of generality, we may assume the $m$-cycle to be $u_0u_1 \cdots u_{m-1}u_0$ and the $n$-cycle to be $u_0u_mu_{m+1} \cdots u_{n+m-2}u_0$. We denote this graph by $U(m, n)$.

**Definition 3.** A *cycle with a long chord* is a simple graph obtained from an $m$-cycle, $m \geq 4$, by adding a chord of length $l$ where $l \geq 1$. Let the $m$-cycle be $u_0u_1 \cdots u_{m-1}u_0$. Without loss of generality, we may assume the chord joins $u_0$ with $u_i$, where $2 \leq i \leq m-2$. That is, $u_0u_mu_{m+1} \cdots u_{m+l-2}u_i$ is the chord. We denote this graph by $C_m(i; l)$.

**Definition 4.** A *long dumbbell graph* is a simple graph obtained from two cycles $C_m$ and $C_n$, by joining a path of length $l$ for $m, n \geq 3$ and $l \geq 1$. Without loss of generality, we may assume

$$C_m = u_0u_1 \cdots u_{m-1}u_0, \quad P_l = u_{m-1}u_m \cdots u_{m+l-1}$$

$$\text{and } C_n = u_{m+l-1}u_{m+l} \cdots u_{m+n+l-2}u_{m+l-1}.$$

We denote this graph by $D(m, n; l)$.

**Theorem 3.2.** *Let $G$ be a $(p, p+1)$-bicyclic graph without pendant. Then, $G$ contains only one $2^+$-vertex if and only if $G$ is a one-point union of two cycles.*

*Proof.* Suppose $G$ contains only one $2^+$-vertex. Let $d$ be the degree of the $2^+$-vertex. Since $2(p-1)+d = 2(p+1)$, $d = 4$. Since $G$ contains one 4-vertex and $(p-1)$ 2-vertices, $G$ is eulerian and contains two cycles. Hence, $G$ is a one-point union of two cycles. The converse is clear. □

**Theorem 3.3.** *Let $G$ be a $(p, p+1)$-bicyclic graph without pendant. Then, $G$ contains two $2^+$-vertices if and only if $G$ is either a long dumbbell graph or a cycle with a long chord.*

*Proof.* Suppose $G$ contains only two $2^+$-vertices. Let $d$ be the sum of the degrees of the $2^+$-vertices. Since $2(p-2)+d = 2(p+1)$, $d = 6$. Since the degree of the $2^+$-vertices is greater than 2, the two $2^+$-vertices must be 3-vertices. Since $G$ contains two 3-vertices and $(p-2)$ 2-vertices, $G$ is edge-traceable. The two 3-vertices are connected to each other by joining either one path or three disjoint paths. If the 3-vertices are connected by one path, then two disjoint cycles are incident with these 3-vertices respectively. Hence, $G$ is a long dumbbell graph. If the vertices are connected by three paths, then $G$ is a cycle with a long chord. The converse is clear. □

**Corollary 3.4.** *A bicyclic graph without pendant is either a one-point union of two cycles, a long dumbbell graph or a cycle with a long chord.*

**Lemma 3.5.** *Let $P$ be a path of odd order $p$. Then, any graph obtained from $P$ by adding two extra edges is 0-edge-graceful.*

*Proof.* If the edges of $P$ are labeled with $1, 2, \dots, p-1$ consecutively, then the induced labels on the vertices are $1, 3, 5, \dots, p-2, 0, 2, \dots, p-1$. The set of these labels is congruent to $\mathbb{Z}_p$. Since the induced labels of the vertices do not change after adding an edge labeled 0 or $p \equiv 0 \pmod{p}$, the graph obtained from an odd path by adding two edges is 0-edge-graceful. □

**Corollary 3.6.** *Let $G$ be a $(p, p+1)$-bicyclic graph without pendant, where $p$ is odd. Then, $Egsp(G) = \{sp \mid s = 0, 1, 2, \dots\}$.*

*Proof.* By Theorems 3.2 and 3.3, $G$ is either a one-point union of two cycles, a long dumbbell graph or a cycle with a long chord. Each of these graphs can be obtained from an odd path by adding two suitable edges. Thus by Lemma 3.5, $G$ is 0-edge-graceful and hence, $\mathrm{Egsp}(G) = \{sp \mid s = 0, 1, 2, \dots\}$. □

# 4 The edge-graceful spectra of $(4n, 4n+1)$-bicyclic graphs without pendant

**Definition 5.** *A tadpole graph* is a simple graph obtained from an $m$-cycle by attaching a path of length $l$, where $m \geq 3$ and $l \geq 1$. Let the $m$-cycle be $u_0u_1 \cdots u_{m-1}u_0$. Without loss of generality, we may assume the path is attached at $u_0$ and that the attached path is $u_0u_mu_{m+1} \cdots u_{m+l-1}$. We denote this graph by $T_{m,l}$.

