Directed Covering with Block Size 5 and v and λ Odd

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Abstract.
A directed covering design, $DC(v,k,\lambda)$, is a $(v,k,2\lambda)$ covering design in which the blocks are regarded as ordered k-tuples and in which each ordered pair of elements occurs in at least λ blocks. Let $DE(v,k,\lambda)$ denote the minimum number of blocks in a $DC(v,k,\lambda)$. In this paper the values of the function $DE(v,k,\lambda)$ are determined for all odd integers $v \geq 5$ and λ odd, with the exception of $(v,\lambda) = (53,1),(63,1),(73,1),(83,1)$. Further, we provide an example of a covering design that can not be directed.

1 Introduction

A transitively ordered k-tuple $(a_1,\ldots,a_k)$ is defined to be the set $\{(a_i,a_j) : 1 \leq i \leq j \leq k\}$. Let v, k, and λ be positive integers. A directed covering (packing) design, denoted by $DC(v,k,\lambda)$ ($DP(v,k,\lambda)$) is a pair (X,A) where X is a set of points and A is a collection of transitively ordered k-tuples of X, called blocks, such that every ordered pair of X appears in at least (at most) λ blocks. Let $DE(v,k,\lambda)$ ($DD(v,k,\lambda)$) denote the minimum (maximum) number of blocks in a $DC(v,k,\lambda)$ ($DP(v,k,\lambda)$). A $DC(v,k,\lambda)$ with $|A| = DE(v,k,\lambda)$ is called a minimum directed covering design and a $DP(v,k,\lambda)$ with $|A| = DD(v,k,\lambda)$ is called a maximum directed packing design. If we ignore the order of the blocks, a $DC(v,k,\lambda)$ ($DP(v,k,\lambda)$) is a standard $(v,k,2\lambda)$ covering (packing) design. Therefore, the following bounds, known as the Schonheim bounds, hold [22].

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$$DE(v,k,\lambda) \geq \lceil \frac{v}{k} \lceil \frac{v-1}{k-1} 2\lambda \rceil \rceil = DL(v,k,\lambda)$$

$$DD(v,k,\lambda) \leq \lfloor \frac{v}{k} \lfloor \frac{v-1}{k-1} 2\lambda \rfloor \rfloor = DU(v,k,\lambda)$$

Here $\lceil x \rceil$ is the smallest and $\lfloor x \rfloor$ is the largest integer satisfying $\lfloor x \rfloor \leq x \leq \lceil x \rceil$. The above bound has been sharpened in certain cases by Hanani [19].

Theorem 1.1. *(i) If* $2\lambda(v-1) \equiv 0$ *(mod* $(k-1)$*) and* $2\lambda v(v-1)/(k-1) \equiv -1$ *(mod* k*), then* $DE(v,k,\lambda) \geq DL(v,k,\lambda)+1$.

(ii) If $2\lambda v(v-1) \equiv 0$ *(mod* $(k-1)$*) and* $2\lambda v(v-1)/(k-1) \equiv 1$*(mod* k*), then* $DD(v,k,\lambda) \leq DU(v,k,\lambda)-1$.

Therefore, let $DE(v,k,\lambda) = DL(v,k,\lambda)+1$ if $2\lambda(v-1) \equiv 0$ (mod $(k-1)$) and $2\lambda v(v-1)/(k-1) \equiv -1$ (mod k) and $DE(v,k,\lambda) = DL(v,k,\lambda)$, otherwise.

When $DE(v,k,\lambda) = DL(v,k,\lambda)$, the directed covering design is called minimal. Similarly when $DD(v,k,\lambda) = DU(v,k,\lambda)$, the directed packing design is called optimal. A directed balanced incomplete block design, $DB[v,k,\lambda]$, is a $DC(v,k,\lambda)$ where every ordered pair of points appears in exactly λ blocks. If a $DB[v,k,\lambda]$ exists then it is clear that $DE(v,k,\lambda) = 2\lambda v(v-1)/(k-1) = DL(v,k,\lambda) = DD(v,k,\lambda)$. In the case $k=5$, Street and Wilson [26] have shown the following:

Theorem 1.2. *Let* λ *and* $v \geq 5$ *be positive integers. The necessary and sufficient conditions for the existence of a* $DB[v,k,\lambda]$ *are that* $(v,\lambda) \neq (15,1)$ *and that* $\lambda(v-1) \equiv 0$ *(mod 2) and* $\lambda v(v-1) \equiv 0$ *(mod 10).*

Lemma 1.1. *If there exists a* $DB[v,k,\lambda]$ *and* $DE(v,k,\lambda') = DL(v,k,\lambda')$ *then* $DE(v,k,\lambda+\lambda') = DL(v,k,\lambda+\lambda')$.

In [23]-[25], Skillicorn discussed the functions $DE(v,4,1)$ and $DD(v,4,1)$ and developed many other results including applications of directed designs to computer network and data flow machine architecture. Further, Assaf et al. [10] have determined the values of $DE(v,4,\lambda)$ and $DD(v,4,\lambda)$ for all positive integers v and λ. The values of $DE(v,5,\lambda)$ have been determined for all $v \geq 5$ and positive λ except for v and λ odd, [5, 6]. It is our purpose here to discuss the function $DE(v,5,\lambda)$ for every odd λ and odd $v \geq 5$.

Theorem 1.3. *Let $v \geq 5$ and λ be odd integers . Then $DE(v,5,\lambda) = DL(v,5,\lambda)$ with the exceptions of $DE(v,5,1) = DL(v,5,1)+1$ for $v = 9,13,15$ and the possible exceptions of $(v,\lambda) = (53,1),(63,1),(73,1),(83,1)$.*

2 Recursive Constructions

To describe our recursive constructions we need the notions of transversal designs, group divisible designs, pairwise balanced designs, for general information see [6]. We shall adopt the following notation: a (v,K,λ)-PBD stands for a pairwise balanced design of size v with index λ and block size from K. A $T[k,\lambda,m]$ stands for a transversal design with block size k, index λ and group size m. A (K,λ)-GDD stands for a group divisible design with block sizes from K and index λ. When $K = \{k\}$ we simply write k for K. The group type of a (K,λ)-GDD is a listing of the group sizes using exponential notations, i.e. $1^i 2^j 3^k ...$ denotes i groups of size 1, j groups of size 2, etc. Further, an incomplete pairwise balanced design with index λ denoted by (v,K,λ)-$IPBD(h)$ is a triple (X,H,A) where X is a finite set of order v, $H \subseteq X$ of order h and A is a collection of k-subsets of X, $k \in K$, called blocks, such that:

1) Each pair of distinct points $x,y \in X$ in which at least one of x and y does not lie in H occurs in exactly λ blocks.

2) No block contains two distinct points of H.

We would like to remark that the notions of transversal designs, group divisible designs, incomplete pairwise balanced design can be easily extended to the directed case. In the sequel we write DT, $DGDD$ and $IDPBD$ with the appropriate parameters.

The following theorem will be used extensively in this paper. The proof of this result may be found in [1]-[3], [14] -[16], [19], [21], [27].

Theorem 2.1. *There exists a $T[6,1,m]$ for all positive integers m, $m \notin \{2,3,4,6\}$ with the possible exception of $m \in \{10,14,18,22\}$.*

Another notion that is used in this paper is the notion of modified group divisible designs. Let k,λ,v and m be positive integers. A modified group divisible design (k,λ)-$MGDD$ of type m^n is a quadruple (V,β,γ,δ) where V is a set of points with $|V| = mn$, $\gamma = \{G_1,G_2,...,G_n\}$ is a partition of V into n sets, called groups, $\delta = \{R_1,R_2,...,R_m\}$ is a partition of V into m sets, called rows, and β is a family of k-subsets of V, called blocks, with the following properties:

1) $|B \cap G_i| \leq 1$ for all $B \in \beta$ and $G_i \in \gamma$.

2) $|B \cap R_i| \leq 1$ for all $B \in \beta$ and $R_i \in \delta$.
3) $|G_i| = m$ for all $G_i \in \gamma$.
4) $|G_i \cap R_j| = 1$ for all $G_i \in \gamma$ and $R_j \in \delta$.
5) Every 2-subset $\{x, y\}$ of V such that x and y are neither in the same group nor same row is contained in exactly λ blocks.

