A Note on Non-Regular Planar Graphs

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Abstract

We give constructive and combinatorial proofs to decide why certain families of slightly irregular graphs have no planar representation and why certain families have such planar representations. Several non-existence results for infinite families as well as for specific graphs are given. For example the nonexistence of the graphs with $n = 11$ and degree sequence $(5, 5, 5, .., 4)$ and $n = 13$ and degree sequence $(6, 5, 5, ..., 5)$ are shown.

1 Introduction

The following definitions and theorems can be found in many textbooks and papers, for example, see Behjad and Chartrand[1], Harris, Hirst and Mossinghoff[5], and Johnsonbaugh[7]. Unless otherwise stated, all graphs are assumed to be simple and connected.

1.1 Planar Graphs

A ***planar graph*** is a graph that can be drawn in the plane in such a way that no two edges intersect except at a vertex.

In 1930, Kuratowski gave the necessary and sufficient conditions for a graph to be planar, see Harary[4].

Theorem 1.1. *A graph G is nonplanar if and only if it contains a subgraph homeomorphic with either K_5 or $K_{3,3}$.*

A connected planar representation of a graph divides the plane into contiguous regions called ***faces***. The ***unbounded face*** includes all the plane up to the outer boundary of the graph. Let F denote the set of faces of graph G. The following theorem, proven by Leonard Euler in 1752 gives the Euler's formula.

Theorem 1.2. *If a graph G is connected and planar, then*

$$|F| = |E| - |V| + 2.$$

Theorem 1.3. *If a graph G is connected and planar with $|V| \geq 3$, then*

$$|E| \leq 3|V| - 6.$$

Corollary 1.1. *In a planar graph G on n vertices,*

$$\frac{\sum_{i=1}^{n} d_i}{n} \leq 6.$$

The condition, $|E| \leq 3|V| - 6$, is necessary, but not sufficient for planarity. An example is $K_{3,3}$.

Theorem 1.4. *For a planar connected graph with $|V| \geq 4$, there exists a vertex $v \in V$ such that*

$$d(v) \leq 5.$$

Theorem 1.5. *Given a face in a planar, connected graph with more than two vertices, there exists a representation of that graph in which that face is the unbounded (exterior) face.*

For the proof see Theorem 11.3 in Harary[4].

1.2 Degree Sequences

An ordered sequence $d_1 \geq d_2 \geq ... \geq d_n$ of non-negative integers is called the ***degree sequence*** if there exists a corresponding graph G with vertices $v_1, v_2, ..., v_n$ such that the degree of v_i is d_i for all i. A sequence is said to be graphic if there exists a graph corresponding to it. Harary [4] has given a complete list of non-isomorphic graphs up to 6 vertices.

Theorem 1.6. *A sequence $d_1, d_2, ..., d_n$ of non-negative integers with $d_1 \geq d_2 \geq ... \geq d_n$ and $n \geq 2$ is graphic if and only if the sequence $d_2 - 1, d_3 - 1, ..., d_{d_1+1} - 1, d_{d_1+2}, ..., d_n$ is graphic.*

See for reference, page 11 of Bondy and Murty[2]. Naturally these and other similar results should be satisfied for the degree sequences we want to study for planarity.

1.2.1 Regular Degree Sequences

A ***regular graph*** has all vertices of equal degree. A ***regular degree sequence*** is the degree sequence of such a graph.

The following result, due to Limaye, Sarvate, Stănică and Young [8], concerns the existence of planar graphs for regular degree sequences.

Theorem 1.7. *For every degree sequence of length p whose elements are all r, with $r \leq 5$, pr even, and $r < p$, there exists a planar graph with that degree sequence, except when $p = 7$ and $r = 4$, and when $p = 14$ and $r = 5$, in which case no planar graphs exist.*

2 Slightly Irregular Degree Sequences

In this paper we consider degree sequences which fail to be regular by only a few vertices.

