On the Edge-graceful Spectra of the Cylinder Graphs (I)

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Abstract. Let G be a (p,q)-graph and $k \geq 0$. A graph G is said to be ***k-edge-graceful*** if the edges can be labeled by k,k+1,...,k+q-1 so that the vertex sums are distinct, modulo p. We denote the set of all k such that G is k-edge graceful by egS(G). The set is called the **edge-graceful spectrum** of G. In this paper, we are concerned with the problem of exhibiting sets of natural numbers which are the edge-graceful spectra of the cylinder $C_n x P_m$ for certain values of n and m.

1. Introduction.

Given an integer $k \in N = \{1,2,3,\ldots\}$, a graph G = (V, E) with p vertices and q edges is said to be k- edge-graceful if there is a bijection

$$f : E \rightarrow \{k,k+1, k+2,..., k+q-1\}$$

such that the induced mapping $f^+: V \rightarrow Z_p$, given by

$$f^+(u) = \Sigma\{f(u,v): (u,v) \text{ in } E\} \pmod p$$

is a bijection.

Theorem 1.1 (Lo's condition [18]). If a (p,q)-graph G is k-edge-graceful, then it satisfies the condition

$$q(q+2k-1) \equiv \frac{p(p-1)}{2} \pmod{p}.$$

Graphs can be 1–edge-graceful but may not be k-edge-graceful for some k>1, and vice versa.

Example 1. Figure 1 shows that K_4 is k-edge graceful for all $k \geq 1$.

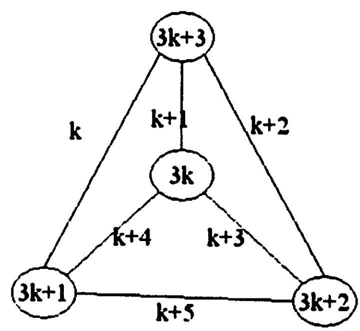


Figure 1.

Example 2. We see in Figure 2 two trees of order 4 which are 2-edge graceful but not 1-edge-graceful.

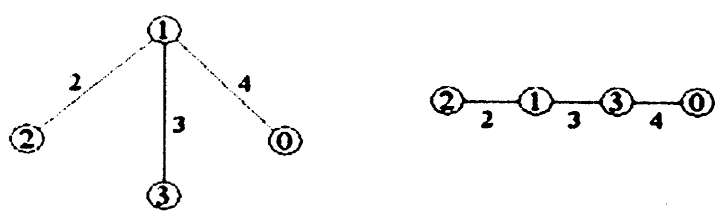


Figure 2.

The set of all integers k≥0 such that G is k-edge-graceful is denoted by egS(G) and is called the **edge-graceful spectrum** of G.

Supposing that G is d-regular graph with m vertices, and H is k-regular graph with n vertices such that GCD(d,n)=GCD(k,m) =1, and assuming that G and H are both odd-order and 1-edge-graceful graphs, Schaffer and the first author [22] showed that GxH is 1-edge-graceful. In particular, they showed that the Cartesian product of two cycles of odd order is 1-edge-graceful.

A cylinder graph $C_n \times P_m$ is the Cartesian product of the cycle C_n and the path P_m. In this paper we are concerned with the problem of exhibiting sets of natural numbers which are the edge-graceful spectra of cylinder graphs $C_n \times P_m$.

1-edge-graceful graphs are investigated in [1,3,4,5,6,7,8,9,10,11,12,13, 14,18,19,20,21,22,23,24]. Some k-edge graceful graphs are considered in

[16,17]. A good account on other graph labeling problems can be found in the dynamic survey of Gallian [2].

2. k-Edge-graceful cylinder graphs

The cylinder graph $C_n \times P_m$ is a (p, q) – graph with p = nm vertices and $q=2nm - n$ edges. Suppose that $C_n \times P_m$ is k-edge graceful. Then Lo's condition reads (2nm – n)(2nm – n + 2k-1) ≡ nm(nm-1) 2 (mod nm), and is equivalent to:

Condition 2.1: n^2-2nk+n ≡ nm(nm-1) / 2 (mod nm).
(i) Suppose that n is even and m is odd. Then condition 2.1 becomes:

Condition 2.2: nm | (n^2-2nk+n-(n/2)m(nm) + (n/2)m), hence nm | (n(n-2k+1) +m(n/2)) and hence m | n-2k+1 +m/2. This implies m is even, a contradiction.

(ii) Suppose that n and m are both odd. Then nm-1 is an even integer and condition 2.1 becomes nm|(n^2 - 2nk+n), m|(n-2k+1). Since n+1-2k is even, this means that n-2k+1=(2t)m for some integer t, n=2tm+2k-1 for some t.

Conversely, let n=2tm+2k-1 for some t. Then n-2k+1=2tm, n-2k+1≡ 0(mod m), hence n(n-2k+1) ≡ 0 (mod nm). Now nm((nm-1)/2) ≡ 0 (mod nm). Therefore, n(n-2k+1) ≡ nm((nm-1)/2)(mod nm), and condition 2.1 is satisfied.

(iii) Suppose that n and m are both even. So nm -1 is odd. Then condition 2.1 becomes condition 2.2, from which we deduce nm | (n^2 -2nk + n + n(m/2), i.e. m | (n-2k+1+m/2).

Since (m/2) | m, we must have (m/2) | (n-2k+1); moreover, as 2|m, we must also have 2|(n-2k+1+m/2), i.e., 2|(1+m/2), i.e., m/2 is odd. This boils down to saying that there exists an odd integer s such that n-2k+1 = s(m/2) with m/2 odd, i.e., there exists t such that n = (2t-1)(m/2)+2k-1 with m/2 odd.

Conversely, suppose n = (2t-1)(m/2)+2k-1 for some t. Then n-2k+1+m/2 = tm ≡ 0 (mod m), so n^2 – 2kn + n + nm/2 ≡ 0 ≡ $n^2m^2/2$ (mod nm). Therefore, condition 2.1 is satisfied.

(iv) Suppose that n is odd and m is even. Then condition 2.1 becomes n^2-2nk+n ≡ -(m/2)n (mod nm), hence m|(n-2k+1+m/2).