A resolvable *MGDD* (*RMGDD*) is one the blocks of which can be partitioned into parallel classes. It is clear that a $(5,1)$-*RMGDD* of type 5^m is the same as $RT[5,1,m]$ with one parallel class of blocks singled out, and since $RT[5,1,m]$ is equivalent to $T[6,1,m]$ we have the following existence theorem.

Theorem 2.2. *There exists a $(5,1)$-RMGDD of type 5^m for all positive integers m, $m \notin \{2,3,4,6\}$, with the possible exception of $m \in \{10,14,18,22\}$.*

The following theorem is a generalization of Theorem 2.6 of [8].

Theorem 2.3. *If there exist a $(5,1)$-RMGDD of type 5^m, a $(5,\lambda)$-DGDD of type $2^m s^1$, and a $(10+h, 5, \lambda)$-IDPBD(h), then there exists a $(10m + 2u + h + s, 5, \lambda)$-IDPBD$(2u + h + s)$, where $0 \leq u \leq m-1$.*

Lemma 2.1. *If there exists a $(v+h, k, \lambda)$-IDPBD(h) and $DE(h,k,\lambda) = DL(h,k,\lambda)$, then $DE(v+h,k,\lambda) = DL(v+h,k,\lambda)$.*

We would like to mention that, for large v, instead of constructing a $DC(v,5,\lambda)$ we will construct a $(v,5,\lambda)$-*IDPBD*(h), $h > 5$ and then on the hole we construct a $DC(h,5,\lambda)$. Finally, about the notations, a block of the form $\langle\ k \quad k+m \quad k+n \quad k+j \quad f(k)\ \rangle$ (mod v) where $f(k) = a$ if k is even and $f(k) = b$ if k is odd is denoted by $\langle\ 0\ m\ n\ j\ \rangle \cup \{a,b\}$. Further, a block $\langle 0 \quad m \quad f(k) \quad n \quad j \rangle$ mod (v) ,where $f(k) = a$ if k is even and $f(k) = b$ if k is odd, is denoted by $\langle 0 \quad m \quad - \quad n \quad j \rangle$.

3 Incomplete Pairwise Balanced Designs

In this section, we construct a $(v,5,1)$-*IDPBD*(k), then we invoke Lemma 2.1 to prove our result. We would like to mention that, the *IDPBD* designs for $v = 29, 77, 93$ are taken from [17], then we directed them and for $v = 23, 27, 33, 43, 113$ are from [20].

Lemma 3.1. *Let $v \equiv 7$ (mod 10), $v \geq 47$ be an integer. Then there exists a $(v,5,1)$-IDPBD(k) where $k = 7$ when $v \equiv 7$ (mod 20) and $k = 17$ when*

$v \equiv 17$ *(mod 20). Furthermore, there exists a* $(57,5,1)$-$IDPBD(7)$ *and* $(77,5,1)$-$IDPBD(19)$

Proof. For $v \equiv 7 \pmod{20}$, $v \geq 47$, the construction is as follows:
1) Take a $B[v-6,5,1]$ in increasing order.
2) Take a $(v+6,5,1)$-$IPBD(13)$ in decreasing order, [18]. Place the points $\{v+6, v+5, v+4, v+3, v+2, v+1\}$ at the end of the blocks in which they appear. Then replace $v+6$, by v, $v+5$ by $v-1$, $v+4$ by $v-2$, $v+3$ by $v-3$, $v+2$ by $v-4$, and $v+1$ by $v-5$. Then it is readily checked that the above two steps yield a $(v,5,1)$-$IDPBD(7)$ for $v \equiv 7 \pmod{20}$, $v \geq 47$.

For $v \equiv 17 \pmod{20}$, $v \geq 97$, the construction is as follows:
1) Take a $(v-8,5,1)$-$IPBD(9)$ in decreasing order, [18].
2) Take a $B[v+8,5,1]$ with a hole of size 25 in increasing order, [12]. Further, place the points $\{v+8, v+7, v+6, v+5, v+4, v+3, v+2, v+1\}$ at the end of the blocks in which they appear. Then replace $v+8$ by v, $v+7$ by $v-1$, $v+6$ by $v-2$, $v+5$ by $v-3$, $v+4$ by $v-4$, $v+3$ by $v-5$, $v+2$ by $v-6$, $v+1$ by $v-7$. Then it is readily checked that the above two steps yield the blocks of a $(v,5,1)$-$IDPBD(17)$ for $v \equiv 17 \pmod{20}$, $v \geq 97$.

For $v = 57$, apply Theorem 2.3 with $u = 3$, $s = 0$, $m = 5$ and $h = 1$.

For $v = 77$, let $X = \mathbb{Z}_3 \times \mathbb{Z}_{19} \cup \{\infty\} \cup H_{19}$, we first construct a 2-$RB[58,4,2]$, then for each parallel class we adjoin a new point at the indicated place. The parallel classes are the following blocks mod $(-,19)$.

$\langle -\ \infty(0,0)(1,0)(2,0)\rangle$ $\langle(2,0)(1,0)(0,0)\infty\ -\rangle$.
$\langle(0,1)(1,17)-(1,2)(0,18)\rangle$ $\langle(1,1)(2,17)-(2,2)(1,18)\rangle$
$\langle(2,1)(0,17)-(0,2)(2,18)\rangle$. $\langle(1,7)(0,6)-(0,13)(1,12)\rangle$
$\langle(2,7)(1,6)-(1,13)(2,12)\rangle$ $\langle(0,7)(2,6)-(2,13)(0,12)\rangle$.
$\langle(1,13)(0,3)-(0,16)(1,6)\rangle$ $\langle(2,13)(1,3)-(1,16)(2,6)\rangle$
$\langle(0,13)(2,3)-(2,16)(0,6)\rangle$. $\langle(0,2)(1,4)-(1,15)(0,17)\rangle$
$\langle(1,2)(2,4)-(2,15)(1,17)\rangle$ $\langle(2,2)(0,4)-(0,15)(2,17)\rangle$.
$\langle(0,10)(1,18)-(1,1)(0,9)\rangle$ $\langle(1,10)(2,18)-(2,1)(1,9)\rangle$
$\langle(2,10)(0,18)-(0,1)(2,9)\rangle$. $\langle(1,14)(0,12)-(0,7)(1,5)\rangle$
$\langle(2,14)(1,12)-(1,7)(2,5)\rangle$ $\langle(0,14)(2,12)-(2,7)(0,5)\rangle$.
$\langle(1,16)(0,11)-(0,8)(1,3)\rangle$ $\langle(2,16)(1,11)-(1,8)(2,3)\rangle$
$\langle(0,16)(2,11)-(2,8)(0,3)\rangle$. $\langle(1,8)(0,15)-(0,4)(1,11)\rangle$
$\langle(2,8)(1,15)-(1,4)(2,11)\rangle$ $\langle(0,8)(2,15)-(2,4)(0,11)\rangle$.
$\langle(0,5)(1,9)-(1,10)(0,14)\rangle$ $\langle(1,5)(2,9)-(2,10)(1,14)\rangle$
$\langle(2,5)(0,9)-(0,10)(2,14)\rangle$.

□

Lemma 3.2. *Let $v \equiv 9$ (mod 10), $v \geq 29$ be an integer. Then there exists:*
i) A $(v, 5, 1)$-$IDPBD(29)$ for $v \equiv 9$ (mod 20), $v \geq 129$,
ii) A $(v, 5, 1)$-$IDPBD(19)$ for $v \equiv 19$ (mod 20), $v \geq 99$,
iii) A $(v, 5, 1)$-$IDPBD(7)$ for $v = 29, 39, 49, 59, 79$, a $(v, 5, 1)$-$IDPBD(17)$ for $v = 69, 89$, and a $(109, 5, 1)$-$IDPBD(19)$.

Proof. For the case (i); see [13].

