A Complete Characterization of Balanced Graphs

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Abstract. Let G be a graph with vertex set V(G) and edge set E(G). For a labeling f: V(G) $\rightarrow$ A = {0,1}, define a partial edge labeling f*: E(G) $\rightarrow$ A such that, for each edge xy $\in$ E(G), f*(xy) = f(x) if, and only if, f(x) = f(y). For i $\in$ A, let $v_f(i) = |\{v \in V(G) : f(v) = i\}|$ and $e_{f^*}(i) = |\{e \in E(G) : f^*(e) = i\}|$. A labeling f of a graph G is said to be friendly if $|v_f(0) - v_f(1)| \leq 1$. If a friendly labeling f induces a partial labeling f* such that $|e_{f^*}(0) - e_{f^*}(1)| \leq 1$, then G is said to be balanced. In this paper, a necessary and sufficient condition for balanced graphs is established. Using this result, the balancedness of several families of graphs are also proven.

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1. Introduction

A graph labeling problem called cordial graph labeling was introduced by Cahit [2] in 1987. Consider a graph G with vertex set V(G) and edge set E(G). A binary vertex labeling of G is a mapping from V(G) into the set A = {0,1}. For each vertex labeling f of G, Cahit considered an induced binary edge labeling $f^{\#}: E \rightarrow \{0,1\}$ defined by $f^{\#}(uv) = |f(u) - f(v)|$, $uv \in E(G)$. Let $v_0(G)$, and $v_1(G)$, denote the number of vertices in V(G) that are labeled with 0, and 1, under the labeling f, respectively. Likewise, let $e_0(G)$, and $e_1(G)$, denote the number of edges in E(G) that are labeled with 0, and 1, under the induced labeling $f^{\#}$, respectively. Cahit called a graph cordial if it satisfies the following properties:

(i) $|v_0(G) - v_1(G)| \leq 1$,

(ii) $|e_0(G) - e_1(G)| \leq 1$.

Several classes of cordial graphs, including the Cartesian product, composition of graphs and tensor products, are considered in [1,2,3,4,6,7,9,10,11,12,13,14,16,18,19,20,21,22]. In [5], Cairnie and Edwards have determined the computational complexity of cordial labeling and Z_k-cordial labeling. They proved that the problem of deciding whether a graph G admits a cordial labeling is NP-complete. Other new and unsolved problems related to cordial labeling can also be found in [4,7,8].

Lee, Liu and Tan introduced another graph labeling problem in [17], called the balanced labeling problem. For any binary vertex labeling f, a partial edge labeling f* of G is defined for each edge $uv \in E(G)$ by

$$f^*(u,v) = \begin{cases} 0, & \text{if } f(u) = f(v) = 0, \\ 1, & \text{if } f(u) = f(v) = 1. \end{cases}$$

Note that if $f(u) \neq f(v)$, the edge uv is not labeled by f*. Thus, f is a partial function defined from E(G) into the set {0, 1}. We shall refer f* as the induced partial function of f.

Let $v_0(G)$, $v_1(G)$, $e_0(G)$, and $e_1(G)$ be defined as above. Hence,

$$v_0(G) = |\{u \in V(G) : f(u) = 0\}|,$$
$$v_1(G) = |\{u \in V(G) : f(u) = 1\}|,$$
$$e_0(G) = |\{\{u, v\} \in E(G) : f(\{u, v\}) = 0\}|,$$
$$e_1(G) = |\{\{u, v\} \in E(G) : f(\{u, v\}) = 1\}|.$$

With these notations, we now introduce the notion of a balanced graph.

Definition 1.1: Let $G = (V,E)$ be a graph. G is a balanced graph, or G is balanced, if there is a binary vertex labeling f of G satisfying the following conditions:

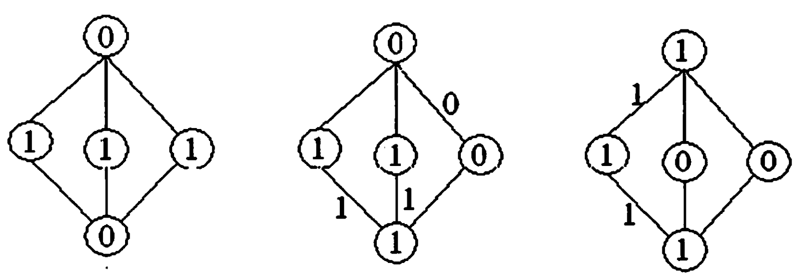
(i) $|v_0(G) - v_1(G)| \leq 1$ and

(ii) $|e_0(G) - e_1(G)| \leq 1$.