Here n+1-2k is even. if it happens that m/2 is odd, we have that n+1-2k+m/2 is odd. This is not possible since the even integer m cannot divide an odd number. Therefore 4|m. For m|(n+1-2k+m/2), we deduce that (m/2) | (n+1-2k) and 2|((n+1-2k)/(m/2)+1), whereupon (n+1-2k)/(m/2) is odd, so there exists an integer t such that n+1-2k=(m/2)(2t-1), i.e.

n = (m/2)(2t-1) + 2k-1 with 4|m.

Conversely, suppose that $4|m$ and $n=(m/2)(2t-1)+2k-1$ for some t; then condition 2.1 holds true, the proof being the same as in part (iii).

From all three remarks, we deduce the following results. Let us start with describing the integer n for which condition 2.1 holds true.

Theorem 2.1. Condition 2.1 is verified if and only if there exists an integer n such that for some integer t, we have

$n = 2tm+2k-1$ with m odd and n odd

$n = (2t-1)(m/2)+2k-1$ with $m \equiv 2 \pmod 4$ and n even; or with $m \equiv 0 \pmod 4$ and n odd.

We can now describe the integers m for which condition 2.1 holds true.

Theorem 2.2. Condition 2.1 is verified if and only if either there exists an odd integer m such that $m \mid ((n+1)/2 - k)$ for odd n, or there exists an even integer m such that $(m/2) \mid (n+1-2k)$ for odd n when $4|m$, or for even n when $4 \nmid m$.

We can also describe the integer k for which condition 2.1 holds true.

Theorem 2.3. Condition 2.1 is verified if and only if there exists an integer k such that for some integer t, we have

$k = (n+1)/2 - tm$, for m odd and n odd,

$k = ((n+1) - (2t-1)(m/2))/2$, for m even with $4 \nmid m$, and n even,

$k = (n+1)/2 - (2t-1)(m/4)$, for m even with $4|m$, and n odd

Remark that in Theorem 2.3 we could have written

$k = (n+1)/2 - tm$, for m odd and n odd,

$k = (n+1 - (2t-1)(m/2))/2$, for m even,

since the parity of m/2 dictates the parity of n.

From the preceding four remarks, we deduce the following results.

Theorem 2.4. (A) The cylinder graph C_nxP_m is not k-edge-graceful if at least one of the following conditions is satisfied:

(a) n even and m odd;

(b) n and m both odd and $(2m) \nmid (n+1-2k)$;

(c) n and m both even and either $4 \mid m$ or $(m/2) \nmid (n+1-2k)$;

(d) n odd and m even and either $4 \nmid m$ or $(m/2) \nmid (n+1-2k)$.

(B) If the cylinder graph C_nxP_m is k-edge-graceful, then at least one of the following conditions is satisfied:

(i) m is odd with $n = 2tm + 2k - 1$ for some t;

(ii) m is even with $n = (2t-1)(m/2) + 2k - 1$ for some t.

Particular cases of the preceding theorem are the next two theorems.

Theorem 2.5. For all odd $n \geq 3$ the edge-graceful spectrum of the cylinder graph C_nxP_m is empty for $m \equiv 2 \pmod 4$.

Theorem 2.6. The edge-graceful spectrum of the cylinder graph C_nxP_m is empty for n even and m ≡ 0,1, or 3 (mod 4).

Theorem 2.1, 2.2 and 2.3 are powerful results because they provide necessary and sufficient conditions for Lo's condition to be satisfied. Unfortunately, Lo's condition is not (so far) an equivalent necessary condition for a (p, q)-graph to be k-edge-graceful.

3. Some k-edge-graceful cylinder graphs

In the following table we list for a given pair (n,m) the values of k for C_nxP_m to satisfy Lo's condition, namely Condition 2.1.

k \ n, m	3	4	5	6	7	8	9	10	11	12
2	-	N	-	N	-	N	-	N	-	N
3	3t+2	-	3t+3	-	3t+1	-	3t+2	-	3t+3	-
4	2t+1	-	2t+2	-	2t+1	-	2t+2	-	2t+1	-
5	5t+2	-	5t+3	-	5t+4	-	5t+5	-	5t+1	-
6	-	3t+1	-	3t+2	-	3t+3	-	3t+1	-	3t+2
7	7t+2	-	7t+3	-	7t+4	-	7t+5	-	7t+6	-
8	4t+4	-	4t+1	-	4t+2	-	4t+3	-	4t+4	-
9	9t+2	-	9t+3	-	9t+4	-	9t+5	-	9t+6	-
10	-	5t+5	-	5t+1	-	5t+2	-	5t+3	-	5t+4

The table above may be summarized and extended in the following chart, for n odd or even, and m congruent to either 2,3,0, or 1, mod 4. Note that this table applies when m = 2, if we take "mod m/2" in this case to indicate that k may be any positive integer in N. When m is odd, the representation of k follows directly from Theorem 2.3.

When m is even and divisible by 4, letting m = 4s for some integer s, Theorem 2.3 implies that

$$k = (1/2)(n+1) - (2t-1)m/4 = (n+1)/2 - t(m/2) + s \equiv (n+1)/2 + s \pmod{m/2}$$

When m is even and not divisible by 4, it is of the form m = 4s + 2, for some integer s, and n must be even. Then

$$k = 1/2((n+1) - (2t-1)m/2) = n/2 - t(m/2) + (2 + m)/4 = n/2 - t(m/2) + s + 1$$
$$\equiv n/2 + s + 1 \pmod{m/2}$$

k \ n, m	n odd	n even
m = 4s + 2	-	$k \equiv n/2 + s + 1 \pmod{m/2}$
m = 4s + 3	$k \equiv (n+1)/2 \pmod{m}$	-
m=4s	$k \equiv (n+1)/2 + s \pmod{m/2}$	-
m = 4s + 1	$k \equiv (n+1)/2 \pmod{m}$	-

The formulation using s in this chart allows us, given m and n, to arrive quickly at the smallest positive value of k to check in searching for a k-edge-graceful labeling of $C_n x P_m$.