For $v \equiv 19$ (mod 20), $v \geq 99$, the construction is as follows:
1) Take a $(v - 6, 5, 1)$-$IPBD(13)$ in increasing order, [18].
2) Take a $B[v + 6, 5, 1]$ with a hole of size 25 [12] in increasing order. Further, place the points $v + 6$, $v + 5$, $v + 4$, $v + 3$, $v + 2$, $v + 1$ at the end of the blocks in which they appear. Then replace $v + 6$ by v, $v + 5$ by $v - 1$, $v + 4$ by $v - 2$, $v + 3$ by $v - 3$, $v + 2$ by $v - 4$, and $v + 1$ by $v - 5$. Then it is readily checked that the above two steps yield the blocks of a $(v, 5, 1)$-$IDPBD(19)$, where the hole is $\{v - 18, v - 17, \ldots, v\}$.

For $v = 29$, let $X = \mathbb{Z}_3 \times \mathbb{Z}_7 \cup \{\infty\} \cup H_7$, we first construct a 2-$RB[22, 4, 2]$, then for each parallel class we adjoin a new point at the indicated place. The parallel classes are the following blocks mod$(-, 7)$.

$\langle \infty(0,0)(1,0)(2,0)- \rangle$ $\langle -(2,0)(1,0)(0,0)\infty \rangle$.
$\langle (0,6)(1,5) - (1,2)(0,1) \rangle$ $\langle (1,4)(0,5) - (0,2)(1,3) \rangle$.
$\langle (0,4)(1,1) - (1,6)(0,3) \rangle$ $\langle (1,6)(2,5) - (2,2)(1,1) \rangle$.
$\langle (1,2)(2,4) - (2,3)(1,5) \rangle$ $\langle (2,6)(1,3) - (1,4)(2,1) \rangle$.
$\langle (0,2)(2,1) - (2,6)(0,5) \rangle$ $\langle (2,2)(0,3) - (0,4)(2,5) \rangle$.
$\langle (2,3)(0,1) - (0,6)(2,4) \rangle$.

For $v = 39, 49, 59, 79, 89$, see the table below.

For $v = 69$, if we take a $RB[52, 4, 1]$ and adjoin a point to each of its parallel classes, we obtain a $(69, 5, 1)$-$IPBD(17)$. Then, by taking two copies of the $(69, 5, 1)$-$IPBD(17)$ in opposite order, we obtain a $(69, 5, 1)$-$IDPBD(17)$.

For $v = 109$, apply Theorem 2.3 with $h = 1$ and $m = 9$, $s = 2$, $u = 8$.

v	Point set	Base Blocks
39	$\mathbb{Z}_{32} \cup H_7$	⟨ 24 10 4 0 31 ⟩ ⟨ 0 25 - 10 13 ⟩ $\cup \{h_1, h_2\}$ ⟨ 0 11 - 15 2 ⟩ $\cup \{h_3, h_4\}$ ⟨ 0 5 - 29 6 ⟩ $\cup \{h_5, h_6\}$ ⟨ 2 18 h_7 0 16 ⟩ orbit length 2 ⟨ 3 19 h_7 17 1 ⟩ orbit length 14
49	$\mathbb{Z}_2 \times \mathbb{Z}_{21}$ $\cup H_7$	$\langle (0,0)(1,1)h_1(1,2)(0,1) \rangle$ $\langle (1,17)(0,0)h_2(0,2)(1,0) \rangle$ $\langle (0,0)(1,7)h_3(1,15)(0,3) \rangle$ $\langle (1,11)(0,0)h_4(1,17)(0,4) \rangle$ $\langle (0,0)(1,10)h_5(0,7)(1,12) \rangle$ $\langle (1,16)(0,0)h_6(0,8)(1,11) \rangle$ $\langle (0,0)(1,6)h_7(0,9)(1,18) \rangle$ $\langle (0,5)(0,11)(0,3)(1,19)(0,0) \rangle$ $\langle (0,10)(1,14)(0,4)(0,9)(0,0) \rangle$ $\langle (1,9)(0,7)(1,6)(1,20)(0,0) \rangle$ $\langle (1,4)(1,11)(1,0)(1,9)(1,3) \rangle$
59	$\mathbb{Z}_{52} \cup H_7$	⟨16 4 0 28 20⟩ ⟨13 44 0 19 14⟩ ⟨9 3 14 0 32⟩ ⟨45 2 - 0 35⟩ $\cup \{h_1, h_2\}$ ⟨0 37 - 2 15⟩ $\cup \{h_3, h_4\}$ ⟨42 11 - 0 45⟩ $\cup \{h_5, h_6\}$ ⟨27 1 h_7 0 26⟩ orbit length 25 ⟨26 0 h_7 25 51⟩ orbit length 1
79	$\mathbb{Z}_{72} \cup H_7$	⟨ 4 0 h_1 36 40 ⟩ orbit length 4 ⟨ 8 40 h_1 4 44 ⟩ orbit length 32 ⟨ 5 0 - 28 45 ⟩ $\cup \{h_2, h_3\}$ ⟨ 23 8 - 0 39 ⟩ $\cup \{h_4, h_5\}$ ⟨ 24 11 - 0 53 ⟩ $\cup \{h_6, h_7\}$ ⟨ 3 7 21 0 1 ⟩ ⟨ 0 7 37 19 9 ⟩ ⟨ 16 0 15 21 41 ⟩ ⟨ 38 26 48 9 0 ⟩ ⟨ 25 0 3 11 38 ⟩
89	$\mathbb{Z}_{72} \cup H_{17}$	⟨ 0 36 h_{17} 38 2 ⟩ orbit length 34 ⟨ 34 70 h_{17} 36 0 ⟩ orbit length 2 ⟨ 32 20 14 4 0 ⟩ ⟨ 8 35 30 56 0 ⟩ ⟨ 3 38 - 0 15 ⟩ $\cup\{h_1, h_2\}$ ⟨ 0 9 - 33 20 ⟩ $\cup\{h_3, h_4\}$ ⟨ 1 0 - 2 5 ⟩ $\cup\{h_5, h_6\}$ ⟨ 0 55 - 6 13 ⟩ $\cup\{h_7, h_8\}$ ⟨ 7 0 - 50 25 ⟩ $\cup\{h_9, h_{10}\}$ ⟨ 27 0 - 41 8 ⟩ $\cup\{h_{11}, h_{12}\}$ ⟨ 9 40 - 0 19 ⟩ $\cup\{h_{13}, h_{14}\}$ ⟨ 26 55 - 11 0 ⟩ $\cup\{h_{15}, h_{16}\}$

□

Lemma 3.3. *There exists a* $(23,5,1)$-*IDPBD*(3).

Proof. The construction consists of the following two steps:
1) Take a $B[21,5,1]$ in increasing order.
2) Take a $B[25,5,1]$ in decreasing order and delete the block ⟨ 25 24 23 22 21 ⟩. Further place the points 24 and 25 at the end of the blocks in which they appear. Then replace 25 by 23 and 24 by 22.

□

Lemma 3.4. *Let $v \equiv 3$ (mod 10) be an integer. Then*
1) There exists a $(v,5,1)$-$IDPBD(33)$ when $v \equiv 13$ (mod 20), $v \geq 133$.
2) There exists a $(v,5,1)$-$IDPBD(23)$ when $v \equiv 3$ (mod 20), $v \geq 93$.

Proof. For $v \equiv 13$ (mod 20), $v \geq 133$, see [13].
For $v = 93$, let $X = Z_3 \times Z_{23} \cup \{\infty\} \cup H_{23}$. we first construct a 2-$RB[70,4,2]$, then for each parallel class we adjoin a new point at the indicated place. The parallel classes are the following blocks mod $(-,23)$.