A balanced graph G is said to be strongly vertex-balanced if $v_0(G) = v_1(G)$. It is strongly edge-balanced if $e_0(G) = e_1(G)$. And if G is both strongly vertex-balanced and strongly edge-balanced, then G is strongly balanced.

Definition 1.2: A labeling f of a graph G is said to be friendly if $|v_0(G) - v_1(G)| \leq 1$. For any given friendly labeling of G, the balance index set of G, BI(G), is defined by $\{|e_f(0) - e_f(1)|\}$.

Example 1.1: Figure 1 shows a graph with BI(G) = {0,1,2}.



$|e(1)\text{-}e(0)| = 0$ $|e(1)\text{-}e(0)| = 1$ $|e(1)\text{-}e(0)| = 2$

Figure 1

Example 1.2: The balance index set of Θ-graph Θ(2,2,3) is {0,1}.

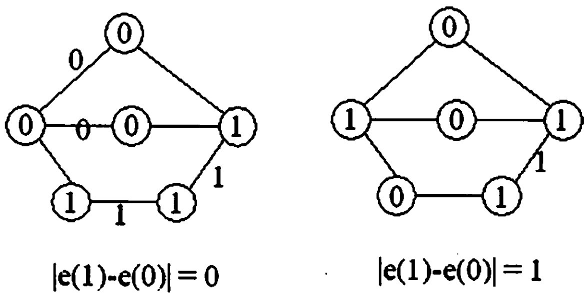


Figure 2

Example 1.3: The following (4,5)-graph G is balanced with two different labelings.

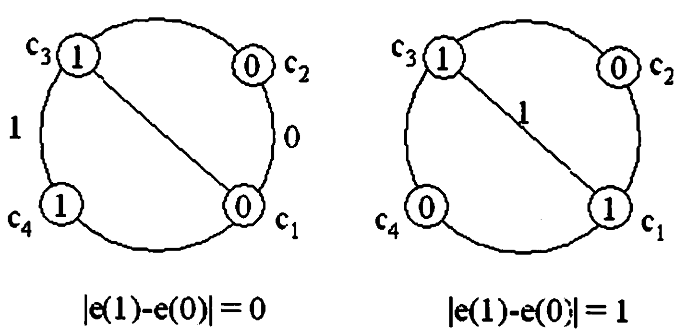


Figure 3

The following results were established in [17].

Theorem 1.1: Let G be a k-regular graph with p vertices,

(i) G is strongly balanced if and only if p is even;

(ii) G is balanced if and only if p is odd and k = 2.

Corollary 1.2: Every cycle Cm is a balanced graph.

Corollary 1.3: For complete graph on n vertices Km,

(i) Km is a strongly balanced graph if m is even;

(ii) If m is odd, Km is balanced if and only if m = 3.

Theorem 1.4: Every path Pm is balanced for $m \geq 1$; it is strongly balanced if m is even.

Theorem 1.5: The complete bipartite graph Km,n is balanced if, and only if, one of the following conditions holds:

(i) both m and n are even;
(ii) both m and n re odd and $|m-n| \leq 2$;
(iii)one of m and n, say m, is odd, $n = 2t$ and $t = -1, 0$, or 1 (mod $|m-n|$).

Other results on balanced labeling and balanced graphs can also be found in [15,20]. In this paper, a necessary and sufficient condition for a graph to be (strongly) balanced is established. Based on this result, the (strongly) balancedness of several families of graphs are determined.