Let us first consider sequences with highest degree greater than 5. In view of Theorem 1.7, for a sequence $d_1, d_2, d_3, ..., d_m, ..., d_{n-2}, d_{n-1}, d_n$ on n vertices, there exists m such that $d_j \geq 6$ for $1 \leq j \leq m$, and $d_j \leq 5$ for some $m + 1 \leq j \leq n$ in order to be planar. The following lemma describes the restrictions on such a sequence.

Lemma 2.1. *In a degree sequence $d_1, d_2, ..., d_m, d_{m+1}, ..., d_{n-1}, d_n$ on n vertices, where $d_j \geq 6$ for $1 \leq j \leq m$, $d_j \leq 5$ for $m + 1 \leq j \leq n$, and $r = \sum_{p=m+1}^{n} d_p$, if the sequence has a planar representation then*

$$6(n - m) - r \geq 12.$$

Proof. Let $s = \sum_{p=1}^{m} d_p$. The number of edges in a graph with this sequence is $\frac{s+r}{2}$. Assume that this sequence has a planar representation. Then, by Theorem 1.6, $s + r \leq 6n - 12$. Since $6m \leq s$, implies $6m + r \leq 6n - 12$. Hence the result. □

If the regularity fails by only one or two vertices, then we have two cases: (i) $(n-m) = 1$ or $(n-m) = 2$ and (ii) $(n-m) = (n-1)$ or $(n-m) = (n-2)$. First case turns out to be of little interest because when $(n-m) = 1$, the inequality in Lemma 2.1 can not be satisfied. When $(n-m) = 2$, the inequality is satisfied only when degree sum of vertices with $d_i < 6$ is zero which gives a non-planar graph as there is no vertex of degree less than 6.

Thus, it is more interesting to study case (ii) when $n - m = n - 1$ or $n - 2$, that is the degree sequences of the form $n - j, d, d, ..., d$ and $n - j, n - k, d, d, ..., d$ where n is the number of vertices and $d \leq 5$. We first discuss the existence of planar graphs with degree sequence $(n - j, d, d, d..., d)$. Here is an important observation:

Lemma 2.2. *(i) A planar graph with degree sequence $(n - j, 2t, 2t, ...2t)$ may exist only when n and j have same parity and does not exist if n and j have opposite parity. (ii) A planar graph with degree sequence $(n-j, 2t+1, 2t+1, ..., 2t+1))$ may exist only when j is odd and does not exist if j is even.*

This lemma is used repeatedly in the following discussion:

Case $d = 1$: For $j = 1$, such a planar graph is a ***star***. For connected planar graph, $j = 1$ is the only possible value of j.

Case $d = 2$: For $(n - j, 2, 2, ..., 2)$, n and j must be of the same parity and $n - j$ must be even. To construct such a planar graph, form $(n - j)/2$ triangles with a common vertex of degree $n - j$.

Case $d = 3$: For $j = 1$, such a graph is a ***wheel***. For general j, the sum of the degrees is $3(n-1) + n - j$. Therefore for even j such a graph does not exist.

Theorem 2.8. *(1) A connected planar graph with degree sequence $(n - j, 1, 1, ..., 1)$ exists iff $j = 1$. (2) A planar graph with degree sequence $(n-j, 2, 2, ..., 2)$ exists iff n and j have the same parity. (3) A planar graph with degree sequence $(n - j, 3, 3, ..., 3)$ exists iff j is odd.*

Theorem 2.9. *A planar graph with degree sequence $(n - 1, d, d, d..., d)$ does not exist for $d = 4$ and $d = 5$.*