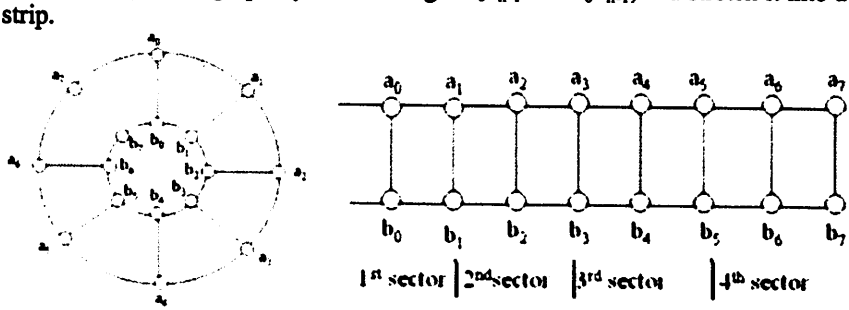
It is interesting to note that from Theorem 2.3, we see that for any even n ≥ 4, Condition 2.1 is satisfied when m =2 (that is m_1 =1). In 1990, the first author and Eric Seah [10] showed that C_{2t} x P_2 is 1-edge-graceful.

We show here C_{2t} x P_2 is in fact k-edge-graceful for any k.

Theorem 3.1. The edge-graceful spectrum of C_n x P_2 is N for all even n ≥ 4.

Proof. The cylinder graph C_n x P_2 has p =2n vertices and q =3n edges.

Assume n =2t. As seen in Figure 3 (for simplicity, we have n=8, i.e. t=4), we cut the cylinder graph by the two edges a_0a_{n-1} and b_0b_{n-1}, and stretch it into a strip.

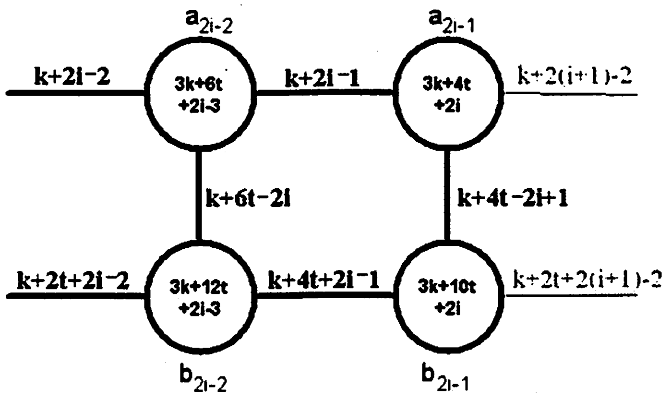


Cylinder graph C_8xP_2 C_8xP_2 stretched onto a strip

Figure 3.

We further divide the strip into t sectors, each of which consists of the 6 bold edges (see Figure 4).

We want to label the edges with {k,k+1,k+2,....,k+6t-1}. The i-th sector has the edges (a_{2i-3},a_{2i-2}),(a_{2i-2},a_{2i-1}),(a_{2i-2},b_{2i-2}),(a_{2i-1},b_{2i-1}),(b_{2i-3},b_{2i-2}), and (b_{2i-2},b_{2i-1}), which we label as in Figure 4.



k-edge graceful labeling for the i-th sector

Figure 4.

It is not difficult to see that the induced vertex labels can be classified into 4 sets A,B,C,D:

$A = \{f^+(a_{2i-2}): i=1,2,\dots,t\} = \{3k+6t+2i-3: i=1,2,\dots,t\} \pmod{4t}$,

$B = \{f^+(a_{2i-1}): i=1,2,\dots,t\} = \{3k+4t+2i: i=1,2,\dots,t-1\} \pmod{4t} \cup \{f^+(a_{n-1}) =3k\}$,

$C = \{f^+(b_{2i-2}): i=1,2,\dots,t\} = \{3k+12t+2i-3: i=1,2,\dots,t\} \pmod{4t}$,

$D = \{f^+(b_{2i-1}): i=1,2,\dots,t\} = \{3k+10t+2i: i=1,2,\dots,t-1\} \pmod{4t} \cup \{f^+(b_{n-1}) =3k+n\}$.

The union of these four sets is $\{3k+2t+2i-3\} \cup \{3k+2i-2\} \cup \{3k+2i-3\} \cup \{3k+2t+2i-2\}$ for $i = 1, 2, \dots, t \pmod{4t}$.

Now modulo 4t, we have

$C\cup B= \{3k-1,3k,3k+1,\dots, 3k+2t-3,3k+2t-2\}$

$A\cup D= \{3k+2t-1, 3k+2t, 3k+2t+1,\dots, 3k+4t-3,3k+4t-2\}$

Since $3k-1 \equiv 3k+4t-1 \pmod{4t}$, we conclude $A\cup D \cup C\cup B$ is the set $\{0,1,\dots,4t-1\} \pmod{4t}$.

Thus f is a k-edge-graceful labeling.□

Remark. Suppose the cylinder $C_n \times P_2$ with 2n vertices and 3n edges is 1-edge-graceful. Then modulo 3n, the union of the sets

$\{ f^+(a_i) : 1 \le i \le n\} \cup \{ f^+(b_i) : 1 \le i \le n\}$

is the set $\{0,1,2,\dots,3n-1\}$. If each edge label is increased by k for any fixed $k \ge 0$, then each value of $f^+(a_i)$ (respectively $f^+(b_i)$) is increased by 3k. Therefore modulo 3n, the previous set $\{0,1,2,\dots,3n-1\}$ becomes $\{3k,3k+1,\dots,3k+3n-1\}$, remains $\{0,1,2,\dots,3n-1\}$. Therefore $C_n \times P_2$ is k-edge-graceful.

We illustrate the above result for $C_4 \times P_2$.

Example 3. Figure 5 shows that $C_4 \times P_2$ is 1-edge-graceful. If we add a fixed integer $k > 0$, to each edge label, we have that $C_4 \times P_2$ is k-edge-graceful.

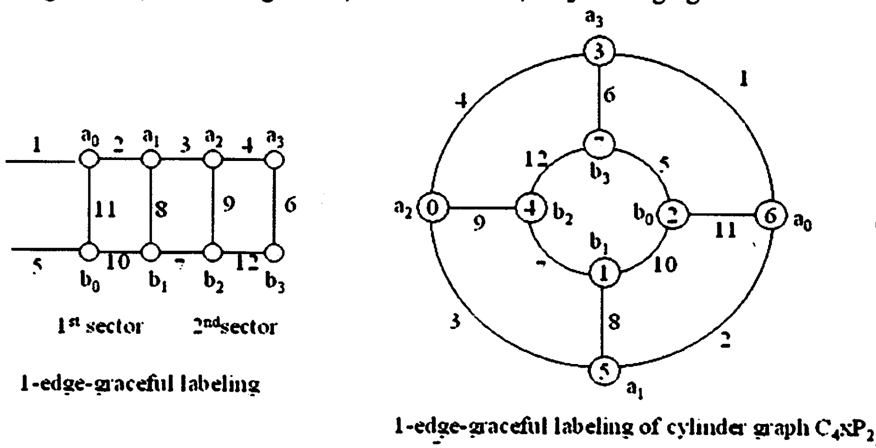


Figure 5.