For the upcoming discussion, we define two sets

$$Q_1 = \{a \in \mathbb{Z} \mid n \le a \le 5n\} \backslash \{4n\}$$

and

$$Q_2 = \{a \in \mathbb{Z} \mid 3n \le a \le 7n\} \backslash \{4n\}.$$

**Lemma 4.1.** *For $n \ge 1$, there is an edge-labeling $f : E(C_{4n}) \to Q_1$ such that the induced mapping $f^+ : V(C_{4n}) \to \mathbb{Z}_{4n}$ is a bijection.*

*Proof.* We use the labels $n$, $n+1$, $\dots$, $3n-1$ on the edges $u_0u_1, u_2u_3$, $\dots, u_{4n-2}u_{4n-1}$ respectively. Then, we use the labels $3n$, $3n+1$, $\dots$, $4n-1$ on the edges $u_{4n-1}u_0$, $u_1u_2$, $\dots$, $u_{2n-3}u_{2n-2}$ respectively. Finally, we use the labels $4n+1$, $4n+2$, $\dots$, $5n$ on the edges $u_{2n-1}u_{2n}$, $u_{2n+1}u_{2n+2}$, $\dots$, $u_{4n-3}u_{4n-2}$ respectively. Thus, we have

$$f(u_iu_{i+1}) = \begin{cases} n + \frac{i}{2} & i = 0, 2, \dots, 4n-2, \\ 3n + \frac{i+1}{2} & i = 1, 3, \dots, 2n-3, \\ 3n + 1 + \frac{i+1}{2} & i = 2n-1, 2n-3, \dots, 4n-3, \end{cases}$$

and

$$f(u_{4n-1}u_0) = 3n.$$

This yields the induced vertex labeling

$$f^+(u_i) \equiv \begin{cases} i & i = 0, 1, \dots, 2n-2, \\ i+1 & i = 2n-1, 2n, \dots, 4n-2, \\ 2n-1 & i = 4n-1. \end{cases} \pmod{4n}$$

Clearly, $f^+$ is a bijection. □

Figure 1 is an edge-labeling of $C_{16}$ which illustrates the proof of Lemma 4.1.

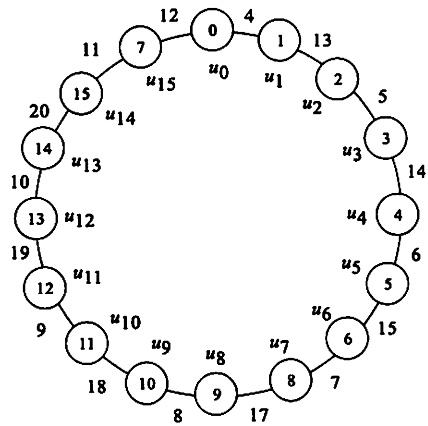


Figure 1: Labeling of $C_{16}$.

For Lemmas 4.2, 4.3, 4.4, 4.5, 4.7 and 4.8, we will continue to use the notation established in Lemma 4.1 and in its proof.

**Lemma 4.2.** *For* $3 \leq m \leq n$, *let* $T = T_{m,l} = C_{4n} - u_{2k+1}u_{2k+2} + u_{2k+1}u_{n+k}$ *where* $m + l = 4n$ *and* $k = m + n - 2$. *Then, there is an edge-labeling* $g : E(T) \to Q_1$ *such that the induced mapping* $g^+ : V(T) \to \mathbb{Z}_{4n}$ *is a bijection.*

*Proof.* We define $g(e) = f(e)$ for each $e \in E(C_{4n}) \cap E(T)$ and $g(u_{n+k}u_{2k+1}) = f(u_{2k+1}u_{2k+2})$. Only the three vertices $u_{2k+1}, u_{2k+2}$ and $u_{n+k}$ need to be considered since the induced labels of the other vertices are not changed.