Proof. The result is true when n is even and $d = 4$ because sum of the degrees is odd in that case. Let n be an odd integer. Name the vertex with degree $(n - 1)$ as central vertex, and denote by C. Consider another vertex x in the graph, then there are three more vertices with which x is connected say x_1, x_2, x_3. These three vertices are connected to C as well as to x, forming three co-axial triangles (i.e. three triangles which share an edge.). See Figure 1. If all the vertices lie outside this structure then one of the vertices inside has degree at most three. If a vertex x_4 lies inside one of the triangles then it gives rise the similar structure embedded in the original structure leaving at least one vertex with degree three. Thus a planar graph with $d = (n-1, 4, 4, 4..., 4)$ does not exist. Consider the case when $d = 5$, applying Theorem 1.6, the inequality yields $\frac{5(n-1)+(n-1)}{2} \leq 3n - 6$, which reduces to $3n \leq 3n - 3$. This is false, therefore no planar graphs exist for d=5. □

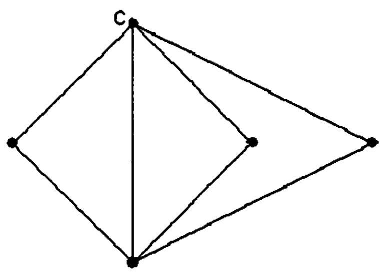

Figure 1: Three co-axial triangles

Theorem 2.10. *A planar graph with degree sequence $(n-2, d, d, d..., d)$ does not exist for $d = 1, 3, 4$ and 5 except for $n = 6$ and $d = 4$.*

Proof. We have discussed $d = 1$ and $d = 3$ cases in Theorem 2.13. Consider $d = 4$, when $n > 6$, three co-axial triangles arise, leaving at least one vertex of degree three and hence the corresponding graphs do not exist. (This situation is similar to above theorem). Consider $d = 5$, then $\sum d_i = 6n - 7 > 6n - 12$ which is upper bound for degree sum of a planar graph, hence the corresponding graph does not exist.

□

Theorem 2.11. *For $n \geq 8$ a planar graph with degree sequence $(n - j, 4, 4, 4..., 4)$ always exists, whenever the necessary conditions $n - j$ is an even number, and $0 \leq n - j \leq n - 3$ are satisfied. For $n = 6$, a planar graph exists except for $(n - j) = 0, 2$. For $n = 7$, for $n - j = 0, 2$ the corresponding planar graphs exist, but for $n - j = 4$ it does not. For $n = 8$, for $n - j = 2, 4$, the corresponding planar graphs exist.*

Proof. As the graph can not have a single vertex of odd degree, $n - j$ must be and $j \geq 3$ is necessary because when j is 1 or 2, the corresponding graphs do not exist by Theorems 2.9 and 2.10. The claims for $n = 6, 7$, and 8 can be checked easily. For $n = 9$ and $n = 10$, the graphs are given in Example D and Example E.

The basic building blocks in this constructive proof for $n \geq 11$ are two planar graphs G_1 and G_2 of degree sequences $(4, 3, 3, ..., 3, 2, 2)$ on odd number of vertices labeled 1,2,...,2t-1; and of degree sequence $(4, 3, ..., 3, 2)$ on even number of vertices labeled 0,1,2,...,2t-1 respectively. The planar representation of these graphs are drawn in Figure 2 with their mirror images G_1' and G_2' on vertices $1', 2', ..., (2t-1)'$ and $0', 1', 2', ..., (2t-1)'$.

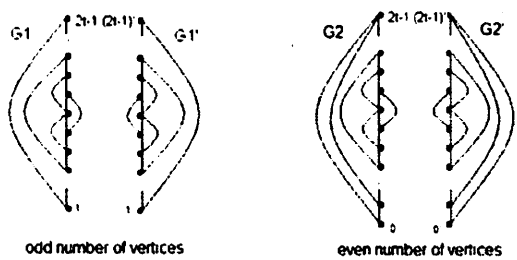

Figure 2: basic building blocks

Case $n = 4t$, $t \geq 3$

Draw the building blocks G_1 and G_2' on $2t-1$ and $2t$ vertices. Note G_1 has two end vertices labeled 1 and $2t-1$ with degree 2 and G_2' has only one vertex labeled $0'$ of degree 2. Join the two end vertices 1 and $2t-1$ of G_1 to the vertex $0'$ of degree 2 of G_2'. Join vertex i with i' to create parallel edges for $i = 1, 2, ..., 2t-1, i \neq t$. Now combine the middle $(n-j)/2$ parallel edges to create a vertex of degree $(n-j)$. For an illustration, see Example A where $n = 12$ and $n - j = 4$.