Example 4. Figure 6 shows that $C_8 \times P_2$ is 1-edge-graceful. If each edge label is increased by k, we have that $C_8 \times P_2$ is k-edge-graceful.

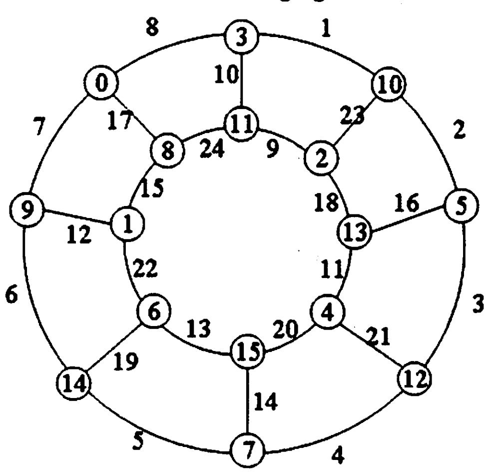


Figure 6.

4. Cylinder graphs $C_n \times P_m$, $m = 3$.

By Theorem 2.3 and Theorem 2.4 (B), if $C_n x P_m$ is k-edge-graceful with m odd, then n is odd and $k \equiv (n+1)/2 \pmod{m}$. The whole section is devoted to the proof of the following theorem.

Theorem 4.1. The cylinder graph C_nxP_3 is k-edge-graceful if and only if n is odd and $k \equiv (s+1)/2 \pmod 3$, where $s \in \{1, 3, 5\}$ verifies $s \equiv n \pmod 6$.
Proof. The graph C_nxP_3 has 3n vertices and 5n edges. For the part (=>), since n is odd, we have $n = 6t+s$ for some $s \in \{1, 3, 5\}$. Hence $k \equiv (n+1)/2 \equiv (s+1)/2 \pmod 3$.

Let us prove now the part (<=). We want to exhibit a k-edge-graceful labeling when $n = 6t+1$ (respectively, $n = 6t+3$, $n=6t+5$) with $k \equiv 1 \pmod 3$ (respectively, $k \equiv 2 \pmod 3$, $k \equiv 3 \pmod 3$).

First case: Consider $n = 6t+1$ and $k =1$. Example 4 exhibits a 1-edge-graceful labeling of $C_7x P_3$.

Example 4. Figure 7 shows that $C_7 x P_3$ is 1-edge- graceful.

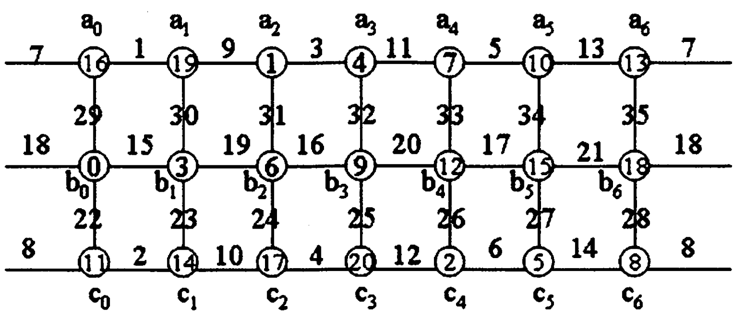


Figure 7.

The exhibited 1-edge-graceful labeling of C_7xP_3 is a particular case of the labeling of Figure 8 showing a 1-edge-graceful labeling of C_nxP_3 where n=6t+1. Note that the labels will not be written modulo 5n in order to make sure that we indeed see a bijection between the edges and the labels. However, it will not hurt to write the values of the vertex sum $f^+(u)$ modulo 3n. This policy will be adopted in the rest of the paper.

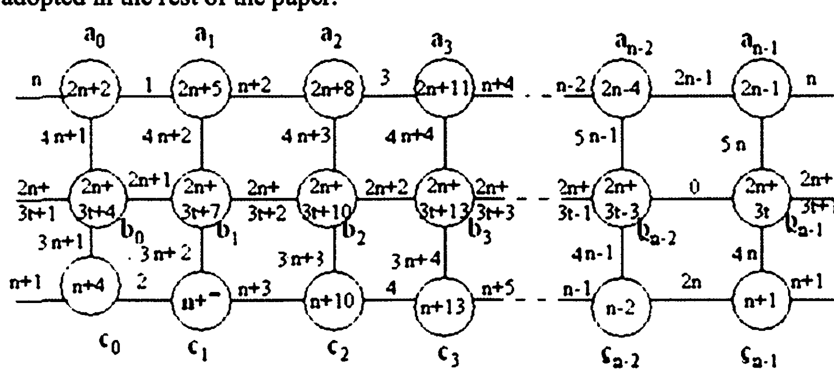


Figure 8.

Therefore modulo 3n, we have

$A = \{ f^+ (a_{2i-2}): i = 1, 2, \ldots, 3t+1\} = \{5n+6i-4: i = 1,2,\ldots,3t+1\}$,

$B = \{ f^+ (a_{2i-1}): i = 1, 2, \ldots, 3t\} = \{ 5n+6i-1: i = 1,2,\ldots,3t\}$,

$C = \{ f^+ (b_{2i-2}): i = 1,2,\ldots, 3t+1\} = \{11n+ 3t+6i-2: i = 1,2,\ldots, 3t+1\}$,

$D = \{ f^+ (b_{2i-1}): i = 1,2,\ldots, 3t\} = \{11n+ 3t+6i+1: i = 1,2,\ldots,3t\}$,

$E = \{ f^+ (c_{2i-2}): i = 1, 2,\ldots, 3t+1\} = \{4n+6i-2: i = 1,2,\ldots, 3t+1\}$,

$F = \{ f^+ (c_{2i-1}): i = 1, 2,\ldots, 3t\} = \{ 4n+6i+1: i = 1,2,\ldots,3t\}$.