2. A Necessary and Sufficient Condition for Balanced Graphs

To establish a necessary and sufficient condition for balanced graphs, we follow the edge-counting proof similar to the proof of Theorem 1.1 as found in [17]. Suppose a graph G has p vertices and q edges. Assume that p_i of the vertices are of degree r_i, for $i = 1, 2, \ldots, n$, and r_i are arranged in such a way that $r_1 < r_2 < \ldots < r_n$. We note that the integers r_i's are not necessarily distinct but, for our purpose, we assume they are. Thus, when n is 1, we have an r_1–regular graph. Counting the number of vertices and edges of G, we have the following simple relationships:

$$p = \sum_{i=1}^{n} p_i, \text{ and } q = \sum_{i=1}^{n} p_i r_i / 2.$$

For each i, $1 \leq i \leq n$, we partition the p_i vertices of degree r_i into two sets A_i and B_i, where all vertices in A_i are labeled by 0 and all vertices in B_i are labeled by 1. Let $a_i = |A_i|$, then $|B_i| = p_i - a_i$. It follows immediately from above that

$$V_0(G) = \sum_{i=1}^{n} a_i, \text{ and } V_1(G) = \sum_{i=1}^{n} (p_i - a_i) = p - \sum_{i=1}^{n} a_i.$$

In order to obtain an expression for $e_0(G)$ and $e_1(G)$, we define $c(u,C,D)$ to be the number of edges connecting a given vertex $u \in C$ to vertices in set D. Hence,

$$c(u,C,D) = |\{\{u, v\} : u \in C, v \in D, \{u,v\} \in E(G)\}|.$$

We have

$$e_0(G) = \sum_{i=1}^{n} \sum_{v \in A_i} c(v,A_i,A_j) + \sum_{i=1}^{n} \sum_{v \in A_i} c(v,A_i,A_j)/2,$$

$$e_1(G) = \sum_{i=1}^{n} \sum_{v \in B_i} c(v,B_i,B_j) + \sum_{i=1}^{n} \sum_{v \in B_i} c(v,B_i,B_j)/2, \ i < j.$$

Let $S_v = v_1(G) - v_0(G)$ and $S_e = e_1(G) - e_0(G)$. Then,

$$S_v = (p - \sum_{i=1}^{n} a_i) - \sum_{i=1}^{n} a_i = \sum_{i=1}^{n} (p_i - 2a_i), \qquad (1)$$

$$S_e = \sum_{i=1}^{n} \sum_{v \in B_i} c(v,B_i,B_j) + \sum_{i=1}^{n} \sum_{v \in B_i} c(v,B_i,B_j)/2 \qquad (2)$$

$$- \sum_{i=1}^{n} \sum_{v \in A_i} c(v,A_i,A_j) + \sum_{i=1}^{n} \sum_{v \in A_i} c(v,A_i,A_j)/2, \ i < j.$$

It is not difficult to see that the number of edges of the graph, q, can also be expressed in terms of $c(u,C,D)$, as follows:

$$\begin{aligned} q = & \sum_{i=1}^{n} \sum_{v \in A_i} c(v,A_i,A_j)/2 + \sum_{i=1}^{n} \sum_{v \in B_i} c(v,B_i,B_j)/2 \\ & + \sum_{i=1}^{n} \sum_{v \in A_i} c(v,A_i,B_j) + \sum_{i=1}^{n} \sum_{v \in B_i} c(v,A_i,B_j) \\ & + \sum_{i=1}^{n} \sum_{v \in A_i} c(v,A_i,A_j) + \sum_{i=1}^{n} \sum_{v \in B_i} c(v,B_i,B_j) \qquad (3) \\ = & \sum_{i=1}^{n} p_i r_i / 2. \end{aligned}$$

Substituting the terms $\sum_{i=1}^{n}\sum_{v\in B_i} c(v,B_i,B_j) + \sum_{i=1}^{n}\sum_{v\in B_i} c(v,B_i,B_j)/2$ in (2) by the same terms in (3) and simplifying, we have

$$S_e = \sum_{i=1}^{n} p_i r_i/2 - \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{v\in A_i} c(v,A_i,A_j) - \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{v\in A_i} c(v,A_i,B_j)$$

$$= \sum_{i=1}^{n} p_i r_i/2 - [\sum_{i=1}^{n}\sum_{v\in A}\sum_{j=1}^{n} c(v,A_i,A_j) + c(v,A_i,B_j)]$$

$$= \sum_{i=1}^{n} p_i r_i/2 - \sum_{i=1}^{n}\sum_{v\in A_i} r_i$$

$$= \sum_{i=1}^{n} p_i r_i/2 - \sum_{i=1}^{n} a_i r_i$$

$$= \sum_{i=1}^{n} (p_i - 2a_i) r_i/2. \qquad (4)$$

We have the following result.