$$g^+(u_{2k+2}) = f(u_{2k+2}u_{2k+3}) = n + k + 1 = f^+(u_{n+k})$$

$$\begin{aligned} g^+(u_{2k+1}) &= g(u_{n+k}u_{2k+1}) + f(u_{2k+1}u_{2k+2}) \\ &= f(u_{2k}u_{2k+1}) + f(u_{2k+1}u_{2k+2}) \\ &= f^+(u_{2k+1}) \end{aligned}$$

$$\begin{aligned} g^+(u_{n+k}) &= f^+(u_{n+k}) + g(u_{n+k}u_{2k+1}) \\ &= (n + k + 1) + (k - n + 2) = 2k + 3 \\ &= f^+(u_{2k+2}) \end{aligned}$$

Thus, the induced labels of vertices $u_{2k+2}$ and $u_{n+k}$ are swapped according to $f^+$. □

**Lemma 4.3.** *For $n \geq 2$ and $n+1 \leq m \leq 2n-1$, let $T = T_{m,l} = C_{4n} - u_{2k}u_{2k+1} + u_{2k}u_{3n+k}$ where $m + l = 4n$ and $k = m - n - 1$. Then, there is an edge-labeling $g : E(T) \to Q_1$ such that the induced mapping $g^+ : V(T) \to \mathbb{Z}_{4n}$ is a bijection.*

*Proof.* We define $g(e) = f(e)$ for each $e \in E(C_{4n}) \cap E(T)$ and $g(u_{2k}u_{3n+k}) = f(u_{2k}u_{2k+1})$. Only the three vertices $u_{2k}, u_{2k+1}$ and $u_{3n+k}$ need to be considered since the induced labels of the other vertices are not changed.

$$g^+(u_{2k+1}) = f(u_{2k+1}u_{2k+2}) = 3n + k + 1 = f^+(u_{3n+k})$$

$$\begin{aligned} g^+(u_{2k}) &= g(u_{3n+k}u_{2k}) + f(u_{2k-1}u_{2k}) \\ &= f(u_{2k-1}u_{2k}) + f(u_{2k}u_{2k+1}) \\ &= f^+(u_{2k}) \end{aligned}$$

$$\begin{aligned} g^+(u_{3n+k}) &= f^+(u_{3n+k}) + g(u_{3n+k}u_{2k}) \\ &= (3n + k + 1) + (n + k) \\ &\equiv 2k + 1 = f^+(u_{2k+1}) \pmod{4n} \end{aligned}$$

Thus, the induced labels of vertices $u_{2k+1}$ and $u_{3n+k}$ are swapped according to $f^+$. □

**Lemma 4.4.** *For $n \geq 2$ and $2n \leq m \leq 3n-1$, let $T = T_{m,l} = C_{4n} - u_{2k}u_{2k+1} + u_{2k+1}u_{3n+k-1}$ where $m + l = 4n$ and $k = 3n - m - 1$. Then, there is an edge-labeling $g : E(T) \to Q_1$ such that the induced mapping $g^+ : V(T) \to \mathbb{Z}_{4n}$ is a bijection.*

*Proof.* We define $g(e) = f(e)$ for each $e \in E(C_{4n}) \cap E(T)$ and $g(u_{2k+1}u_{3n+k-1}) = f(u_{2k}u_{2k+1})$. Only the three vertices $u_{2k}, u_{2k+1}$ and $u_{3n+k-1}$ need to be considered since the induced labels of the other vertices are not changed.

$$g^+(u_{2k}) = f(u_{2k-1}u_{2k}) = 3n + k = f^+(u_{3n+k-1})$$

$$\begin{aligned} g^+(u_{2k+1}) &= g(u_{3n+k-1}u_{2k+1}) + f(u_{2k+1}u_{2k+2}) \\ &= f(u_{2k}u_{2k+1}) + f(u_{2k+1}u_{2k+2}) \\ &= f^+(u_{2k+1}) \end{aligned}$$

$$
\begin{aligned}
g^+(u_{3n+k-1}) &= f^+(u_{3n+k-1}) + g(u_{3n+k-1}u_{2k+1}) \\
&= (3n+k) + (n+k) \\
&\equiv 2k = f^+(u_{2k}) \pmod{4n}
\end{aligned}
$$

Thus, the induced labels of vertices $u_{2k}$ and $u_{3n+k-1}$ are swapped according to $f^+$. □

**Lemma 4.5.** *For $n \geq 2$ and $3n \leq m \leq 4n-2$, let $T = T_{m,l} = C_{4n} - u_{2k+1}u_{2k+2} + u_{2k+2}u_{n+k-1}$ where $m+l = 4n$ and $k = 5n-m-2$. Then, there is an edge-labeling $g : E(T) \to Q_1$ such that the induced mapping $g^+ : V(T) \to \mathbb{Z}_{4n}$ is a bijection.*