Hence modulo 3n, we have

$A \cup B = \{2n+2, 2n+5, \ldots, 2n-1\}$.

We used the fact that $5n+6(3t+1)-4 = 5n+18t+2=5n+3(6t+1)=8n-1 \equiv 2n-1 \pmod{3n}$.

Similarly, modulo 3n, we have

$C \cup D = \{2n+3t+4, 2n+3t+7, \ldots, 2n+3t+1\}$,

$E \cup F = \{n+4, n+7, \ldots, n+1\}$.

We conclude that $A \cup B \cup C \cup D \cup E \cup F$ is modulo 3n the set $\{0,1,2,\ldots,3n-1\}$, and this proves that indeed $C_m x P_3$ is 1-edge-graceful.

Note that all the elements of $A \cup B$ are congruent to $2n+2 \equiv 1$, modulo 3, all the elements of $C \cup D$ are congruent to 0, modulo 3, and all the elements of $E \cup F$ are congruent to 2, modulo 3.

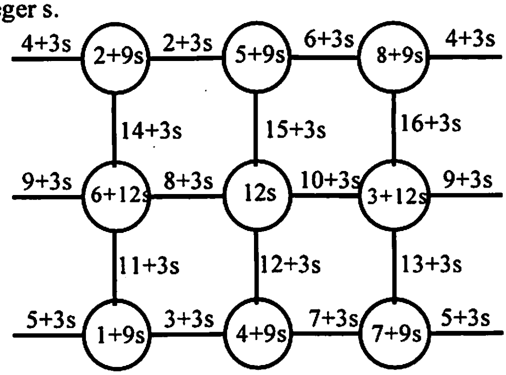
Now we can produce a 4-edge-graceful labeling from this 1-edge-graceful labeling by simply adding 3 to the value on each edge. The edge labels will then run 4, 5, 6, ..., 3n+3. Each vertex a_i has vertex sum 3+3+3=9 more than it had in the 1-edge-graceful labeling. Because the a_i vertex sums under the 1-edge-graceful labeling consist of all the elements of $\{1, 2, 3, \ldots, 3n\}$ which are congruent to 1, modulo 3, the values of $A \cup B$ are not changed by adding 9 to each integer of $A \cup B$, when we work modulo 3n. Since the elements of $C \cup D$ are all congruent to 0, modulo 3, the values of $C \cup D$ are not changed modulo 3n by adding 4(3)=12 to each element of $C \cup D$. The process also adds 9 to each vertex sum in E and F, and the same conclusion holds true for $E \cup F$. This proves that $C_n \times P_3$ is 4-edge-graceful.

It is clear now that a 7-edge-graceful labeling can be obtained by adding 3 to each edge label in the 4-edge-graceful labeling. This allows to conclude that for all $k \equiv 1 \pmod 3$ we have exhibited a k-edge-graceful labeling of a $C_n \times P_3$ when n=6t+1.

Second case:

<u>Theorem 4.2.</u> $egS(C_3 \times P_3) = 2 + 3N$.

Proof. Figure 9 shows that $C_3 \times P_3$ is 2-edge-graceful for s=0; in a manner identical to that in the proof of Theorem 4.1, $C_3 \times P_3$ is 2+3s-edge-graceful for any positive integer s.



$C_3 \times P_3$ is 2+3s-edge graceful

Figure 9.

Figure 10 shows similar 2-edge-graceful labelings of $C_n \times P_3$ for n=6t+3.

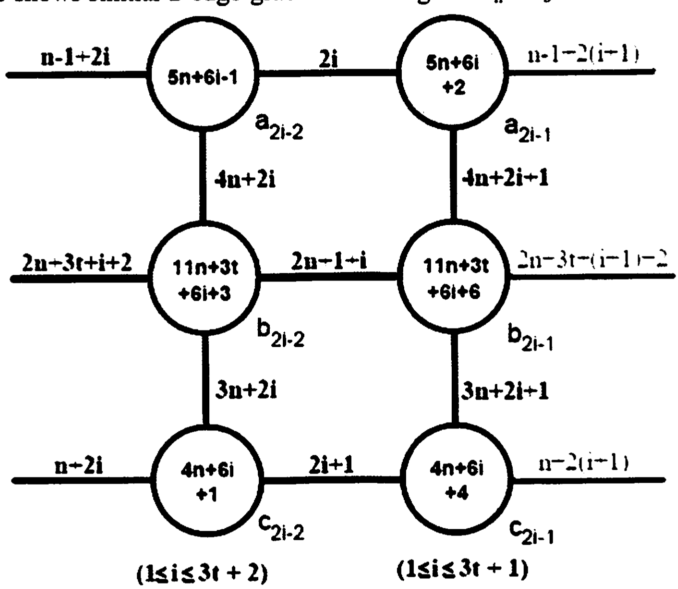


Figure 10.

Therefore modulo 3n, we have

$$A = \{ f^+(a_{2i-2}) : 1 \le i \le 3t+2\} = \{5n+6i-1 : 1 \le i \le 3t+2\},$$
$$B = \{ f^+(a_{2i-1}) : 1 \le i \le 3t+1\} = \{5n+6i+2 : 1 \le i \le 3t+1\},$$
$$C = \{ f^+(b_{2i-2}) : 1 \le i \le 3t+2\} = \{11n+3t+6i+3 : 1 \le i \le 3t+2\},$$
$$D = \{ f^+(b_{2i-1}) : 1 \le i \le 3t+1\} = \{11n+3t+6i+6 : 1 \le i \le 3t+1\},$$

$$E = \{ f^+(c_{2i-2}) : 1 \le i \le 3t + 2\} = \{4n + 6i + 1 : 1 \le i \le 3t + 2\},$$
$$F = \{ f^+(c_{2i-1}) : 1 \le i \le 3t + 1\} = \{4n + 6i + 4 : 1 \le i \le 3t + 1\}.$$