$\langle -\ \infty(0,0)(1,0)(2,0)\rangle$ $\langle(2,0)(1,0)(0,0)\infty\ -\rangle$
$\langle(0,1)(1,2)-(1,21)(0,22)\rangle$ $\langle(1,1)(2,2)-(2,21)(1,22)\rangle$
$\langle(2,1)(0,2)-(0,21)(2,22)\rangle$ $\langle(0,8)(1,16)-(1,7)(0,15)\rangle$
$\langle(1,8)(2,16)-(2,7)(1,15)\rangle$ $\langle(2,8)(0,16)-(0,7)(2,15)\rangle$
$\langle(0,3)(1,17)-(1,6)(0,20)\rangle$ $\langle(1,3)(2,17)-(2,6)(1,20)\rangle$
$\langle(2,3)(0,17)-(0,6)(2,20)\rangle$ $\langle(0,18)(1,13)-(1,10)(0,5)\rangle$
$\langle(1,18)(2,13)-(2,10)(1,5)\rangle$ $\langle(2,18)(0,13)-(0,10)(2,5)\rangle$
$\langle(0,9)(1,5)-(1,18)(0,14)\rangle$ $\langle(1,9)(2,5)-(2,18)(1,14)\rangle$
$\langle(2,9)(0,5)-(0,18)(2,14)\rangle$ $\langle(1,12)(0,6)-(0,17)(1,11)\rangle$
$\langle(2,12)(1,6)-(1,17)(2,11)\rangle$ $\langle(0,12)(2,6)-(2,17)(0,11)\rangle$
$\langle(1,15)(0,4)-(0,19)(1,8)\rangle$ $\langle(2,15)(1,4)-(1,19)(2,8)\rangle$
$\langle(0,15)(2,4)-(2,19)(0,8)\rangle$ $\langle(0,21)(1,19)-(1,4)(0,2)\rangle$
$\langle(1,21)(2,19)-(2,4)(1,2)\rangle$ $\langle(2,21)(0,19)-(0,4)(2,2)\rangle$
$\langle(0,11)(1,22)-(1,1)(0,12)\rangle$ $\langle(1,11)(2,22)-(2,1)(1,12)\rangle$
$\langle(2,11)(0,22)-(0,1)(2,12)\rangle$ $\langle(0,7)(1,14)-(1,9)(0,16)\rangle$
$\langle(1,7)(2,14)-(2,9)(1,16)\rangle$ $\langle(2,7)(0,14)-(0,9)(2,16)\rangle$
$\langle(0,10)(1,20)-(1,3)(0,13)\rangle$ $\langle(1,10)(2,20)-(2,3)(1,13)\rangle$
$\langle(2,10)(0,20)-(0,3)(2,13)\rangle$

For $v = 113$, let $W = \{[j,h], j \in \mathbb{Z}_2, h \in \mathbb{Z}_9\} \cup \{[g], g \in \mathbb{Z}_5\}$ and $V = \{(j,i) : j \in \mathbb{Z}_2, i \in \mathbb{Z}_{45}\} \cup W$. Let σ be the mapping given by $\sigma((j,i)) = (j,i+1)$, $\sigma([j,h]) = [j,h]$ and $\sigma([g]) = [g+1]$. And τ be the mapping given by $\tau((j,i)) = (j+1,i)$, $\tau([j,h]) = [j+1,h]$ and $\tau[g] = [g]$. G is the group generated by $\langle\sigma,\tau\rangle$. The base blocks are the following:

$\langle(1,5)(0,5)[0](0,0)(1,0)\rangle$ $\langle(0,12)(1,3)[0](0,1)(1,14)\rangle$
$\langle(0,1)(1,4)[0](0,13)(0,22)\rangle$ $\langle(1,4)(0,0)[0,0](1,19)(0,1)\rangle$
$\langle(1,20)(0,1)[0,1](0,0)(1,5)\rangle$ $\langle(1,1)(0,0)[0,2](1,23)(0,2)\rangle$
$\langle(0,2)(1,24)[0,3](0,0)(1,8)\rangle$ $\langle(0,0)(1,10)[0,4](0,3)(1,14)\rangle$
$\langle(0,3)(1,10)[0,5](0,0)(1,34)\rangle$ $\langle(1,16)(0,4)[0,6](0,0)(1,32)\rangle$
$\langle(0,0)(1,30)[0,7](0,5)(1,17)\rangle$ $\langle(0,14)(1,6)[0,8](1,29)(0,0)\rangle$
$\langle(0,0)(0,25)(0,17)(0,8)(0,35)\rangle$ $\langle(0,6)(0,12)(0,26)(0,19)(0,0)\rangle$

By applying the maps $\{\sigma^0,\sigma^1,...,\sigma^{44}\}$ on the first base block , we will get 45 blocks. And by applying the elements of $G = \{\sigma^0, \sigma^1, ..., \sigma^{44}, \tau, \tau \circ \sigma^1, ..., \tau \circ \sigma^{44}\}$ on the other 13 base blocks, we will get 90 blocks from each base block. So we will have a total of 1215 blocks.

For $v \equiv 3 \pmod{20}$, $v \geq 3$, take a $(5,1)$ -*DGDD* of type 20^k where $k = \frac{(v-3)}{20}$, see [11], add three points to the groups and on all the groups except one we construct a $(23,5,1)$-$IDPBD(3)$, then take these three points with that group to be the hole.

□

4 Directed Covering With Index One

Lemma 4.1. *Let* $v \equiv 7 \pmod{10}$ *be a positive integer. Then* $DE(v,5,1) = DL(v,5,1)$.

Proof. In view of Lemma 2.1, Lemma 3.1, and Lemma 4.1, we only need to consider the cases $v = 7, 17, 27, 37$.
For v =7, let $X = \mathbb{Z}_7$. Then the blocks are
$\langle 2\,5\,3\,1\,4 \rangle$, $\langle 6\,4\,5\,2\,1 \rangle$,
$\langle 3\,2\,5\,6\,0 \rangle$, $\langle 4\,0\,1\,6\,3 \rangle$,
$\langle 1\,0\,2\,5\,4 \rangle$.

For $v = 17$, let $X = \mathbb{Z}_3 \times \mathbb{Z}_5 \cup \{a,b\}$. Then the construction is as follows:
1) On $\mathbb{Z}_3 \times \mathbb{Z}_5$ construct a $(5,1)$-*DGDD* of type 3^5, [12].
2) On each group with $\{a,b\}$ we construct a $DB[5,5,1]$.
Then it is easily checked that the above construction yields a minimal $DC(17,5,1)$.

For $v = 27$, let $X = Z_5 \times Z_5 \cup \{a,b\}$. Then the blocks are.

$\langle (0,4)(0,3)(0,2)(0,1)(0,0) \rangle$
$\langle (3,4)(0,0)(0,4)(4,4)(4,1) \rangle$
$\langle (3,0)(0,0)(0,1)(4,0)(4,2) \rangle + (-,i), i \in \mathbb{Z}_4$
$\langle (4,3)(2,0)(0,0)(0,2)(3,0) \rangle + (-,i), i \in \mathbb{Z}_3$
$\langle (4,1)(2,3)(0,0)(0,3)(3,3) \rangle + (-,i), i \in \mathbb{Z}_2$
$\langle (0,0)(2,1)(2,2)(3,4)(2,0) \rangle$ mod $(-,5)$
$\langle (2,1)(4,0)(0,0)(1,0)(2,3) \rangle$ mod $(-,5)$
$\langle (4,2)(1,0)(3,3)(0,0)(1,1) \rangle$ mod $(-,5)$
$\langle (1,1)(3,2)(3,1)(0,0)(1,3) \rangle$ mod $(-,5)$
$\langle (1,3)(1,2)(4,4)(0,0)(2,4) \rangle$ mod $(-,5)$
$\langle (2,2)(0,0)(1,4)(1,2)(3,1) \rangle$ mod $(-,5)$
$\langle (3,3)(3,0)(4,1)(3,1)(2,0) \rangle$ mod $(-,5)$

$\langle(2,2)(1,0)(4,0)(4,3)(4,4)\rangle$ mod $(-,5)$
$\langle(2,4)(3,0)(4,4)(1,0)b\rangle$ mod $(-,5)$
$\langle(4,3)(3,1)a(1,0)(2,4)\rangle$ mod $(-,5)$
$\langle\, b\ (1,4)(2,4)\ a\ (0,0)\rangle$ mod $(-,5)$
$\langle(0,0)\ a\ b\ (4,3)(3,2)\rangle$ mod $(-,5)$.