*Proof.* We define $g(e) = f(e)$ for each $e \in E(C_{4n}) \cap E(T)$ and $g(u_{n+k-1}u_{2k+2}) = f(u_{2k+1}u_{2k+2})$. Only the three vertices $u_{2k+1}$, $u_{2k+2}$ and $u_{n+k-1}$ need to be considered since the induced labels of the other vertices are not changed.

$$
g^+(u_{2k+1}) = f(u_{2k}u_{2k+1}) = n + k = f^+(u_{n+k-1})
$$

$$
\begin{aligned}
g^+(u_{2k+2}) &= g(u_{n+k-1}u_{2k+2}) + f(u_{2k+2}u_{2k+3}) \\
&= f(u_{2k+1}u_{2k+2}) + f(u_{2k+2}u_{2k+3}) \\
&= f^+(u_{2k+2})
\end{aligned}
$$

$$
\begin{aligned}
g^+(u_{n+k-1}) &= f^+(u_{n+k-1}) + g(u_{n+k-1}u_{2k+2}) \\
&= (n+k) + (k-n+2) = 2k+2 \\
&= f^+(u_{2k+1})
\end{aligned}
$$

Thus, the induced labels of vertices $u_{2k+1}$ and $u_{n+k-1}$ are swapped according to $f^+$. □

Figure 2 gives edge-labelings of $T_{10,6}$, $T_{6,10}$ $T_{13,3}$ and $T_{3,13}$ which illustrate the proof of Lemmas 4.2 to 4.5.

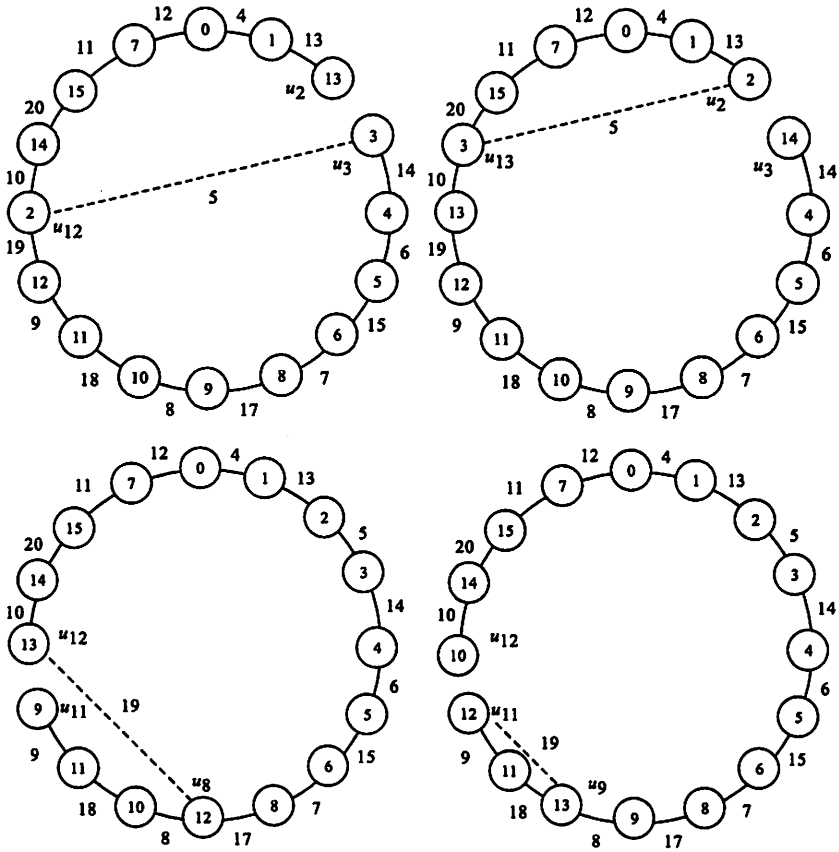


Figure 2: Labelings of $T_{10,6}$, $T_{6,10}$, $T_{13,3}$ and $T_{3,13}$.

Combining Lemmas 4.2 to 4.5, we have the following result. For $3 \leq m \leq 4n-2$, there is an edge-labeling $g : E(T_{m,l}) \to Q_1$ such that the induced mapping $g^+ : V(T_{m,l}) \to \mathbb{Z}_{4n}$ is a bijection, where $m + l = 4n$.