Theorem 2.1. Let G be a (p, q)-graph with S_v and S_e as defined in (1) and (4) respectively. G is balanced if, and only if, there exists a set of integers $\{a_i : 0 \le a_i \le p_i, i = 1,2, \dots, n\}$ such that $|S_v| \le 1$ and $|S_e| \le 1$. Furthermore, G is strongly balanced if, and only if, there exists a set of integers $\{a_i : 0 \le a_i \le p_i, i = 1, \dots, n\}$ such that $S_v = 0$,and $S_e = 0$.

3. Applications

By using the result from Theorem 2.1, we can determine the balancedness of the following families of graphs.

Example 3.1: A k-regular graph G on p vertices.
Here n = 1. If we let $p_1 = p$, $r_1 = k$ and $a_1 = a$, we have $S_v = p - 2a$ and $S_e = (p - 2a)k/2$. If p is even, we set $a = p/2$ and, it is obvious that G is strongly balanced. For p odd, G cannot be strongly balanced since it is impossible for $S_v = 0$. Thus, for G to be

balanced, a must either be $(p - 1)/2$ or $(p + 1)/2$, and k must be 2.

Example 3.2: A path P_m on m vertices. Here $n = 2$. Let $r_1 = 2$, $r_2 = 1$, $p_1 = m - 2$, $p_2 = 2$, we have $S_v = m - 2a_1 - 2a_2$ and $S_e = m - 1 - 2a_1 - 2a_2$. For Pm to be strongly balanced, $S_v = 0$ and $S_e = 0$. Hence, $a_1 = (m - 2)/2$ and $a_2 = 1$. Note that in this case m must be even. If m is odd, let $a_1 = (m - 1)/2$ and $a_2 = 0$, and it is obvious that P_m is balanced.

Ho, Lee and Shee [11] proved $C_{4m} \times P_t$ is cordial for all m and odd t. Seoud and Abdel [23] proved certain cylinder graphs are cordial. By applying the above characterization, we have the following result.

Theorem 3.1: A cylinder graph $C_s \times P_t$ is

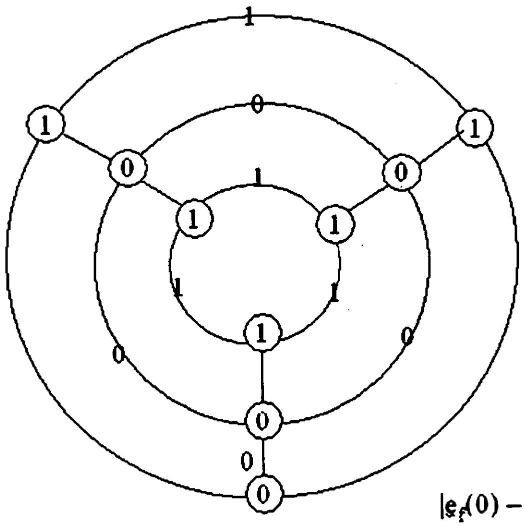
(1) strongly balanced if either s or t is even;

(2) balanced if both s and t are odd.