Case $n = 4t + 1$, $t \geq 3$

Start with the building blocks G_2 and G_2' on $2t$ vertices. They are identical and vertices labeled 0 and $0'$ are of degree two. Join these two vertices to a new vertex x. Also add edges (i, i'), $i = 0, 1, ..., 2t-1$ while keeping planar property intact. Now replace edges (i, i') from $i = 1, 2, ..., t-1, t+1, ..., (n-j-2)/2$ by edges (i, x) and (i', x) to get the required graph.

Case $n = 4t + 2$, $t \geq 3$

Start with the building blocks G_1 and G_1' on $2t-1$ vertices. Add four new vertices $0, 0', -x, x$ and edges $(0, 0')$,$(0, 1)$,$(0', 1')$, $(0, -x)$, $(0', -x)$, $(0, x)$,$(0', x)$, $(-x, 2t-1)$ and $(-x, (2t-1)')$ and parallel edges (i, i') for $i = 0, ..., 2t-1$ while keeping planarity intact. Note we already have a planar graph with $n - j = 2$. Now replace edges (i, i') with edges (i, x) and

(i', x) for $i = 1, ..., t-1, t+1, ..., (n-j)/2$ to get the required planar graph.

Case $n = 4t + 3$, $t \geq 2$

Start with the building blocks G_1 and G_1' on $2t + 1$ vertices. Add the parallel edges (i, i') for $i = 1, 2, ..., 2t + 1$. There are two cases to consider: first $n - j = 4t$ and then $2 \leq n - j \leq 4t$. For the first case: replace the edges (i, i') for $i = 2, 3, ..., t-1, t+1, ..., 2t$ with edges (i, x) and (i', x) and also draw edges $(1, x), (1', x), (2t+1, x), ((2t+1)', x)$, where x, the new vertex of degree $4t$, is placed in the midway between G_1 and G_1' so as to preserve planarity of the graph. See Example B (Figure 4) after the proof for $n = 15$ and $n - j = 12$. For the second case, add edges $(2t-1, 2)$ and $((2t-1)', 2')$ and $(1, x)$ and $(1', x)$ while preserving planarity. Remove the edge (2,2'). This gives the required graph for $n - j = 2$. For other values of $n - j$, we replace the edges (i, i') by (i, x) and (i', x) for $i = 3, 4, ..., t-1, t+1, ...(n-j)/2$. See Example C, for $n = 15$ and $n - j = 6$.

□

Example A : $n = 12$, $n - j = 4$ (see Figure 3).

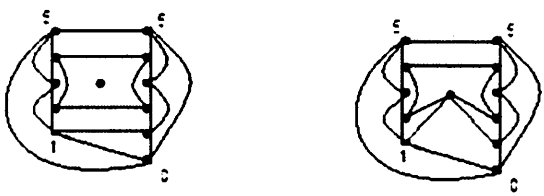

Figure 3: Example A n=12

Example B : $n = 15$, $n - j = 12$ (see Figure 4). Example C : $n = 15$, $n - j = 6$ (see Figure 5). Example D : $n = 9$, $n - j = 0, 2, 4, 6$ (see Figure 6). Example E : $n = 10$, $n - j = 0, 2, 4, 6$ (see Figure 7).

Theorem 2.12. *A planar graph with degree sequence* $(n - j, 5, 5, 5..., 5)$ *does not exist for* $j \leq 6$.

Proof. Note that the degree sum $6n - (5 + j)$ must be less than or equal to $6n - 12$, i.e. $5 + j \geq 12$ or $j \geq 7$ for such planar graph to exist.