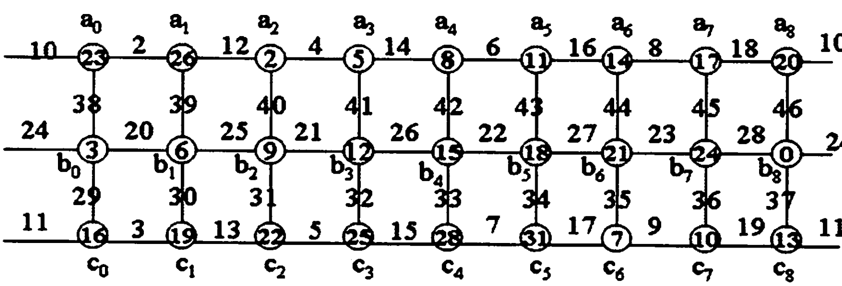
Modulo 3n, i.e. modulo 18t+9, we have

$$A \cup B = \{2n + 5, 2n + 8, \ldots, 2n - 1, 2n + 2\} = \{2,5,8,\ldots,3n-4, 3n-1\},$$
$$C \cup D = \{2n + 3t + 9, 2n + 3t + 12, \ldots, 2n + 3t + 3, 2n + 3t + 6\}$$
$$= \{0,3,6,\ldots,3n-6,3n-3\},$$
$$E \cup F = \{n + 7, n + 10, \ldots, n + 1, n + 4\} = \{1,2,3,\ldots,3n-5, 3n-2\}$$

whereupon $A \cup B \cup C \cup D \cup E \cup F = \{0, 1, 2, 3, \ldots, 3n - 1\}$ and $C_{6t+3} \times P_3$ is 2-edge-graceful.

For reasons similar to those given in the first case, the labelings can be extended to any k of the form 3s + 2 (by adding respectively 3, 6, 9, …, etc. to each edge in a given labeling). This allows us to conclude that $C_n \times P_3$ with n=6t+3 is k-edge-graceful for all $k \equiv 2 \pmod 3$.

Example 5. Figure 11 shows that $C_9 \times P_3$ is 2-edge-graceful.



C_9xP_3 is 2-edge-graceful.

Figure 11.

Third case:

Figure 12 shows a similar 0-edge-graceful labeling for n=5 which generalizes in the same way for all cases n=6t+5, as shown in Figure 13. Again, these labelings can be extended to k-edge-graceful labelings for k=3, by simply adding 3s to each edge label. The proofs that these labelings work in general are similar to the case n=6t+1.

Example 6. Figure 12 illustrates a 0-edge-graceful labeling of $C_5 \times P_3$.

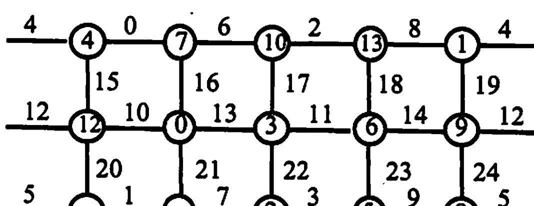


Figure 12.

Figure 13 shows the general solution for C_n x P_3 for n=6t+5. As before this labeling partitions the vertex labels, so that the a_i receive labels congruent to 1, modulo 3, the b_i receive labels congruent to 0, modulo 3, and the c_i receive labels congruent to 2, modulo 3. Again, we can modify this 0-edge-graceful labeling by adding 3s to each edge, creating a 3s-edge-graceful labeling. This completes the proof of Theorem 4.1

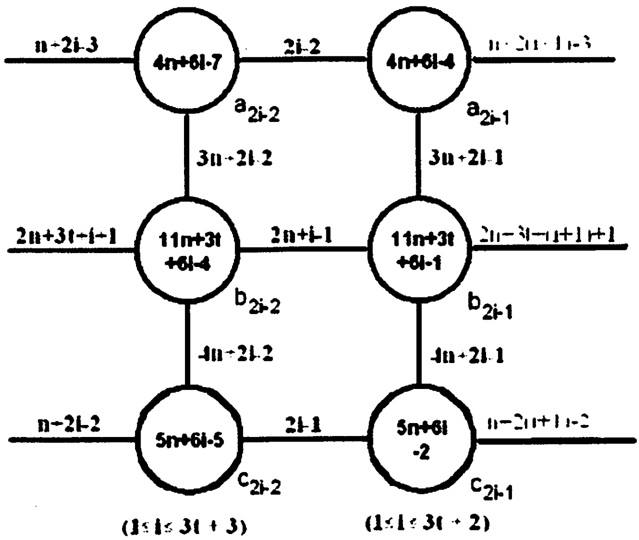


Figure 13.

5. Cylinder graphs C_n x P_m, m =4.

Figure 14 shows that C_3 x P_4 is 1-edge-graceful, for s=0. For other values of s, the situation is similar to that in the proof of Theorem 4.1. If we add 2 to each edge label in the 1-edge graceful labeling, then the vertex labels for the a_i and d_i will have 6 added to them, while the vertex labels for the b_i and c_i will have 8 added. Thus the vertex labels for the b_i and c_i will be unchanged, modulo 4, while the labels for the a_i and d_i will "switch"; for example, the 3-edge graceful labeling arrived at in this manner will have vertex labels for the a_i which run through all values from 1 to 12 that are congruent to 2 (mod 4), namely 2, 6 and 10, while the labels for these vertices under the 1-edge graceful labeling are the values congruent to 0 (mod 4), namely 0,4, and 8. Similarly, the labels for the d_i switch from 2 (mod 4) values to 0 (mod 4). When we again add 2 to each edge label to produce a 5-edge graceful labeling, these values switch again.

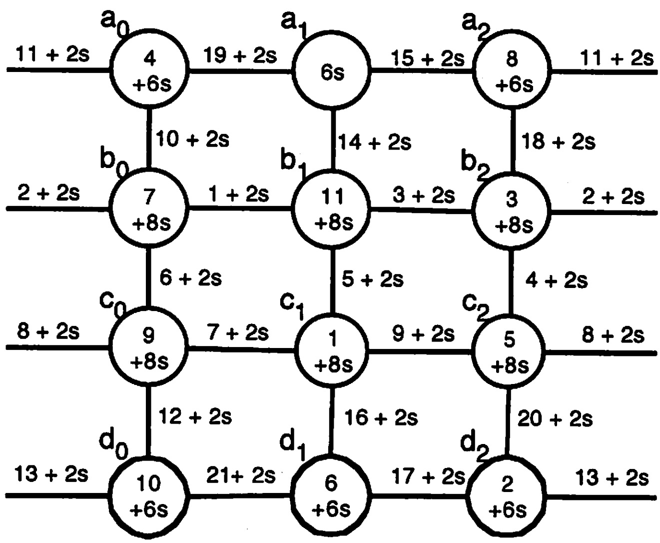


Figure 14.