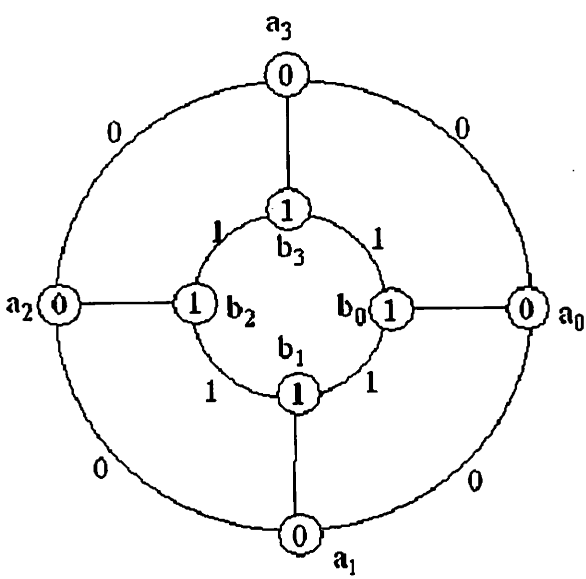
Proof. A cylinder graph $C_s \times P_t$ has st vertices and $2st - s$ edges. Here $n = 2$. Let $r_1 = 3$, $r_2 = 4$, $p_1 = 2s$, $p_2 = (t - 2)s$, we have $S_v = st - 2a_1 - 2a_2$ and $S_e = 2st - s - 3a_1 - 4a_2$. If either s or t is even, the graph is strongly balanced, since we can set $a_1 = s$ and $a_2 = (t - 2)s/2$. If both s and t are odd, let $a_1 = s - 2$ and $a_2 = [(t - 2)s + 3]/2$ and we see that the labeling is balanced.

Example 3.3: We show that $C_3 \times P_3$ is balanced and $C_4 \times P_2$ is strongly balanced in Figure 4.

Let G, H be two graphs, and let G have p vertices. The corona of G with H is the graph obtained by taking one copy of G and p copies of H and then joining the ith vertex of G to each vertex in the ith copy of H, for each i from 1 to p. We will use G©H to denote the corona of G with H.



$C_3 \times P_3$ $|e_f(0) - e_f(1)| = 0$



$C_4 \times P_2$ $|e_f(0) - e_f(1)| = 0$

Figure 4

Example 3.4: The sun graph $C_m©K_1$.
Here n = 2. Let $r_1 = 1$, $r_2 = 3$, $p_1 = m$, $p_2 = m$, we have $S_v = 2m - 2a_1 - 2a_2$ and $S_e = 2m - a_1 - 3a_2$. In order for $Cm©K_1$ to be strongly balanced, $S_v = 0$ and $S_e = 0$. Hence, $a_1 = m/2$ and $a_2 = m/2$. If m is odd, let $a_1 = (m - 1)/2$ and $a_2 = (m + 1)/2$ and, it is obvious that $C_m©K_1$ is balanced (See Figure 5).

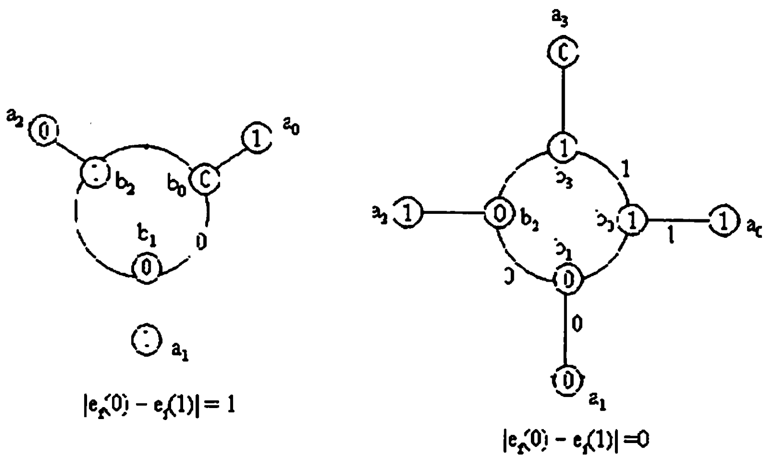


Figure 5

By applying the above characterization, we have the following result.

Theorem 3.2: The sun graph $Cm©K_1$ is
(1) strongly balanced if m is even
(2) balanced if m > 3.

4. Conclusion

As can be seen from above, the condition is particularly useful if the number of distinct degrees is small. In the event that a large number of distinct degrees are involved, the problem can be solved by using dynamic programming or other integer programming algorithms.

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