Powers of directed Hamiltonian paths as feedback arc sets

Darren A. Narayan
School of Mathematical Sciences
Rochester Institute of Technology, Rochester, NY 14623 USA

Abstract

Given an acyclic digraph D, we seek a smallest sized tournament T having D as a minimum feedback arc set. The reversing number of a digraph is defined to be $r(D) = |V(T)| - |V(D)|$. We use integer programming methods to obtain new results for reversing number where D is a power of a directed Hamiltonian path. As a result we establish that known reversing numbers for certain classes of tournaments actually suffice for a larger class of digraphs.

Keywords: Linear ordering, tournament, feedback arc set, digraph, reversing number

AMS Classification: Primary: 05C20, Secondary: 90C47

1 Introduction

A *tournament* is a digraph where the underlying undirected graph is complete. A *minimum feedback arc set* of a digraph is a smallest sized set of arcs whose reversal makes the resulting digraph acyclic. Given an acyclic digraph D we seek a smallest sized tournament T that has D as a minimum feedback arc set. It was shown in [2] that for any acyclic digraph D there exists some such T, and the *reversing number of a digraph,* $r(D)$ was defined to be $|V(T)| - |V(D)|$. Reversing numbers for several classes of digraphs were determined in [1]-[9] and it was shown in [2] that $0 \leq r(D) \leq 2n - 4$. In particular, the case where D is the acyclic tournament on n vertices T_n was investigated in [2] and [4].

For given n and $k < n$, P_n^k will denote the k-th power of a directed Hamiltonian path on n vertices. That is, P_n^k is the digraph containing the directed Hamiltonian path on vertices $v_1, v_2, ..., v_n$ and having the arc set consisting of all arcs (v_i, v_j) where $i < j$ and $|i - j| \leq k$. Of course, P_n^{n-1}

is simply the tournament, T_n. In this paper we investigate the reversing number of P_n^k.

Reversing numbers for various tournaments were investigated in [4]. This problem can be viewed in the context of player rankings. If the players in a tournament T_n are ranked to minimize inconsistencies then $r(T_n)$ is a smallest sized tournament for which there are n players that are all ranked inconsistently with respect to each other. In the investigation of $r\left(P_n^k\right)$ we have a similar correspondence to rankings but will require most, but not all of the players be ranked inconsistently. Here we consider rankings with many inconsistencies, but avoid inconsistencies where one team defeats another and the winner receives a significantly lower rank than the loser.

We use $A(D)$ to denote the arc set of a digraph D. However when $A(D)$ forms a feedback arc set of T we may simply say 'D is a feedback arc set of T' when there is no ambiguity. We next restate a lemma from [2] that will be frequently used throughout the paper.

Lemma 1 *Let D and D' be digraphs on n vertices. Then $D' \subseteq D \Rightarrow r(D') \leq r(D)$.*

We continue by reviewing some known results. As an immediate consequence of the above lemma, $r\left(P_n^k\right) \leq r(T_n)$ for all k and n. It was shown in [2] that $r(P_n) = n - 1$, and $2n - 4\log_2 n \leq r(T_n) \leq 2n - 4$. Precise values for $r\left(P_n^k\right)$ for $k \leq 7$ were established in [7]. It was shown in [8] that $\left(\frac{2k+1}{k+1}\right) n - c(k) \leq r(P_n^{2k}) \leq 2n - 4$ for all $n \geq 4k$ where $c(k)$ is a positive constant depending only on k. This result shows that even for small values of k, $r\left(P_n^k\right)$ will side with $r(T_n)$ rather than $r(P_n)$, for large values of n. This suggests that $r\left(P_n^k\right)$ and $r(T_n)$ may coincide for values of k that are close to $n-1$.

As mentioned reversing numbers for T_n were determined for some values of n by Isaak [4]. We restate this result as our next theorem.

Theorem 2 *Let $0 \leq t \leq s$. Then $r(T_{2^s - 2^t}) = 2^{s+1} - 2^{t+1} - s - 1$.*

Since $A\left(P_n^k\right) \subseteq A\left(T_n\right)$ we can combine Lemmas 1 and 2 with known upper bounds for $r\left(T_n\right)$ to immediately obtain upper bounds for $r\left(P_n^k\right)$. For example $r\left(P_{32}^{16}\right) \leq r(T_{32}) = 57$. Surprisingly despite the fact that P_{32}^{16} has 120 fewer arcs than T_{32} they have the same reversing number. One consequence of our main result will show that when n is a power of 2, it is possible to remove approximately $\frac{1}{4}$ of the arcs, with no change to the reversing number.