□

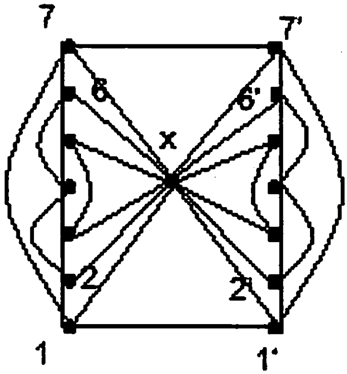

Figure 4: Example B n=15

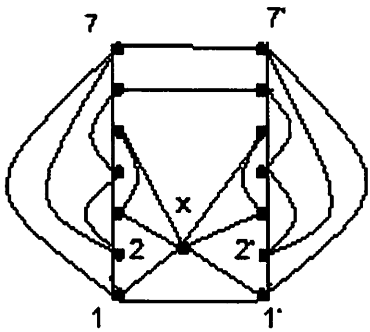

Figure 5: Example C n=15

3 Maxplanar graphs with slightly irregular degree sequences

A planar graph is called maxplanar (or maximal planar graph) when it has exactly $3(n-2)$ edges. If a face has more than three sides we can add an edge between nonadjacent vertices on the boundary of this face and get a planar graph, therefore in a maxplanar graph all faces are triangles. Construction of such graphs is also known as triangulation.

Theorem 3.13. *A maxplanar graph is regular only for* $n = 3, 4, 6,$ *and* 12. *Furthermore, all these planar graphs exist.*

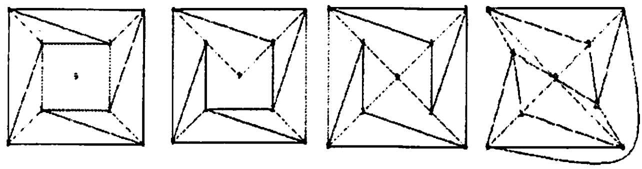

Figure 6: Example D n=9

Figure 7: Example E n=10

Proof. A necessary condition for a graph to be maxplanar is that $\sum d_i = 6(n-2)$ and is regular when $6(n-2)$ is divisible by n. Combining these two we must have $6(n-2) = \alpha n$ where α is some constant integer. This is satisfied when α takes values $2, 3, 4$ or 5 resulting into corresponding values of n to be $3, 4, 6$ and 12 with $d = 2, 3, 4$, and 5 respectively. It is well known that all these planar graphs exist. □

We call a maxplanar graph slightly irregular if max d_i−min$d_i \leq 1$ and obtain a general complete result.

Theorem 3.14. *A maxplanar graph with degree sequence such that max d_i−min $d_i \leq 1$ is always constructible except for $n = 11$ and $n = 13$.*

Proof. From Theorem 1.7 and the hypothesis, for slightly irregular maxplanar graphs the possible pairs $(min\ d_i, max\ d_i)$ are $(1,1), (1,2), (2,2), (2,3), (3,3), (3,4), (4,4), (4,5), (5,5)$, and $(5,6)$. First we consider each case with same min and max degrees. The planar graphs of type $(1,1)$ are regular planar graphs of degree 1 that do not exist. The cases $(2,2), (3,3), (4,4)$ and $(5,5)$are that of regular planar graphs as described in theorem 3.18.

Recall, for a planar graph to be maxplanar $\sum d_i = 6(n-2)$. Let x_1 be the number of vertices with degree 1 and x_2 be the number of vertices with degree 2. Solving two equations $6(n-2) = x_1 + 2x_2$ and $x_1 + x_2 = n$ gives $x_1 = 12 - 4n$. The only valid value n for which $max\ d_i = 2$ is 3 which gives $x_1 = 0$, thus there does not exist a maxplanar graph of type $(1,2)$. Similarly for the maxplanar graph of type $(2,3)$, we have two equations $6(n-2) = 2x_1 + 3x_2$, and $x_1 + x_2 = n$, implying $x_1 = 12 - 3n$. The only valid value for n is $n = 4$ for which a maxplanar regular graph K_4 exists, hence a maxplanar graph of type $(2,3)$ does not exist.

