Judgment Aggregation And The Greedy Algorithm

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Abstract

The judgment aggregation problem is an extension of the group decision making problem, wherein each voter votes on a set of propositions which may be logically interrelated (such as p, $p \to q$, and q). The simple majority rule can yield an inconsistent set of results, so more complicated rules must be developed. Here the problem is cast in terms of matroids, and the Greedy Algorithm is modified to obtain a "best" result. An NP-completeness result is also presented for this particular formulation of the problem.

The fundamental political question of individuals making decisions as a group has come under increasing mathematical scrutiny in the past century as researchers have become aware of the incredible difficulties inherent in the problem. The most famous of such difficulties is Arrow's Theorem, which proves the nonexistence of voting rules satisfying certain fairness criteria (see [1] or the more elementary [6]). Ever since then, endless variations of this theorem have been created, exploring the limits of aggregating individual preferences into group preferences. Recently, a different problem has arisen, that of judgment aggregation, which is the aggregation of individual beliefs into group beliefs.

Judgment aggregation suffers from many of the same problems as preference aggregation, but it is more general in scope. Indeed, preference aggregation is merely the problem of the aggregation of individual beliefs about a certain order relation (see [4]). Work by List and Dietrich has highlighted the various relationships between fairness criteria one might impose on an aggregation rule, including several impossibility and possibility theorems (see, for example, [2, 3, 5]).

In the present article, the author intends to add to this literature by developing a new aggregation rule based on the idea of maximizing group

JCMCC **66** (2008), pp. 187-194

agreement with the final decision. The details of the problem suggest the use of the Greedy Algorithm, which turns out to be insufficient; however, ideas from matroid theory yield an algorithm which selects the desired aggregate. Unfortunately, far from being as simple as independent set maximization for matroids, the maximization problem used in this aggregation rule is NP-complete.

1 The Discursive Paradox

It is natural to suggest that when a group must make decisions over several propositions, they should vote on each proposition and select those propositions for which more than half of the voters have voted. This is the Majority Rule: believe about each fact what the majority of voters believe. As long as the propositions are independent of each other, this works fine. However, when the propositions are interdependent, this can lead to difficulties. Consider the following example, a case of the discursive paradox.

Suppose a committee of three people (Dr. A, Dr. B, and Dr. C) are deciding on an important matter of department business. They are voting on three propositions

p : "We will have a colloquium series."
q : "We need a cookie budget."
$p \rightarrow q$: "If we have a colloquium series, then we need a cookie budget."

The difficulty arises in the very different belief systems of the participants. In the table below we see each of the participants' beliefs and the majority aggregate.

Proposition:	p	$p \rightarrow q$	q
Dr. A	Yes	No	No
Dr. B	No	Yes	No
Dr. C	Yes	Yes	Yes
Majority	Yes	Yes	No

As a consequence, we see that the Majority Rule cannot be used to collect the beliefs together into a single consistent choice.

One might object that it is simplest to vote for only the atomic propositions p and q. This is a valid approach to resolving the paradox, perhaps even the best approach to this particular matter of department business. However, in other decisions which are formally identical to this, deciding on, for example, p and $p \rightarrow q$ may be preferable. Besides which, not making use of the beliefs about $p \rightarrow q$ ignores the reasons the participants have for believing in the particular way they do. (See List and Dietrich on premise-based and conclusion-based aggregation rules in [3].)

2 Formal Description of the Problem

Now, let us set down formal definitions for the main elements of the problem. This will allow us to see more clearly the structure inherent in the problem. We have the following:

A set of **players** $N = \{1, 2 \dots , n\}$.

An **agenda** X, consisting of propositions and their negations.

The **judgment sets** $A_1, \dots , A_n$ of the players. We assume these are consistent and complete (hence $|A_i| = |X|/2)$.

An **aggregation rule** $F(A_1, \dots , A_n)$, which generates a consistent and complete **collective judgment set**

The aggregation rule which we will develop in this article will actually be a *probabilistic* aggregation rule, in that the result of the rule is a probability distribution over all consistent and complete judgment sets rather than a specific judgment set. A theory of probabilistic aggregation rules has yet to be properly developed. Nevertheless, for the purpose of this article, it is enough to require that an aggregation rule give a definite—if not deterministic—result every time it is applied.

3 A Class of Aggregation Rules

A significant portion of the literature on collective decision making has dealt with the so-called (im)possibility theorems. These theorems are of the form "The only aggregation rule which satisfies a certain list of properties is a rule of the form such-and-such" or "No aggregation rule can satisfy all of a certain list of properties." We will not prove any such theorem here, but we should make note of some of the relevant properties.

An aggregation rule is **anonymous** if it is invariant under permutation of judgment sets. That is, it does not take into account which player holds which judgment set but only the number of times each judgment set is chosen. We will be interested in anonymous aggregation rules, specifically those which arise from the weighting function

$$w(p) = |\{i : p \in A_i\}|.$$

Such aggregation rules take into account only how frequently a particular proposition p is chosen.