For $v = 37$, let $X = \mathbb{Z}_7 \times \mathbb{Z}_5 \cup \{x, y\}$. Then the blocks are

$\langle(i,0)(i,1)(i,2)(i,3)(i,4)\rangle$ for $i = 0, 1$.
$\langle(i,4)(i,3)(i,2)(i,1)(i,0)\rangle$ for $i = 0, 1$.

And the following blocks are of mod $(-,5)$

$\langle(4,1)(2,0)(3,0)(6,1)(5,2)\rangle$	$\langle(6,2)(3,0)(4,2)(5,4)(2,0)\rangle$
$\langle(1,0)(6,0)(3,0)(3,2)(6,4)\rangle$	$\langle(3,0)(6,3)(6,0)(3,4)(1,0)\rangle$
$\langle(1,0)(2,0)(3,1)(5,4)(5,1)\rangle$	$\langle(5,2)(3,2)(2,0)(1,0)(5,3)\rangle$
$\langle(4,1)(1,0)(2,2)(4,4)(2,1)\rangle$	$\langle(2,2)(1,0)(2,4)(4,3)(4,2)\rangle$
$\langle(0,0)(3,0)(4,3)(4,0)(3,1)\rangle$	$\langle(4,0)(3,2)(3,0)(4,1)(0,0)\rangle$
$\langle(0,0)(5,0)(5,3)(3,4)(2,0)\rangle$	$\langle(5,1)(2,0)(5,0)(3,3)(0,0)\rangle$
$\langle(0,0)(2,1)(2,4)(6,3)(6,4)\rangle$	$\langle(6,3)(6,1)(2,2)(2,3)(0,0)\rangle$
$\langle(0,0)(6,1)(5,4)(4,2)(1,0)\rangle$	$\langle(4,4)(5,3)(1,0)(6,2)(0,0)\rangle$
$\langle(1,1)(0,0)(4,1)(5,1)(6,2)\rangle$	$\langle(5,2)(4,2)(6,4)(0,0)(1,2)\rangle$
$\langle(0,0)(5,2)(6,0)(4,4)(1,1)\rangle$	$\langle(1,2)(6,0)(5,4)(4,3)(0,0)\rangle$
$\langle(1,4)(0,0)(3,3)(2,2)\ x\,\rangle$	$\langle\, x\ (2,4)(1,3)(3,1)(0,0)\rangle$
$\langle(0,0)(2,3)(3,2)(1,4)\ y\,\rangle$	$\langle\, y\ (3,4)(2,1)(0,0)(1,3)\rangle$
$\langle(5,4)(4,0)(6,4)\ x\ y\,\rangle$	$\langle\, y\ x\ (6,3)(4,0)(5,3)\rangle$

□

Lemma 4.2. *Let* $v \equiv 9$ *(mod* 10*),* $v \geq 19$ *be an integer. Then* $DE(v,5,1) = DL(v,5,1)$. *Further,* $DE(9,5,1) = DL(9,5,1) + 1$.

Proof. In view of Lemma 2.1 and Lemma 3.2, we only need to consider the cases $v = 9$ and 19.

For $v = 9$, from [20] we know that $DE(9,5,1) = DL(9,5,1) + 1$. Let $X = \mathbb{Z}_9$ then the blocks are $\langle\ 0\ 5\ 4\ 1\ 3\ \rangle$ (mod 9)

For $v = 19$, let $X = \mathbb{Z}_5 \times \mathbb{Z}_3 \cup \{w, x, y, z\}$. Then the blocks are

$\langle(0,0)(3,1)(0,1)(2,2)(1,2)\rangle$ mod $(-,3)$
$\langle(4,0)(2,1)(1,2)(0,1)(0,0)\rangle$ mod $(-,3)$
$\langle(2,2)(0,0)(4,1)(3,2)\ z\,\rangle$ mod $(-,3)$
$\langle(0,0)(4,0)\ y\ (4,2)(1,0)\rangle$ mod $(-,3)$
$\langle(2,1)(3,0)\ y\ (1,0)(3,2)\rangle$ mod $(-,3)$

$\langle(3,2)(4,1)(0,0)\ x\ (3,0)\rangle \bmod (-,3)$
$\langle(3,1)(1,0)(0,0)\ (2,0)\ w\rangle \bmod (-,3)$
$\langle(4,0)\ w\ (3,0)(4,1)(2,0)\rangle \bmod (-,3)$
$\langle\ z\ (1,0)(1,1)(3,1)(4,1)\rangle$ $\langle(1,1)\ z\ (1,2)(3,2)(4,2)\rangle$
$\langle(1,0)(1,2)\ z\ (3,0)(4,0)\rangle$ $\langle(1,0)(2,1)(2,2)\ x\ (4,2)\rangle$
$\langle(1,1)\ x\ (2,0)(2,2)(4,0)\rangle$ $\langle(1,2)(2,0)\ x\ (2,1)(4,1)\rangle$
$\langle w\ \ x\ \ y\ \ z\ \ (0,0)\rangle$ $\langle z\ \ y\ \ x\ \ w\ \ (0,1)\rangle$
$\langle w\ \ x(1,2)(1,1)(1,0)\rangle$ $\langle y\ \ z(2,2)(2,1)(2,0)\rangle$
$\langle z\ \ y\ \ x\ \ w\ \ (0,2)\rangle$

□

Lemma 4.3. *Let $v \equiv 3 \pmod{10}$, $v \geq 13$ be an integer. Then $DE(v,5,1) = DL(v,5,1)$ with the possible exception of $v = 53, 63, 73, 83$. Further $DE(13,5,1) = DL(13,5,1) + 1$.*

Proof. Again, we only need to consider the cases where $v = 13, 23, 33, 43$. For $v = 13$, from [20] we know that $DE(13,5,1) = DL(13,5,1) + 1$. To construct such design, let $X = \mathbb{Z}_{13}$. Then the blocks, are

⟨0 1 2 3 4⟩ ⟨6 5 2 1 0⟩ ⟨7 3 0 8 5⟩
⟨8 4 0 6 7⟩ ⟨0 9 10 11 12⟩ ⟨12 11 10 9 0⟩
⟨3 7 9 1 10⟩ ⟨4 1 5 8 9⟩ ⟨1 7 6 11 12⟩
⟨12 11 8 10 1⟩ ⟨8 3 2 11 12⟩ ⟨10 9 7 4 2⟩
⟨11 12 2 5 7⟩ ⟨2 6 10 9 8⟩ ⟨12 11 6 4 3⟩
⟨9 10 5 3 6⟩ ⟨5 4 12 11 10⟩

For $v = 23$, let $X = \mathbb{Z}_7 \times \mathbb{Z}_3 \cup \{x, y\}$. Then the blocks, taken mod $(-,3)$, are

$\langle(6,0)(4,1)(0,0)(0,1)(2,0)\rangle$ $\langle(6,2)(1,1)(1,0)(4,1)(3,1)\rangle$
$\langle(0,0)(1,0)(2,1)(4,2)(6,2)\rangle$ $\langle(5,0)(2,1)(3,2)(1,0)(2,2)\rangle$
$\langle(1,1)(0,0)(5,2)(3,0)(5,0)\rangle$ $\langle(0,1)y(1,0)\ x\ (0,0)\rangle$
$\langle(3,2)(0,0)(1,1)(5,1)(6,1)\rangle$ $\langle\ x\ (2,0)(3,0)(1,0)\ y\ \rangle$
$\langle(5,1)(4,2)(5,0)(1,2)(0,0)\rangle$ $\langle(2,0)(3,2)(5,0)\ x\ (4,0)\rangle$
$\langle(2,1)(2,0)(5,2)(0,0)(6,0)\rangle$ $\langle(6,0)(4,0)\ x\ (6,1)(5,2)\rangle$
$\langle(3,1)(3,0)(6,1)(2,2)(0,0)\rangle$ $\langle(4,0)\ y\ (5,0)(2,0)(4,2)\rangle$
$\langle(0,0)(3,1)(4,0)(4,1)(3,2)\rangle$ $\langle(6,0)(5,0)\ y\ (6,2)(3,0)\rangle$
$\langle(4,2)(1,0)(6,1)(2,0)(1,1)\rangle$