For the case of graph of type $(3,4)$, the equations are $6(n-2) = 3x_1 + 4x_2$, and $x_1 + x_2 = n$, which gives $x_1 = 12 - 2n$. This has two valid solutions: $n = 5$ gives $x_1 = 2$, and a maxplanar graph with degree sequences $(4,4,4,3,3)$, which exists. The other solution, namely $n = 6$ gives $x_1 = 0$, which gives regular planar graph on 6 vertices.

For the case of graphs of type $(4,5)$, the equations are $6(n-2) = 4x_1 + 5x_2$, and $x_1 + x_2 = n$, which gives $x_1 = 12 - n$. This has six valid solutions, $n = 7,8,9,10,11,12$. Each of these values gives rise to different degree sequences as given below:

when $n = 7$, $x_1 = 5$, the resulting maxplanar graph has degree sequence $(5,5,4,4,4,4,4)$,

when $n = 8$, $x_1 = 4$, the resulting maxplanar graph has degree sequence $(5,5,5,5,4,4,4,4)$,

when $n = 9$, $x_1 = 3$, the resulting maxplanar graph has degree sequence $(5,5,5,5,5,5,4,4,4)$,

when $n = 10$, $x_1 = 2$, the resulting maxplanar graph has degree sequence $(5,5,5,5,5,5,5,5,4,4)$, and

when $n = 11$, $x_1 = 1$, the resulting maxplanar graph has degree sequence $(5,5,5,5,5,5,5,5,5,5,4)$.

All graphs with above degree sequences given are constructible except for n=11 and n=13. The non-existence of these graphs have been proven in the following.

Now consider the case of graphs with $(min\ d_i, max\ d_i) = (5,6)$. For such graphs the simultaneous equations are $6(n-2) = 5x_1 + 6x_2$ and $x_1 + x_2 = n$. Solving these two equations give $x_1 = 12$ and $x_2 = n - 12$. Obviously $n \geq 13$. Such maxplanar graphs exist for all n but 13.The degree distribution is as follows : $x_1 = 12$ (number of vertices with degree 5) $x_2 = i$ (number of vertices with degree 6), for $n = 12 + i$ and $i = 2,3,...$

For n =14 and 15 the graphs are given in the Examples F and G. We give a construction for such graphs with $n \geq 16$ as follows:
For $n \geq 16$ a maxplanar graph with degree sequence such that max d_i- min $d_i \leq 1$ is always constructible with min $d_i = 5$ and max $d_i = 6$ with degree distribution: $x_1 = 12$, number of vertices with degree 5, $x_2 = i$, number of vertices with degree 6 for $n = 12 + i$ and $i = 4,5,...$

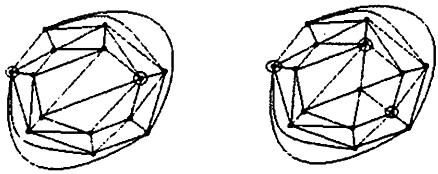

Figure 8: Example F and G

Note that in all such graphs, the number of vertices with degree five is 12. Let G be the graph with $n = 16$ and $r = 5$. It is known that G is unique, see Brinkman and McKay[3]. The maxplanar graph on 16 vertices has degree sum 84 but the degree sum of G is 80. Thus G is 4 degrees short of being maxplanar graph. Alternatively, G is two edges short of being maxplanar graph. Note that G has two quadrangles, draw two diagonals (shown as dash lines, see Figure 9) in these two quadrangles to get the desired graph on 16 vertices with 12 vertices of degree 5 and four vertices of degree 6. Call this graph as G_1. In G_1 the vertices of degree six are adjacent in pairs. See Figure 9. The graphs for $n > 16$ can be