We will then use methods from integer programming to establish lower bounds for $r\left(P_n^k\right)$ which match these upper bounds. Our main result is an extension of Theorem 2.

Theorem 3 *Let P_n^k be the k-th power of a directed Hamiltonian path P_n, and let T_n be the acyclic tournament on n vertices. Then*

(i) $r\left(P_{2^k-2^t}^{2^{k-1}}\right) = r(T_{2^k-2^t})$ when $t = k-1$ or $k-2$ and

(ii) $r(T_{2^k-2^t}) - 1 \leq r\left(P_{2^k-2^t}^{2^{k-1}}\right) \leq r(T_{2^k-2^t})$ for all $0 \leq t \leq k-3$.

2 Preliminaries

Let $T\left(\overrightarrow{x}, P_n^k\right)$ be a tournament having the k-th power of a directed Hamiltonian path P_n^k as a feedback arc set. Any tournament $T\left(\overrightarrow{x}, P_n^k\right)$ having P_n^k as a feedback arc set will have this form for some set of extra vertices, $\overrightarrow{x}$. Let $V(P_n^k) = \{v_1, v_2, ..., v_n\}$ and $A(P_n^k) = \{(v_j, v_i)\ |0 < j-i \leq k\}$. Then $V\left(T\left(\overrightarrow{x}, P_n^k\right)\right) = V(P_n^k) \cup \{u_{i,j} | 1 \leq i \leq n-1,\ 0 \leq j \leq x_i - 1\}$ and $A\left(T\left(\overrightarrow{x}, P_n^k\right)\right) = A(P_n^k) \cup \{(v_i, v_j) | k < j-i\} \cup \{(u_{i,j}, u_{s,t}) : i < s$ or $i = s$ and $j < t\} \cup \{(v_i, u_{s,t}) \mid i \leq s\} \cup \{(u_{i,j}, v_s) \mid i < s\}$. That is, $V\left(T\left(\overrightarrow{x}, P_n^k\right)\right)$ $= V\left(P_n^k\right)$ along with a set of extra vertices dependent upon P_n^k and the arc set consists of those arcs consistent with the ordering:

$$v_1, u_{1,0}, ..., u_{1,x_1-1}, v_2, u_{2,0}, ..., u_{2,x_2-1}, v_3, ..., v_{n-1}, u_{n-1,0}, ..., u_{n-1,x_{n-1}-1}, v_n$$

except for arcs between vertices v_i and v_j where $0 < j-i \leq k$, which are inconsistent with the ordering.

Given P_n^k we investigate inequalities involving the number of extra vertices specified by $\overrightarrow{x}$. We note that $r\left(P_n^k\right)$ equals the minimum $\sum_{i=1}^{n-1} x_i$ such that a tournament on $n + \sum_{i=1}^{n-1} x_i$ vertices has P_n^k as a minimum feedback arc set. We will focus on two different types of sums involving $\overrightarrow{x}$ and seek lower bounds for them. A sum involving subsets of $\overrightarrow{x}$ having the form $\sum_{i=1}^{\lceil\frac{t-1}{2}\rceil} ix_i + \sum_{i=\lceil\frac{t-1}{2}\rceil+1}^{t-1} (t-i)x_i$ will be referred to as a *sum of type I.* A sum involving subsets of $\overrightarrow{x}$ having the form $\sum_{i=1}^{t-1}(t-i)x_i$ will be referred to as *a sum of type II.*

These expressions were studied in [2] and [4]. We restate a bound from [4] as our next lemma.

Lemma 4 *We have $\sum_{i=1}^{\lceil\frac{t-1}{2}\rceil} ix_{i+a} + \sum_{i=\lceil\frac{t-1}{2}\rceil+1}^{t-1} (t-i)x_{i+a} \geq \binom{t}{2}$ for $2 \leq t \leq k+1$ and $0 \leq a \leq n-t$.*

We note that this bound from [4] was used in the context of tournaments. However since Lemma 4 holds for P_n^k where $k \geq t-1$, it can be applied to part of the family consisting of powers of directed Hamiltonian paths.

Next we examine sums of type II, $\sum_{i=1}^{w-1}(w-i)x_i$ and seek lower bounds for them. We build on results presented in [4] The first few cases are given below.