This labeling generalizes for all $C_n \times P_4$ for which n is of the form 4t+3.

Theorem 5.1. The cylinder graph $C_n \times P_4$ is 1-edge-graceful if and only if n =4t+3. For n=4t+3, the edge-graceful spectrum of $C_n \times P_4$ is 1+2N.

Proof. Figure 15 shows 1-edge-graceful labelings of $C_n \times P_4$ for n=4t+3.

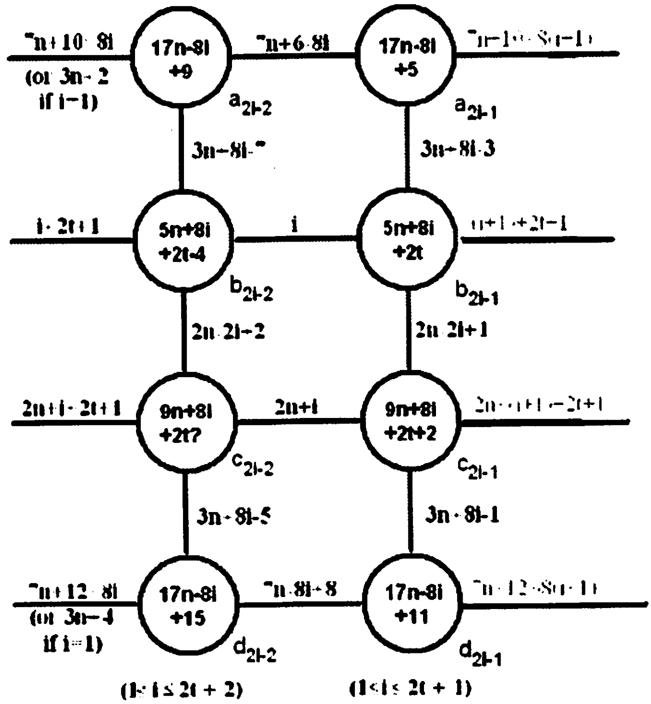


Figure 15. C_nxP_4, n=4t+3.

Note that the edge joining a_{n-1} to a_0 takes label 3n+2, which is 4 less than the label on the edge joining a_{n-2} to a_{n-1}, and similarly for the edge joining d_{n-1} to d_0. However, $3n+2 \equiv 7n+10-8(1) \pmod{4n}$, so the vertex sums remain as in the diagram, and similarly for the edge labeled 3n+4.

Modulo 4n, we have two variations in the overall pattern, depending on whether t is even or odd. If t is even, for example when n=3, then for C ∪ D, the vertex sums of the form

$$5n + 8i + 2t - 4 \equiv 5(4t+3) + 8i + 2t - 4 \equiv 2t+3 \equiv 3 \pmod 4,$$

and similarly for vertex sums of the form 5n + 8i + 2t + 8. If t is odd, for example when n=7, then vertex sums of the form

$$5n + 8i + 2t - 4 \equiv 5(4t+3) + 8i + 2t - 4 \equiv 2t+3 \equiv 1 \pmod 4,$$

and similarly for vertex sums of the form 5n + 8i + 2t. The opposite pattern holds for vertex sums in E ∪ F, so that together

$$(C \cup D) \cup (E \cup F) \equiv \{1, 3, 5, 7, 9, \ldots, 4n + 3\}.$$

Here,

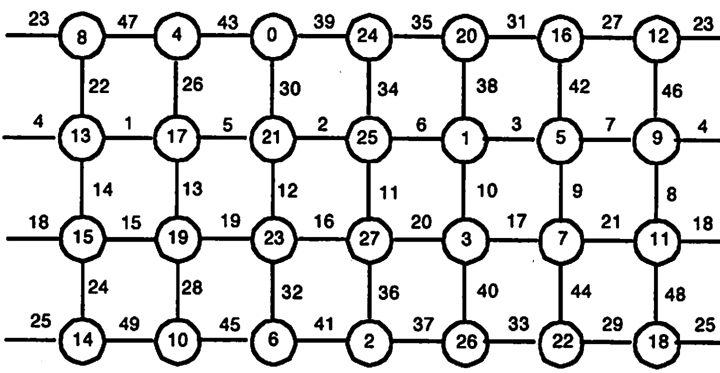
$A \cup B = \{17n - 8i + 9 : 1 \leq i \leq 2t + 2\} \cup \{17n - 8i - 3 : 1 \leq i \leq 2t+1\}$
$\equiv \{0, 4, 8, 12, \ldots, 4n - 4\}$

$C \cup D = \{5n + 8i + 2t - 4: 1 \leq i \leq 2t + 2\} \cup \{5n + 8i + 2t : 1 \leq i \leq 2t+1\}$
$\equiv \{1, 5, 9, \ldots, 4n - 3\}$ if t odd, or $\equiv \{3, 7, 11, \ldots, 4n - 1\}$, if t even,

$E \cup F = \{9n + 2t + 8i - 2: 1 \leq i \leq 2t + 2\} \cup \{9n + 2t + 8i + 2: 1 \leq i \leq 2t+1\}$

$\equiv \{3, 7, 11, \ldots, 4n - 1\}$ if t odd, or $\equiv \{1, 5, 9, \ldots, 4n - 3\}$ if t even,
$G \cup H = \{17n - 8i + 15: 1 \le i \le 2t + 2\} \cup \{17n - 8i + 11: 1 \le i \le 2t+1\}$
$\equiv \{2, 6, 10, 14, \ldots, 4n - 2\}$,
whereupon $A \cup B \cup C \cup D \cup E \cup F \cup G \cup H \equiv \{0, 1, 2, \ldots, 4n - 1\}$. As in the case of $C_3 \times P_4$ we obtain the 2s+1-edge-graceful labeling of $C_{4t+3} \times P_4$ by adding 2s to each edge in the 1-edge-graceful labeling given above.

Example 8. The cylinder graph $C_7 \times P_4$ is 2s+1-edge-graceful. Figure 16 shows the 1-edge-graceful labeling according to Theorem 5.1. To obtain a 2s+1-edge graceful labeling, we add 2s to each edge, as before.