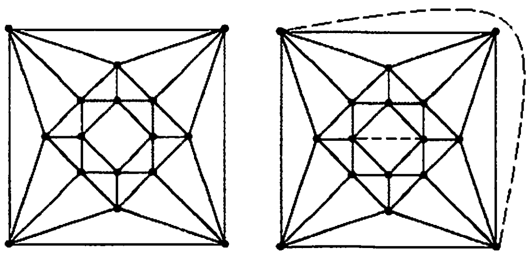

Figure 9: graph with n=16

constructed from G_1 of figure 9 as follows:
In the figure 10 (a) a subgraph on seven vertices of G_1 is shown where degree of vertex a is 6 and all other vertices are of degree 5. It is obvious from figure 9 that such a subgraph exists in G_1. We perform the set of operations of addtion and deletion on Figure 10(a) as described below:
In the Figure 10(a), a new vertex h is added on the edge ac, an edge ad is removed and new vertex h is connected to d and e. The resulting subgraph is shown in Figure 10(b).

This set of operations reduces the degree of a to 5 and increases degrees of b and e to 6. Thus we arrive at a subgraph on vertices a, c, d, e, f, g, h which is isomorphic to the subgraph in Figure 10(a). Adding a new vertex i in this subgraph would result in the figure 10(c), which contains the subgraph isomorphic to Figure 10(a). Repeating the set of operations described above, every time we add a new vertex to the subgraph we get a vertex of degree 6 keeping the degrees of other vertices in the graph G_1 as it is. Thus we can construct the required graph for any $n > 16$ by performing the described set of operations recursively.

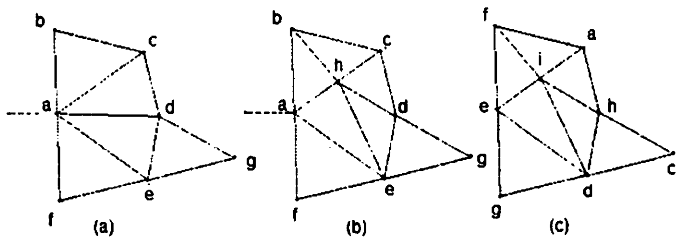

Figure 10: subgraph from figure 9

□

Introducing a further slight irregularity in the degree sequence gives us the following series of maxplanar graphs.

Theorem 3.15. *A planar graph with degree sequence* $(n-2, n-2, d, d, ..., d)$ *exists whenever* $n \geq d+1$ *for* $d = 2, 3, 4$ *and do not exist for* $d = 1, 5$. *Further, these graphs are maxplanar for* $d = 4$

Proof. Clearly to have a graph with least degree d, the graph must have at least $d+1$ vertices. For $d = 4$, consider a wheel with $(n-2)$ spokes, that means on the outer cycle there are $(n-2)$ vertices. Join these $(n-2)$ vertices with a new vertex of degree $(n-2)$ located in the exterior region of the wheel to get the required planar graph. For $d = 2$, delete the edges from the $(n-2)$-cycle of the graph constructed for $d = 4$. For $d = 3$, note that $n = 2t$ for some positive integer t. Delete alternate edges from the $(n-2)$-cycle of the graph for $d = 4$. For $d = 5$, the required planar graph does not exist because the number of edges exceeds the bound $3(n-2)$. For $d = 1$ the sequence is not even a graphic sequence. □

Theorem 3.16. *A maxplanar graph with degree sequence* $(n-7, 5, 5, 5..., 5)$ *does not exist except for* $n = 12$.

Proof. Suppose such a planar graph exists. Let C be the vertex with degree $n-7$. We divide the vertices into two groups. The first group containing $(n-7)$ vertices $x_1, x_2, ..., x_{n-7}$ which are connected to center C forming a wheel with $n-7$ spokes and the second group which containing the remaining six vertices, say, a, b, c, d, e, f, which are not connected to C. None of those six vertices are inside the triangles of the wheel as the graph we are considering is maxplanar and connected and so the degree of the center will increase, a contradiction.