Not every anonymous aggregation rule can be defined by one that depends only on w. For instance, the rule which selects the most popular judgment set (or chooses randomly from the favorites, if there is a tie) cannot be reduced to a function of w. Consider, for instance, the following situation arising from two elections with 41 voters but different beliefs:

Case One	Case Two	a	$a \to c$	b	$b \to c$	c
6 votes	5 votes	Yes	Yes	Yes	Yes	Yes
8 votes	5 votes	Yes	No	Yes	No	No
4 votes	5 votes	Yes	Yes	No	Yes	Yes
2 votes	5 votes	Yes	No	No	Yes	No
4 votes	5 votes	No	Yes	Yes	Yes	Yes
2 votes	5 votes	No	Yes	Yes	No	No
7 votes	6 votes	No	Yes	No	Yes	Yes
8 votes	5 votes	No	Yes	No	Yes	No

In both cases, the weighting function is $w(a) = w(b) = 20$, $w(c) = 21$, $w(a \to c) = w(b \to c) = 31$. However, choosing the most popular judgment set gives a random choice between $\{a, b, \neg c, \neg(a \to c), \neg(b \to c)\}$ and $\{\neg a, \neg b, \neg c, a \to c, b \to c\}$ in the first case, but in the second case, the choice is $\{\neg a, \neg b, c, a \to c, b \to c\}$. That is, this anonymous aggregation rule is not a function of w.

The aggregation method studied here arises from the consideration of the disenfranchisement of each player, the number of propositions wherein the player disagrees with the aggregate. We desire to minimize the total disenfranchisement of the players. To do so, we maximize the total "enfranchisement", i.e. the sum of $w(p)$ for each p in a judgment set. A maximizer of this sum is a judgment set with largest total weight, which we call a **maximum judgment set** (MJS).

To the combinatorist, the problem of finding an MJS resembles the problem of finding a maximum independent set in a matroid. The natural technique to try is the Greedy Algorithm. The algorithm works as usual: choose the proposition with the most votes; choose the next-largest if it is consistent with what you have already; repeat until you have exhausted the list. Break ties randomly. The bad news is that it doesn't work. Consider the following example:

	a	$a \to c$	b	$b \to c$	c
3 votes	Yes	Yes	Yes	Yes	Yes
3 votes	No	Yes	No	Yes	No
3 votes	Yes	No	Yes	No	No
1 vote	No	Yes	No	Yes	Yes
for	6	7	6	7	4
against	4	3	4	3	6

Begin by choosing $a \rightarrow c$ (7 votes), and $b \rightarrow c$ (7 votes). Next, randomly select one of a, b, or $\neg c$ (6 votes). Suppose we choose $\neg c$. Then, we must choose $\neg a$ and $\neg b$ last (each 4 votes). The total weight is 28 votes. However, the set $a, b, c, a \rightarrow c, b \rightarrow c$ has weight 30 votes.

4 Matroid Embedding

The Greedy Algorithm fails to solve the MJS problem, which means that the structure of the problem is not matroidal. It is similar, though: the set $\mathcal{C}$ of consistent subsets of X is hereditary. It has a little extra structure, as well.

There is a fixed-point-free involution $n : X \rightarrow X$, specifically proposition negation. The involution is enough to give an embedding of the problem into a partition matroid (see below), which allows us to use the theory of matroids to find a solution. Furthermore, the weighting function satisfies a constant sum rule:

$$w + w \circ n = \text{constant}.$$

Although unnecessary for the solution of the MJS problem, I make note of one additional structural fact of interest. The operation of logical completion is a a closure operation $\mathcal{C} \rightarrow \mathcal{C}$, which enriches the structure under study.

The judgment aggregation structure $(X, \mathcal{C})$ is a substructure of a partition matroid $(X, \mathcal{I})$. A set $I \in \mathcal{I}$ is independent if I contains at most one element of each proposition-negation pair $\{p, \neg p\}$ in X. Note that $\mathcal{C} \subseteq \mathcal{I}$, and complete judgment sets are bases of the partition matroid. Thus, to find an MJS, we can start by finding maximum bases in $(X, \mathcal{I})$. These are the results of the Majority Rule, as one simply chooses the heavier-weighted proposition from each proposition-negation pair.

Since the Majority Rule does not always yield an MJS, we must find a way to work from the Majority Rule to an MJS. To do this, construct a weighted directed graph as follows. The nodes will be all the bases of $(X, \mathcal{I})$, the partition matroid. Direct an edge from a basis B_1 with weight w_1 to a basis B_2 with weight w_2 if B_1 and B_2 differ by exactly one element and $w_2 \leq w_1$. Weight the edge by the difference $w_1 - w_2$. The edges should go "downhill" from higher-weighted bases to lower-weighted bases, trading one proposition at a time for its negation. (This downhill direction is a consequence of the constant sum rule relating the weight and negation.) The undirected version of the graph is called the **matroid basis graph** of the matroid, and in this case it is an n-cube. The Majority Rule choice or choices can be found at the "top" of this graph, and the negation of those choices are at the "bottom". The MJS is the consis tent basis that is closest to the Majority Rule nodes. One can use Dijkstra's Algorithm to find it.