**Theorem 4.6.** *For $n \geq 2$ and $3 \leq m \leq 4n-2$, $T_{m,l} + e$ is $n$-edge-graceful, where $e$ is an extra edge and $m + l = 4n$.*

*Proof.* There is an edge-labeling $g : E(T_{m,l}) \to Q_1$ such that the induced mapping $g^+ : V(T_{m,l}) \to \mathbb{Z}_{4n}$ is a bijection. It is clear that the induced mapping does not change after adding the edge $e$ labeled by $0 \equiv 4n \pmod{4n}$. Thus, $T_{m,l} + e$ is $n$-edge-graceful. $\square$

**Lemma 4.7.** *For $n \geq 1$, there is an edge-labeling $\hat{f} : E(C_{4n}) \to Q_2$ such that the induced mapping $\hat{f}^+ : V(C_{4n}) \to \mathbb{Z}_{4n}$ is a bijection.*

*Proof.* Define $\hat{f}(e) = 8n - f(e)$ for each $e \in E(C_{4n})$. $\hat{f}^+(v) = 16n - f^+(v) \equiv 4n - f^+(v) \pmod{4n}$ for each 2-vertex $v \in V(C_{4n})$, and $\hat{f}^+(v) = 24n - f^+(v) \equiv 4n - f^+(v) \pmod{4n}$ for each 3-vertex $v \in V(C_{4n})$. Clearly, $\hat{f}^+$ is a bijection. □

Figure 3 gives an edge-labeling of $C_{16}$ which illustrates the proof of Lemma 4.7.

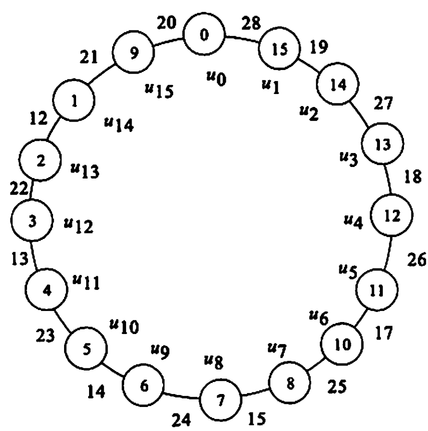


Figure 3: Labeling of $C_{16}$.

**Lemma 4.8.** *For $n \geq 2$ and $3 \leq m \leq 4n-2$, there is an edge-labeling $g : E(T_{m,l}) \to Q_2$ such that the induced mapping $g^+ : V(T_{m,l}) \to \mathbb{Z}_{4n}$ is a bijection, where $m + l = 4n$.*

*Proof.* The proof of this result uses Lemma 4.7 and is very similar to the proofs of Lemmas 4.2 to 4.5. □

Figure 4 gives edge-labelings of $T_{10,6}$, $T_{6,10}$, $T_{13,3}$ and $T_{3,13}$ which illustrate the proof of Lemma 4.8.