$w = 2: \quad x_1 \geq 1$

$w = 3: \quad 2x_1 + x_2 \geq 4$

$w = 4: \quad \sum_{i=1}^{3}(4-i)x_i = 3x_1 + 2x_2 + x_3$
$= (x_1 + 2x_2 + x_3) + 2x_1 \geq 6 + 2 = 8$

$w = 5: \quad \sum_{i=1}^{4}(5-i)x_i = 4x_1 + 3x_2 + 2x_3 + x_4$

$= (x_1 + 2x_2 + 2x_3 + x_4) + (2x_1 + x_2) + x_1 \geq 15$

We observe a recursive pattern:

$\sum_{i=1}^{4}(5-i)x_i = \sum_{i=1}^{1}(2-i)x_i + \sum_{i=1}^{2}(3-i)x_i + (x_1 + 2x_2 + 2x_3 + x_4).$

Similarly, $\sum_{i=1}^{5}(6-i)x_i = 2\left(\sum_{i=1}^{2}(3-i)x_i\right)$
$+ (x_1 + 2x_2 + 3x_3 + 2x_4 + x_5).$

In general we decompose $\sum_{i=1}^{w-1}(w-i)x_i$ into three parts:

$\sum_{i=1}^{\lceil \frac{w-1}{2} \rceil} ix_i + \sum_{i=\lceil \frac{w-1}{2} \rceil+1}^{w-1}(w-i)x_i,$

$\sum_{i=1}^{\lceil \frac{w}{2} \rceil - 1} \left(\lceil \frac{w}{2} \rceil - i\right) x_i$

and $\sum_{i=1}^{\lfloor \frac{w}{2} \rfloor - 1} \left(\lfloor \frac{w}{2} \rfloor - i\right) x_i,$

and apply a lower bound for each part. A lower bound for the first part is given in Lemma 4, and bounds for the other parts can be obtained recursively ending with the given base cases. We generalize these methods in the following lemma by Isaak [4].

Lemma 5 *We have* $\sum_{i=1}^{w-1}(w-i)x_i \geq \binom{w}{2} + \sum_{i=1}^{\lceil \frac{w}{2} \rceil - 1} \left(\lceil \frac{w}{2} \rceil - i\right) x_i$
$+ \sum_{i=1}^{\lfloor \frac{w}{2} \rfloor - 1} \left(\lfloor \frac{w}{2} \rfloor - i\right) x_i.$

Proof. The proof follows from the above and Lemma 4. ■

We consider cases involving small values of k in our next lemma.

Lemma 6 *Let* $2 \leq k \leq 4$. *Then* $r\left(P_{2^k}^{2^{k-1}}\right) = r(T_{2^k}) = 2^{k+1} - k - 2$.

Proof. We prove separate cases for $k = 2,3$, and 4.

(i) $k = 2$

By Theorem 2 $r(P_4^2) \leq r(T_4) = 2^3 - 2 - 2 = 4$. Application of Lemmas 4 and 5 give the following inequalities:

$$\begin{array}{ccccccc} x_1 & + & x_2 & & & \geq & 3 \\ & & x_2 & + & x_3 & \geq & 3 \\ x_1 & & & + & x_3 & \geq & 2 \end{array}$$

Summing the above inequalities yields $2\sum_{i=1}^{3} x_i \geq 8 \Rightarrow \sum_{i=1}^{3} x_i \geq 4 = 2^3 - 2 - 2$.

Hence $r(P_4^2) \geq \sum_{i=1}^{3} x_i = 4$.

(ii) $k = 3$

By Theorem 2 $r(P_8^4) \leq r(T_8) = 2^4 - 3 - 2 = 11$. Application of Lemmas 4 and 5 give the following inequalities:

$$\begin{array}{cccccccc} x_1 + & 2x_2 & +2x_3 & + x_4 & & & & \geq 10 \\ & x_2 & +2x_3 & +2x_4 & + x_5 & & & \geq 10 \\ & & + x_3 & +2x_4 & +2x_5 & + x_6 & & \geq 10 \\ & & & + x_4 & +2x_5 & +2x_6 & + x_7 & \geq 10 \\ x_1 + & 2x_2 & +x_3 & & + x_5 & +2x_6 & + x_7 & \geq 12 \\ x_1 + & x_2 & & & & + x_6 & + x_7 & \geq 6 \\ x_1 & & & & & & + x_7 & \geq 2 \\ 2x_1 & & & & & & +2x_7 & \geq 4 \end{array}$$

Summing the above inequalities yields $6\sum_{i=1}^{7} x_i \geq 64 \Rightarrow r(P_8^4) \geq \sum_{i=1}^{7} x_i \geq 11$.

(iii) $k = 4$

By Theorem 2, $r(P_{16}^8) \leq r(T_{16}) = 2^5 - 4 - 2 = 26$.