$C_7 \times P_4$ is 1-edge-graceful
Figure 16.

The labeling scheme for $C_n \times P_4$, with n=4t+1, is nearly identical to that for n=4t+3, and this will finish the $C_n \times P_4$ cases.

Theorem 5.2. The cylinder graph $C_n \times P_4$ is 2-edge-graceful if and only if n =4t+1. For n=4t+1, the edge-graceful spectrum of $C_n \times P_4$ is 2+2N.

Proof: to obtain a 2-edge-graceful labeling of $C_n \times P_4$, for n=4t+1, we use the labeling scheme shown in Figure 17.

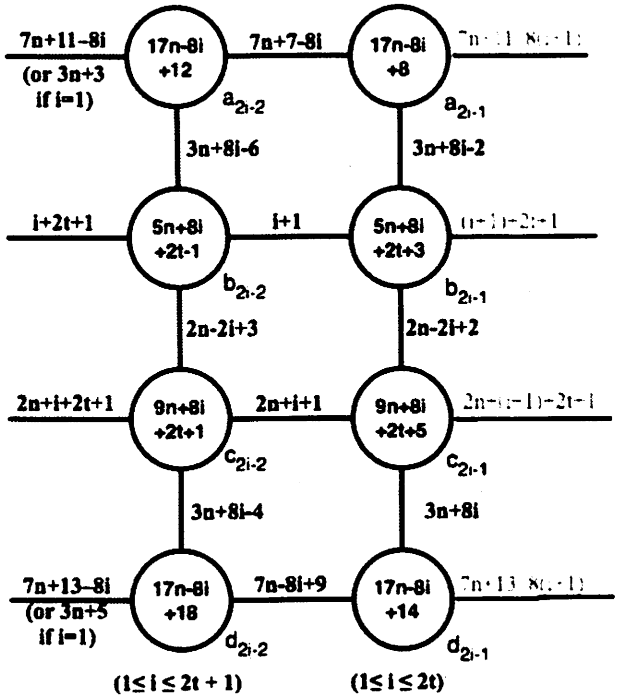


Figure 17.

Because now $n \equiv 1 \pmod 4$, the vertex sums will reduce to:

$A \cup B = \{17n - 8i + 12 : 1 \le i \le 2t + 1\} \cup \{17n - 8i + 8 : 1 \le i \le 2t\}$
$\equiv \{1, 5, 9, \ldots, 4n - 3\}$,

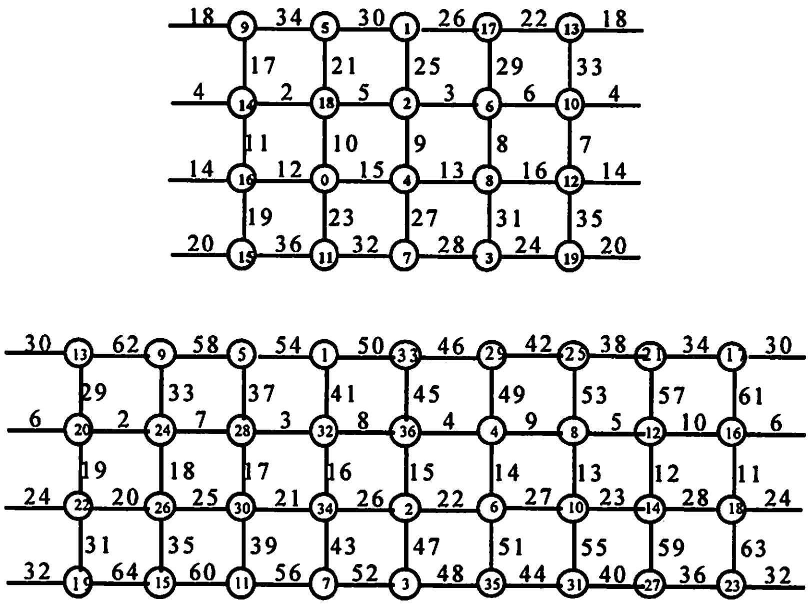
$C \cup D = \{5n + 8i + 2t - 1 : 1 \le i \le 2t + 1\} \cup \{5n + 8i + 2t + 3 : 1 \le i \le 2t\}$
$\equiv \{2, 6, 10, \ldots, 4n - 2\}$ if t odd, or $\equiv \{0, 4, 8, \ldots, 4n - 4\}$, if t even,

$E \cup F = \{9n + 2t + 8i + 1 : 1 \le i \le 2t + 1\} \cup \{9n + 2t + 8i + 5 : 1 \le i \le 2t\}$
$\equiv \{0, 4, 8, \ldots, 4n - 4\}$ if t odd, or $\equiv \{2, 6, 10, \ldots, 4n - 2\}$ if t even,

$G \cup H = \{17n - 8i + 18 : 1 \le i \le 2t + 1\} \cup \{17n - 8i + 14 : 1 \le i \le 2t\}$
$\equiv \{3, 7, 11, \ldots, 4n - 1\}$,

whereupon $A \cup B \cup C \cup D \cup E \cup F \cup G \cup H \equiv \{0, 1, 2, \ldots, 4n - 1\}$. (Note that the edge labels in A,B,G, and H are now odd, and those in C,D,E, and F are even, the reverse of the situation for the labeling given in Theorem 5.1 for the n=4t+3 case.) As in the case of $C_{4t+3} \times P_4$ we obtain the 2s+2-edge-graceful labeling of $C_{4t+1} \times P_4$ by adding 2s to each edge in the 2-edge-graceful labeling given above. As before, when s=1 (or any odd value), this adds 8 to each vertex label in C,D,E, and F, leaving their values the same, modulo 4. It adds 6 to each vertex label in A,B,G, and H, switching the values in $A \cup B$ with those in $G \cup H$.

Example 9. Figure 18 shows the 2-edge-graceful labelings given by Theorem 5.2 for $C_5 \times P_4$ and $C_9 \times P_4$.



$C_5 \times P_4$ and $C_9 \times P_4$ are 2-edge-graceful

Figure 18.

6. Conjecture.

We propose the following conjecture.

Conjecture. The cylinder graph $C_4 \times P_m$ is k-edge-graceful for some k if and only if $m \equiv 2 \pmod 4$.

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