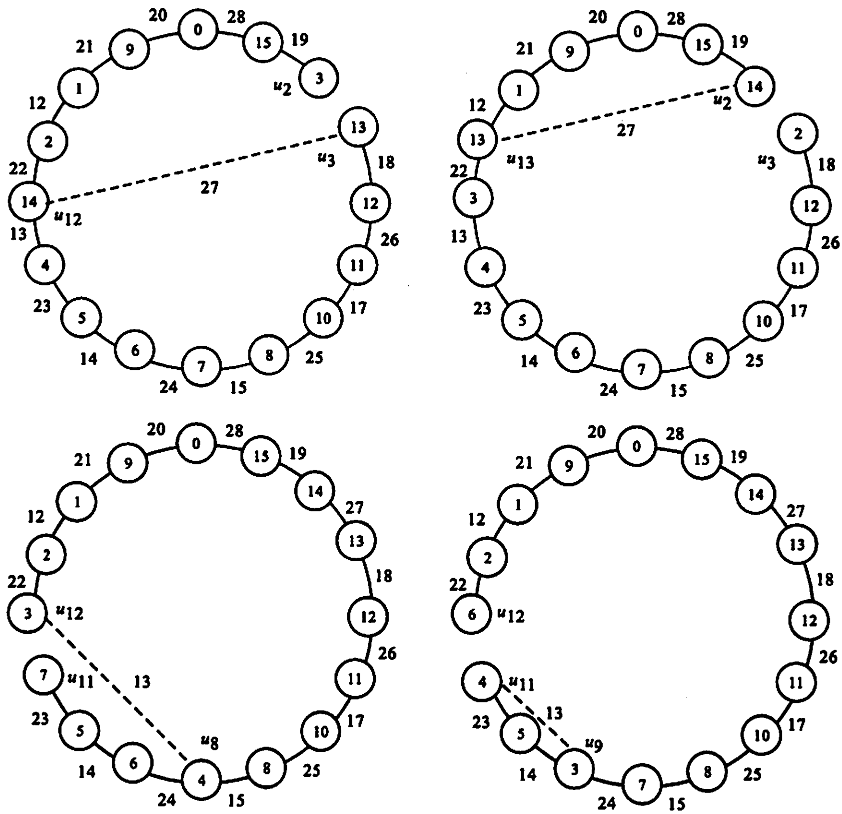

Figure 4: Labelings of $T_{10,6}$, $T_{6,10}$, $T_{13,3}$ and $T_{3,13}$.

**Theorem 4.9.** *For $n \geq 2$ and $3 \leq m \leq 4n-2$, $T_{m,l}+e$ is $3n$-edge-graceful, where $e$ is an extra edge and $m+l=4n$.*

*Proof.* Similar to the proof of Theorem 4.6, we obtain the result. □

**Corollary 4.10.** *For $n \geq 2$ and $3 \leq m \leq 4n-2$, $Egsp(T_{m,l}+e) = \{sn \mid s = 1,3,5,\dots\}$, where $m+l=4n$.*

*Proof.* This follows from Theorems 4.6 and 4.9. □

**Lemma 4.11.** *Let $n \geq 2$. Then, any bicyclic $(4n, 4n+1)$-graph $G$ without pendant can be constructed by adding an edge to a tadpole graph $T_{m,l}$, where $3 \leq m \leq 4n-2$.*

*Proof.* The one-point union of two cycles, $U(m,n)$, is isomorphic to $T_{m,n-1}+u_0u_{m+n-2}$. A cycle with a long chord, $C_m(i;l)$, is isomorphic to $T_{m,l-1}+u_iu_{m+l-2}$. Lastly, a long dumbbell graph, $D(m,n;l)$, is isomorphic to $T_{m,n+l-1}+u_{m+l-1}u_{m+n+l-2}$.

Furthermore, we do not have to consider the case where $m = 4n-1$. Suppose that $G$ contains a $C_{4n-1}$. Let $u$ be the vertex not in $C_{4n-1}$. Then, $\deg(u) = 2$. Let $x$ and $y$ be its neighbors, and $P$ be the shortest $x-y$ path in $C_{4n-1}$. Then, the length of $P$ is less than or equal to $\lfloor \frac{4n-1}{2} \rfloor$. Hence, there is a cycle $C$ of length $c$ less than or equal to $\lfloor \frac{4n-1}{2} \rfloor + 2$, which is less than or equal to $4n-2$. Thus in this case, $G$ can be constructed by adding an edge to $T_{c,l}$, where $c+l=4n$. □

**Theorem 4.12.** *For a bicyclic graph $G$ of order $4n$ without pendant, $Egsp(G) = \{sn \mid s = 1,3,5,\dots\}$.*

*Proof.* The case where $n = 1$ is established in [24]. For $n \geq 2$, the result follows immediately from Corollaries 3.4 and 4.10, and Lemma 4.11. □

Figure 5 gives 12-edge-graceful labelings of $U(10,7)$, $C_{10}(3;7)$ and $D(10,5;2)$ which illustrate the proof of Theorem 4.9

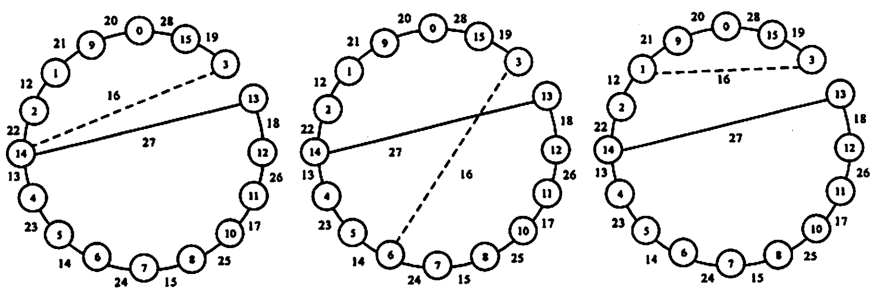

Figure 5: 12-edge-graceful labelings of $U(10,7)$, $C_{10}(3;7)$ and $D(10,5;2)$.

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