For $v = 33$, let $X = \mathbb{Z}_{11} \times \mathbb{Z}_3$. Then the blocks, taken mod $(-,3)$, are

$\langle(5,0)(0,1)(0,0)(6,0)(5,1)\rangle$ $\langle(6,2)(0,0)(7,0)(5,2)(0,1)\rangle$
$\langle(0,0)(3,1)(2,2)(1,0)(4,2)\rangle$ $\langle(7,0)(0,0)(6,1)(2,1)(1,1)\rangle$

$\langle(0,0)(8,0)(2,0)(1,2)(8,1)\rangle$ $\langle(0,0)(4,0)(3,2)(9,0)(9,1)\rangle$
$\langle(0,0)(10,0)(9,2)(7,1)(3,0)\rangle$ $\langle(8,1)(10,0)(0,0)(4,1)(10,2)\rangle$
$\langle(6,0)(10,1)(10,2)(9,2)(0,0)\rangle$ $\langle(7,1)(9,0)(8,0)(8,2)(0,0)\rangle$
$\langle(9,1)(0,0)(7,2)(10,1)(8,2)\rangle$ $\langle(3,2)(5,1)(1,0)(8,0)(9,2)\rangle$
$\langle(8,2)(9,0)(1,0)(6,0)(3,0)\rangle$ $\langle(1,0)(4,1)(9,0)(6,2)(6,1)\rangle$
$\langle(4,1)(8,0)(1,0)(7,2)(7,1)\rangle$ $\langle(5,0)(5,2)(9,2)(1,0)(10,1)\rangle$
$\langle(7,0)(7,1)(1,0)(9,1)(5,1)\rangle$ $\langle(10,1)(6,1)(6,2)(1,0)(8,1)\rangle$
$\langle(2,0)(8,0)(9,0)(5,2)(3,2)\rangle$ $\langle(2,0)(3,0)(10,0)(5,1)(8,2)\rangle$
$\langle(2,0)(6,2)(3,1)(7,1)(10,2)\rangle$ $\langle(7,0)(3,1)(10,0)(6,2)(2,0)\rangle$
$\langle(9,2)(5,0)(2,0)(7,2)(4,2)\rangle$ $\langle(10,1)(2,0)(5,0)(4,1)(7,0)\rangle$
$\langle(8,2)(10,2)(4,1)(5,2)(2,0)\rangle$ $\langle(8,1)(6,1)(7,1)(4,2)(2,0)\rangle$
$\langle(5,2)(7,0)(3,0)(6,0)(4,2)\rangle$ $\langle(6,2)(4,2)(8,0)(5,1)(3,0)\rangle$
$\langle(4,0)(3,0)(5,0)(8,0)(6,2)\rangle$
$\langle(10,2)(1,2)(3,0)(3,1)(7,2)\rangle$ orbit length 2
$\langle(1,0)(1,1)(5,0)(2,2)(10,0)\rangle$ orbit length 2
$\langle(9,2)(4,0)(4,1)(10,0)(1,1)\rangle$ orbit length 2
$\langle(9,1)(2,0)(2,1)(6,1)(6,2)\rangle$ orbit length 2

Together with the following single blocks

$\langle(10,1)(1,1)(3,0)(3,2)(7,1)\rangle$ $\langle(1,0)(1,2)(5,2)(2,1)(10,2)\rangle$
$\langle(9,1)(4,0)(4,2)(10,2)(1,0)\rangle$ $\langle(9,0)(2,0)(2,2)(6,0)(6,1)\rangle$
$\langle(1,2)(1,1)(1,0)(0,1)(0,0)\rangle$ $\langle(2,2)(2,1)(2,0)(0,1)(0,0)\rangle$
$\langle(3,2)(3,1)(3,0)(0,1)(0,0)\rangle$ $\langle(4,2)(4,1)(4,0)(0,1)(0,0)\rangle$
$\langle(3,0)(1,0)(4,0)(2,0)(0,2)\rangle$ $\langle(3,1)(1,1)(4,1)(2,1)(0,2)\rangle$
$\langle(3,2)(1,2)(4,2)(2,2)(0,2)\rangle$

For $v = 43$, let $X = \mathbb{Z}_{13} \times \mathbb{Z}_3 \cup \{w, x, y, z\}$. To shorten the list of blocks we adopt the following notations: If a pair of points in a block are highlighted, say, (a,i) (a,j), then when develop the block mod $(-,3)$ and $j = 2$ and $i = 0$ we write $(a,2)$ $(a,0)$, that is, we write (a,j) before (a,i) in the block. Further, if w, x, y, or z appear in such block, in addition to the previous notation, the letter should appear in the middle in the case of $(a,2)$ $(a,1)$, on the left of both in the case of $(a,2)$ $(a,0)$, and on the right of both in the case of $(a,1)$ $(a,0)$. Now, the blocks are the following mod $(-,3)$.

$\langle(0,0)(1,0)(2,0)(3,0)(4,0)\rangle$ $\langle(5,0)\mathbf{(4,1)\ (4,0)}(1,1)(0,0)\rangle$
$\langle(0,0)(1,1)(5,0)(2,2)(6,0)\rangle$ $\langle(5,1)(6,0)(2,1)(0,0)(7,0)\rangle$
$\langle(0,0)(2,1)(1,2)\mathbf{(9,1)\ (9,0)}\rangle$ $\langle(3,1)(0,0)(12,1)\mathbf{(8,2)\ (8,1)}\rangle$
$\langle(7,0)(0,0)(3,1)(10,0)(5,1)\rangle$ $\langle(6,2)(8,1)\mathbf{(1,1)\ (1,0)}(7,1)\rangle$
$\langle(0,0)(8,0)(10,1)(7,2)(3,2)\rangle$ $\langle(12,1)(4,0)(9,2)\mathbf{(2,1)\ (2,0)}\rangle$
$\langle(0,0)(4,1)(11,0)(5,2)(10,2)\rangle$ $\langle(6,2)(5,2)\mathbf{(3,1)\ (3,0)}(7,0)\rangle$