5 An Algorithm

To summarize, the following steps yield an MJS. Remember to break ties randomly.

1. Invoke the Greedy Algorithm on the partition matroid.
2. Construct matroid basis graph, directed from higher-weighted to lower-weighted bases.
3. Weight the edges according to the difference between basis weights.
4. Invoke Dijkstra's Algorithm to find a shortest path from the Majority point to a basis in C.
5. This nearest element is an MJS.

In the case of the discursive paradox at the beginning of this article, this method gives a solution by randomly choosing one of the three chosen judgment sets, as each of the sets has weight five, hence each set is equidistant from the weight six Majority Rule node.

6 NP-Completeness

A criticism of the algorithm of the previous section is that it is exponential in the size of the agenda: the graph constructed has 2^n vertices whenever the agenda has n propositions. However, finding an MJS in a judgment aggregation structure with the constant sum requirement $w + w \circ n = const$ is an NP-complete problem. Finding a polynomial-time algorithm for the solution may be difficult, at best.

(Note: finding an MJS for the judgment aggregation problem—i.e., given the players' judgment sets, not just the weighting function—might not be NP-complete: the problem transformation given below does not seem to correspond to a situation beginning from judgment sets.)

Given a simple graph G with vertices $v_1, \dots, v_n$, create an agenda

$$X = \{p_i, \neg p_i | i = 1, \dots n\} \cup \{p_i \wedge p_j, \neg(p_i \wedge p_j) | 1 \leq i < j \leq n\}.$$

The atomic proposition p_i represents the vertex v_i in G, and the conjunctive proposition $p_i \wedge p_j$ represents the assertion that v_i and v_j are adjacent. So, a complete (and consistent) judgment set for this agenda consists of a choice of vertices and all of the edges between them, whether or not those edges actually exist.

Now, weight the propositions as follows. The (atomic) propositions will be given weight zero: $w(p_i) = w(\neg p_i) = 0$. The conjunctive propositions will be given weights as follows. If v_i and v_j are adjacent in G, then weight $w(p_i \wedge p_j) = 1$ and $w(\neg(p_i \wedge p_j)) = -1$. If v_i and v_j are not adjacent in G, then weight $w(p_i \wedge p_j) = -n$ and $w(\neg(p_i \wedge p_j)) = n$.

Constructing the agenda and weighting function from the graph can be done in polynomial time. Proof of the following proposition will complete the proof that the MJS problem is NP-complete.

Proposition 6.1. *An MJS for this system corresponds to a maximum clique.*

Proof. Consider an MJS, supposing the vertices and propositions are numbered so that $p_1, \dots, p_k$ are the atomic propositions selected, and $p_{k+1}, \dots, p_n$ are not selected. There are two cases: either $v_1, \dots, v_k$ form a clique, or they do not.

If they do not, then, by relabeling if necessary, we may assume v_{k-1} and v_k are not adjacent. Removing p_k from the MJS, the score decreases by two points for each edge between v_k and another vertex of $v_1, \dots, v_{k-1}$ but increases by $2n$ for each vertex of $v_1, \dots, v_{k-1}$ not adjacent to v_k. Since v_k is not adjacent to v_{k-1}, the score increases by at least $2n - 2(k-1)$. Consequently, the judgment set with $\neg p_k$ rather than p_k has a higher score than the given MJS. This is contradictory, so the MJS must correspond to a clique.

The vertices $v_1, \dots, v_k$ form a clique, which is either a maximum clique, or some vertex v_i, $(i > k)$ is adjacent to each of $v_1, \dots, v_k$. Suppose v_i is adjacent to each of $v_1, \dots, v_k$. Adding p_i to the MJS, the score increases by two points for each of the edges connecting v_i to the original clique, but the score does not decrease. So, this new judgment set has a higher score than the given MJS. This is contradictory, so the MJS must correspond to a maximum clique.

□

7 Conclusion

When making a collective decision over multiple connected propositions, the natural Majority Rule fails in fairly simple cases. We have explored the possibility of trying to minimize disenfranchisement by choosing the maximum judgment set. Considerations of matroid theory yield a straightforward algorithm for finding such a judgment set. However, the algorithm is probabilistic, which could be disconcerting for the participants in the decision-making process. Furthermore, from a theoretical standpoint, probabilistic decision rules are not well-understood. Nevertheless, this method has the benefit of two desirable properties: symmetry and the selection of a broadly-supported decision which is as close to the Majority Rule as possible.

References

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