As the graphs we consider are maxplanar and in a maxplanar graph $min\ d_i \geq 3$, $(n-7)$ has to be greater than or equal to 3.
Without loss of generality, assume the edges on the wheel are (x_i, x_{i+1}) for $i = 1, 2, ..., n-8$ and (x_{n-7}, x_1). There are no other edges among the vertices from the first group. If there were such edges, without loss of genenerality, choose the edge (x_i, x_j) where $j \neq (i+1)$ and there are no other edges (x_r, x_k) with $i < r < k < j$ and $(j-i) \geq 2$. This implies, by triangulation of the polygon $x_i, ...x_j, x_i$, that there exists a vertex x_k for some k which must be of degree less than 5 (in fact of degree 3), a contradiction, unless some of the six vertices are inside the polygon. In this case there are at least three of the six vertices inside the polygon $x_i, ...x_j, x_i$.
All six vertices can not be inside, because then we will have the polygon $x_j, x_{j+1}, ..., x_1, ...x_i, x_j$ and triangulation will give a vertex of degree 3 in the graph, a contradiction.
There can not be five vertices inside the polygon $x_i, ...x_j, x_i$, because then there will be one vertex, say a, outside the polygon. The vertex a must be adjacent to x_i and x_j because of triangulation and we arrive at similar situation where triangulation of a polygon will give us a vertex of degree 3 in the graph.
Now suppose there are exactly two vertices outside $x_i, ...x_j, x_i$ and 4 vertices inside. The four vertices can form a planar K_4 , but then the degree of all vertices can not be five without getting non-planar graph, at the most five edges can be drawn among these four vertices. Therefore to have degree five, $2n_1 \geq 8$ where $n_1 = j - i - 1$ and $n_1 \leq 10$ as two edges will emit from each of n_1 vertices to cover the degree sum 20 of four vertices. In any case, there is at least one vertex out of these four which is joined with three vertices x,y, and z from $x_i, ..., x_j$, but then one of the x,y or z will be of degree 4 or some of the vertex from $x_i, ..., x_j$-x,y,z will remain as degree three vertex. Now suppose there are exactly three vertices inside the polygon $x_i, ...x_j, x_i$, then $2n_1 + 2 \geq 9$, so $n_1 \geq 4$, therefore by same argument as in the case of 4 vertices, there will be a vertex from $x_i, ..., x_j$ which will not be of degree 5. This completes the argument for nonexistence of any other edges among the vertices on the cycle of the wheel.

Next, note that the $n-7$ vertices on the cycle is two degrees short of 5, there are $2(n-7)$ edges coming out of the wheel and joining the remaining six vertices and none of these six vertices are adjacent to more than 2 vertices on the cycle of the wheel. Second part of the last statement is true, because if one of the six vertices is adjacent to three or more vertices of the cycle of the wheel, one of the vertices on the cycle of the wheel can not be of degree 5 as required due to triangulation.
The sum of the degrees of the group of six vertices is 30. A vertex outside the wheel can not be connected to more than two vertices on the wheel. Thus there are at the most 12 edges, two emitting from each of six vertices. Thus these six vertices have to form an outer planar (all the vertices are exterior) graph with degrees three or four.
It can be seen that regular graphs with d=3 or d=4 on six vertices are not outer planar. Thus we are left with only one outer planar graph on six vertices, which is a wheel of size 5. As all vertices other than C are of degree 5, exactly 10 edges emit from this wheel to match up with 2(n-7) edges of the first wheel. Therefore 2(n-7)=10 and hence n=12. This completes the argument why a maxplanar with given degree sequence does not exist except for $n = 12$. □

Remark: The above theorem explains the nonexistence of the graphs with $n = 11$ and degree sequence $(4, 5, .., 5)$ and $n = 13$ with degree sequence $(6, 5, ..., 5)$

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