$\langle(8,1)(10,1)(11,2)(0,0)(4,2)\rangle \quad \langle(9,1)(10,1)(6,2)(5,1)(4,0)\rangle$
$\langle(11,1)(8,0)(9,0)(6,2)(0,0)\rangle \quad \langle(5,0)(9,0)(8,2)(10,1)(7,0)\rangle$
$\langle(10,2)(6,1)(0,0)(11,1)(12,0)\rangle \quad \langle(12,0)(0,0)\mathbf{(6,2)}\ \mathbf{(6,1)}\ y\rangle$
$\langle(7,2)(0,0)(9,2)(12,2)(11,2)\rangle \quad \langle(10,0)(11,0)y(9,1)(0,0)\rangle$
$\langle(3,1)(1,0)(12,0)(6,0)(11,1)\rangle \quad \langle y\ (2,1)(1,0)(3,2)(10,2)\rangle$
$\langle(7,0)(11,2)(9,2)(3,2)(1,0)\rangle \quad \langle(9,0)(7,1)(1,0)\ y\ (4,1)\rangle$
$\langle(7,2)(4,1)(10,1)(8,2)(1,0)\rangle \quad \langle(2,0)\mathbf{(5,2)}\ \mathbf{(5,1)}\ y\ (8,2)\rangle$
$\langle(10,2)(1,0)(4,2)(8,1)(12,2)\rangle \quad \langle(7,1)\ z\ (4,2)(8,2)(0,0)\rangle$
$\langle(9,1)(1,0)(8,2)(5,0)(11,2)\rangle \quad \langle(12,2)(9,2)(0,0)\ z\ (7,1)\rangle$
$\langle(5,1)\mathbf{(12,0)}\ \mathbf{(12,2)}(10,0)(1,0)\rangle \quad \langle(6,0)(1,0)(10,1)\ z\ (2,2)\rangle$
$\langle(1,0)(7,2)(6,1)(12,1)(10,0)\rangle \quad \langle z\ (1,0)(11,0)(5,1)(3,1)\rangle$
$\langle(9,0)(3,1)(2,0)(11,2)(10,2)\rangle \quad \langle(8,1)(2,0)(4,1)(3,2)\ z\rangle$
$\langle(8,2)(12,2)(3,2)(11,1)(2,0)\rangle \quad \langle(12,1)(1,0)(9,0)(8,0)\ w\rangle$
$\langle(4,1)(6,0)(11,2)(10,0)(2,0)\rangle \quad \langle(11,0)\mathbf{(7,1)}\ \mathbf{(7,0)}\ w\ (2,0)\rangle$
$\langle(2,0)(11,1)(4,2)(7,0)(12,1)\rangle \quad \langle\mathbf{(10,1)}\ \mathbf{(10,0)}\ w\ (3,0)(9,0)\rangle$
$\langle(10,1)(2,0)(12,0)(7,1)(5,0)\rangle \quad \langle w\ (11,1)(8,1)(4,0)(6,1)\rangle$
$\langle(8,0)(5,2)(2,0)(9,1)(12,2)\rangle \quad \langle(5,0)\mathbf{(11,1)}\ \mathbf{(11,0)}\ x\ (1,0)\rangle$
$\langle(7,2)(2,0)(11,0)(8,1)(6,2)\rangle \quad \langle(2,0)\ x\ (10,0)(6,0)(8,0)\rangle$
$\langle(4,1)(11,1)(12,2)(3,0)(5,1)\rangle \quad \langle(6,0)(4,0)(9,1)(3,0)\ x\rangle$
$\langle(3,0)(6,0)(9,2)(12,1)(4,2)\rangle \quad \langle(12,1)\ x\ (3,0)(4,1)(7,1)\rangle$
$\langle(3,0)(6,1)(8,2)(9,1)(5,2)\rangle \quad \langle w\ x\ (12,1)(5,2)(0,0)\rangle$
$\langle(4,0)(7,2)(5,2)(9,0)(6,0)\rangle \quad \langle(3,0)(2,0)(1,0)\mathbf{(0,1)}\ \mathbf{(0,0)}\rangle$

Together with the following single blocks

$\langle(3,0)(3,1)(3,2)\ w\ y\rangle \quad \langle w\ (1,0)(1,1)(1,2)\ x\rangle$
$\langle(4,0)(4,1)(4,2)\ y\ w\rangle \quad \langle x\ (2,0)(2,1)(2,2)\ w\rangle$
$\langle y\ (11,0)(11,1)(11,2)\ z\rangle \quad \langle y\ (7,0)(7,1)(7,2)\ x\rangle$
$\langle z\ y\ (12,0)(12,1)(12,2)\rangle \quad \langle(8,0)(8,1)(8,2)\ x\ y\rangle$
$\langle(5,0)(5,1)(5,2)\ w\ z\rangle \quad \langle x\ z\ (9,0)(9,1)(9,2)\rangle$
$\langle z\ (6,0)(6,1)(6,2)\ w\rangle \quad \langle z\ (10,0)(10,1)(10,2)\ x\rangle$
$\langle(0,0)(0,1)(0,2)\ w\ x\rangle$

□

5 Directed Covering with Index 3

Notice that if there exists a $(v,5,1)$ - $IDPBD(k)$, then, for any positive integer λ, there exists a $(v,5,\lambda)$-$IDPBD(k)$. Hence, Lemma 3.1, Lemma 3.2, and Lemma 3.3 hold in the case $\lambda = 3$. Furthermore, there exist a $(37,5,3)$-$IDPBD(9)$. Such design can be constructed by taking a $RB[28,4,1]$, then adding nine new points , a point for each parallel class, yields a $(37,5,1)$-$IPBD(9)$. Now by taking three copies of a $(37,5,1)$-$IPBD(9)$ and another three copies in opposite direction we obtain a $(37,5,3)$-$IDPBD(9)$.

Now we can prove the following Lemma.

Lemma 5.1. *Let $v \equiv 3,7,9 \pmod{10}$, $v \geq 7$ be an integer. Then $DE(v,5,3) = DL(v,5,3)$.*

Proof. From the above discussion, it is clear that we only need to consider the cases where $v = 7,9,13,17,19,23,27,33,43,53,63,73$, and 83. For $v = 7$, let $X = \mathbb{Z}_5 \cup \{a,b\}$, then the blocks are

$\langle\, 1\,2\,3\,4\,0 \,\rangle$-three times $\langle\, 3\,2\,1\,a\,b \,\rangle$
$\langle\, b\,a\,4\,2\,1 \,\rangle$ $\langle\, 4\,3\,2\,a\,b \,\rangle$ $\langle\, b\,a\,0\,3\,2 \,\rangle$
$\langle\, 0\,4\,3\,a\,b \,\rangle$ $\langle\, b\,a\,4\,3\,1 \,\rangle$ $\langle\, 0\,4\,1\,a\,b \,\rangle$
$\langle\, b\,a\,0\,4\,2 \,\rangle$ $\langle\, 0\,2\,1\,a\,b \,\rangle$ $\langle\, b\,a\,0\,3\,1 \,\rangle$

For $v = 9$, let $X = \{1,2,...,9\}$. Then the blocks are

$\langle\, 1\,8\,2\,9\,7 \,\rangle$ twice $\langle\, 3\,7\,9\,4\,8 \,\rangle$ twice
$\langle\, 6\,5\,9\,8\,7 \,\rangle$ $\langle\, 2\,3\,5\,8\,6 \,\rangle$ $\langle\, 8\,7\,6\,3\,2 \,\rangle$
$\langle\, 6\,5\,7\,8\,9 \,\rangle$ $\langle\, 7\,2\,8\,5\,4 \,\rangle$ $\langle\, 2\,9\,1\,5\,4 \,\rangle$
$\langle\, 9\,5\,3\,2\,1 \,\rangle$ $\langle\, 1\,2\,3\,7\,6 \,\rangle$ $\langle\, 1\,3\,6\,4\,9 \,\rangle$
$\langle\, 2\,8\,6\,4\,1 \,\rangle$ $\langle\, 1\,8\,4\,3\,5 \,\rangle$ $\langle\, 4\,7\,1\,5\,6 \,\rangle$
$\langle\, 9\,6\,5\,3\,1 \,\rangle$ $\langle\, 8\,7\,3\,5\,1 \,\rangle$ $\langle\, 4\,9\,6\,2\,3 \,\rangle$
$\langle\, 6\,4\,7\,8\,1 \,\rangle$ $\langle\, 5\,4\,7\,3\,2 \,\rangle$ $\langle\, 4\,5\,9\,6\,2 \,\rangle$

For $v = 17,27$, the construction is as follows:

1. Take an optimal $DP(v,5,2)$, [7]. In this design, there is a set $\{x,y\}$ the ordered pairs of which appear in zero blocks, while each other ordered pair appears in two blocks.
2. Take the minimal $DC(v,5,1)$ in Lemma 4.1. In this design there is a set $\{x,y\}$ the ordered pairs of which appear in five blocks, while each other ordered pair appears exactly in one block. It is clear that the blocks of the above two designs yield the blocks of a $DC(v,5,3)$ for $v = 17,27$.

For $v = 19$, the construction is as follows:

1. Take the blocks of the $DC(19,5,1)$ as presented in Lemma 4.2 and delete the block $\langle\, w\ x\ y\ z\ (0,0) \,\rangle$. Furthermore, replace the three blocks $\langle\, w\ x(1,2)(1,1)(1,0) \,\rangle$ $\langle\, z\ y\ x\ w\ (0,1) \,\rangle$ $\langle\, z\ y\ x\ w\ (0,2) \,\rangle$ by $\langle\, x\ w\ (1,2)(1,1)(1,0) \,\rangle$ $\langle\, w\ x\ y\ z\ (0,1) \,\rangle$ $\langle\, w\ x\ y\ z\ (0,2) \,\rangle$

2. Take the blocks of the $DC(19,5,1)$ as presented in Lemma 4.2, replace w by $(0,0)$ and $(0,0)$ by w , and then delete the block $\langle\, (0,0)\ x\ y\ z$

w ⟩. Further, replace the single block ⟨ $(3,2)(4,1)$ w x $(3,0)$ ⟩ by ⟨ $(3,2)$ $(4,1)$ x w $(3,0)$ ⟩.

3. Take the blocks of the $DC(19,5,1)$ as presented in Lemma 4.2, replace x by $(0,0)$ and $(0,0)$ by x , and then delete the block ⟨ w $(0,0)$ y z x ⟩. Further, replace the single block ⟨ z y $(0,0)$ w $(0,1)$ ⟩ by ⟨ $(0,0)$ z y w $(0,1)$ ⟩.

4. Adjoin the block ⟨ w $(0,0)$ x y z ⟩.
It is easy to check that the above four steps yield a $DC(19,5,3)$.

For $v \equiv 13 \pmod{20}$ a $DC(v,5,3)$ can be constructed by taking two copies, in opposite direction, of a minimal $(v,5,3)$ covering design.

For $v \equiv 3 \pmod{20}$ a $DC(v,5,3)$ can be constructed by taking a minimal $DC(v,5,2)$ [5] and a minimal $DC(v,5,1)$ which exists for all v with the possible exceptions of $v = 63, 83$ (Lemma 4.3).

For $v = 83$ the construction is as follows:
Take a $(\{5,6\},1)$-GDD of type $7^5 6^1$, which is obtained by deleting one point from one group. Inflate the GDD by a factor of two (see [28] for a $(5,3)-GDD$ of type 2^5 and 2^6).
Now adjoin a new point to the groups and on the groups of size 14 we construct, with the new, a $DB[15,5,3]$ and on the group of size 12 with the new point we construct a minimal $DC(13,5,3)$.

For $v = 63$ take a $(5,1)$-$RGDD$ of type 5^5 and inflate the design by a factor 2. To each parallel class of the five parallel classes adjoin two new points and construct a $(5,3)$-$DGDD$ of type 2^6. To the groups we adjoin 3 new points and on each group we construct a $(13,5,3)$-$IDPBD(3)$, then we take these three points with the ten points added to be the hole of size 13. This construction yields a $(63,5,3)$-$IDPBD(13)$.

To complete this construction we need to construct a $(13,5,3)$-$IDPBD(3)$. For this purpose, let $X = \mathbb{Z}_{10} \cup \{x,y,z\}$. Then the required blocks are the following (mod 10)
⟨ 0 3 x 1 5 ⟩ ⟨ 1 0 y 3 7 ⟩
⟨ 0 z 1 3 9 ⟩ ⟨ 2 3 - 0 7 ⟩ $\cup\{x,y\}$
⟨ 0 1 5 6 z ⟩ half orbit. □

6 Conclusion

In this short section we conclude our result.

Theorem 6.1. *Let $v \geq 5$ be an odd integer greater than or equal 5. Then (i) If $v \equiv 1$ or 5 (mod 10), then $DE(v,5,\lambda) = DL(v,5,\lambda)$ with the exception of*
$DE(15,5,1) = DL(15,5,1)+1$.
(ii) If $v \equiv 3$, 7, or 9 (mod 10), then $DE(v,5,\lambda) = DL(v,5,\lambda)$ with the exception of $DE(9,5,1) = DL(9,5,1)+1$, $DE(13,5,1) = DL(13,5,1)+1$ and the possible exceptions $(v,\lambda) = (53,1),(63,1),(73,1)$, or $(83,1)$.

Proof. For (i), see [26]. For (ii), see Lemma 4.1, 4.2, 4.3, and 5.1 for $\lambda = 1,3$, and [5] for $\lambda = 2,4$. For $\lambda = 5$, there exists a $DB[v,5,5]$. For $\lambda > 5$, the result follows from Lemma 1.1 and [5] for $(v,\lambda) = (53,6),(63,6),(73,6),(83,6)$. □

Remark: The following is an example of a $(77,5,2)$ covering design that can not be directed.

Let $X = \mathbb{Z}_2 \times \mathbb{Z}_{30} \cup \{\infty_i\}_{i=1}^{17}$, then the blocks are

$\langle(0,0)(0,6)(0,12)(0,18)(0,24)\rangle + i,\ i \in \mathbb{Z}_6,$ twice
$\langle(1,0)(1,6)(1,12)(1,18)(1,24)\rangle + i,\ i \in \mathbb{Z}_6,$
together with the following blocks, mod $(-,30)$.
$\langle(0,0)(0,14)(1,2)(1,14)(1,15)\rangle$
$\langle(0,0)(0,5)(0,13)(0,20)\rangle \cup \{\infty_1,\infty_2\}$
$\langle(1,0)(1,3)(1,11)(1,20)\rangle \cup \{\infty_1,\infty_2\}$
$\langle(0,0)(0,9)(0,11)(1,22)\rangle \cup \{\infty_3,\infty_4\}$
$\langle(0,0)(1,20)(1,25)(1,27)\rangle \cup \{\infty_3,\infty_4\}$
$\langle(0,0)(0,3)(0,7)(1,24)\rangle \cup \{\infty_5,\infty_6\}$
$\langle(0,0)(1,5)(1,19)(1,26)\rangle \cup \{\infty_5,\infty_6\}$
$\langle(0,0)(0,1)(0,17)(1,29)\rangle \cup \{\infty_7,\infty_8\}$
$\langle(0,0)(1,4)(1,8)(1,23)\rangle \cup \{\infty_7,\infty_8\}$
$\langle(0,0)(0,1)(1,27)(1,29)\ \infty_9\rangle$
$\langle(0,0)(0,2)(1,24)(1,25)\ \infty_{10}\rangle$
$\langle(0,0)(0,3)(1,9)(1,19)\ \infty_{11}\rangle$
$\langle(0,0)(0,4)(1,7)(1,15)\ \infty_{12}\rangle$
$\langle(0,0)(0,5)(1,7)(1,10)\ \infty_{13}\rangle$
$\langle(0,0)(0,8)(1,16)(1,21)\ \infty_{14}\rangle$
$\langle(0,0)(0,9)(1,12)(1,18)\ \infty_{15}\rangle$
$\langle(0,0)(0,10)(1,10)(1,14)\ \infty_{16}\rangle$

$\langle(0,0)(0,11)(1,1)(1,17)\ \infty_{17}\rangle$

Proof. Consider the two blocks
$\langle(1,0)(1,6)(1,12)(1,18)(1,24)\rangle + i,\ i \in \mathbb{Z}_6$, and
$\langle(0,0)(0,9)(1,12)(1,18)\ \infty_{15}\rangle$ mod $(-,30)$.
To direct the above design, the block $\langle(0,0)(0,9)(1,12)(1,18)\ \infty_{15}\rangle$ should have ∞_{15} in the middle with $(0,0)$ or $(0,9)$ on the left or right of ∞_{15} and $(1,12)$ or $(1,18)$ on the left or right of ∞_{15} e.g. $\langle(0,0)(1,18)\ \infty_{15}\ (1,12)(0,9)\rangle$. But if this is the case then look at the blocks

$\langle(0,0)(1,18)\ \infty_{15}\ (0,9)(1,12)\rangle$, $\langle(0,6)(1,24)\ \infty_{15}\ (0,15)(1,18)\rangle$,
$\langle(0,12)(1,0)\ \infty_{15}(0,21)(1,24)\rangle$, $\langle(0,18)(1,6)\ \infty_{15}\ (0,27)(1,0)\rangle$,
$\langle(0,24)(1,12)\ \infty_{15}\ (0,3)(1,6)\rangle$, taken from the blocks
$\langle(0,0)(1,18)\ \infty_{15}\ (0,9)(1,12)$ mod $(-,30)$ and the block $\langle(1,0)\ (1,6)\ (1,12)\ (1,18)\ (1,24)\ \rangle$. Then it is easy to see that the pairs $((1,18),(1,12))$, $((1,24),(1,18))$, $((1,0),(1,24))$, $((1,6),(1,0))$, and $((1,12),(1,6))$ can not be ordered.

□

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