We investigate the coefficients of the terms present in these inequalities in Table 1. The coefficients of x_i are given in the i-th column.

	1	2	3	4	4	3	2	1							
		1	2	3	4	4	3	2	1						
			1	2	3	4	4	3	2	1					
First				1	2	3	4	4	3	2	1				
Set					1	2	3	4	4	3	2	1			
						1	2	3	4	4	3	2	1		
							1	2	3	4	4	3	2	1	
								1	2	3	4	4	3	2	1
	1	2	3	4	3	2	1		1	2	3	4	3	2	1
	1	2	3	3	2	1				1	2	3	3	2	1
Second	1	2	3	2	1						1	2	3	2	1
Set	1	2	2	1								1	2	2	1
	1	2	1										1	2	1
	1	1												1	1
	1														1
Third	6	4	2										2	4	6
Set	4	2												2	4
	2														2

Table 1

The first 8 rows of the table yield

$\sum_{t=0}^{7}(\sum_{i=1}^{4} ix_{i+t} + (5-i)x_{i+4+t}) \geq 8\binom{9}{2}$.

The next set of rows gives us

$\sum_{j=1}^{4}(\sum_{i=1}^{j} i(x_i + x_{16-i}) + (j+1-i)(x_{i+j} + x_{16-i-j}))$

$\quad + \sum_{j=1}^{4}(\sum_{i=1}^{j} i(x_i + x_{16-i}) + (j-i)(x_{i+j} + x_{16-i-j}))$

$\geq 2\sum_{i=2}^{8}\binom{i}{2} = 2\binom{9}{3}$.

Finally, entries in the last three rows correspond to two copies of

$\sum_{i=2}^{4}(\sum_{j=1}^{i-1}(i-j)x_j)$

and two copies of

$\sum_{i=2}^{4}(\sum_{j=1}^{i-1}(i-j)x_{16-j})$.

Taking advantage of symmetry we formulate the bound,

$4(\sum_{i=2}^{4}(\sum_{j=1}^{i-1}(i-j)x_j)) \geq 4(8+4+1) = 52$.

Summing over all rows in the table yields

$20\sum_{i=1}^{15} x_i \geq 8\binom{9}{2} + 2\binom{9}{3} + 52 = 508$

$\Rightarrow r(P_{16}^8) \geq \lceil \frac{508}{20} \rceil = 26 = 2^5 - 4 - 2.$ ■

We use the ideas of the above lemma to show the case when $k = 5$. By Lemma 2 $r(P_{32}^{16}) \leq r(T_{32}) = 2^6 - 5 - 2 = 57$. We obtain the reverse inequality using the same methods that were used in part (iii) in the proof above. Then it will follow that

$$r(P_{32}^{16}) \geq \left\lceil \frac{16\binom{17}{2} + 2\binom{17}{3} + 4\left(\sum_{i=2}^{8}\left(\sum_{j=1}^{i-1}(i-j)x_j\right)\right)}{(8)(9)} \right\rceil = 57. \quad (1)$$

These methods will be described in a general setting in the next section. We will show that we may replace $k = 5$ in inequality (1) with an arbitrary k to get the generalized inequality (2):

$$r\left(P_{2^k}^{2^{k-1}}\right) \geq \left\lceil \frac{2^{k-1}\binom{2^{k-1}+1}{2} + 2\binom{2^{k-1}+1}{3} + 4\left(\sum_{i=2}^{2^{k-2}}\left(\sum_{j=1}^{i-1}(i-j)x_j\right)\right)}{(2^{k-2})(2^{k-2}+1)} \right\rceil$$

$$= 2^{k+1} - k - 2. \quad (2)$$

3 The general case

In this section we present additional results involving $r\left(P_n^k\right)$. In the steps leading up to our main results, our focus shall not be on obtaining bounds for individual expressions $\sum_{j=1}^{i-1}(i-j)x_j$ as in [4], but rather on the development of bounds for collections of these expressions where the values of i form a set of consecutive integers. These lower bounds can be combined with the inequalities corresponding to the first and second sets of rows in Table 1 to create lower bounds for $r(P_n^k)$. For certain n, these lower bounds will be matched with values for $r(T_n)$ from [4] to yield new values for $r(P_n^k)$.

Next we will determine $r\left(P_{2^k}^{2^{k-1}}\right)$. We will follow the construction methods from the proof of Lemma 6 (iii) that were used to generate $r\left(P_{16}^8\right)$. We note that in Table 1 each column sum is 20. That is the sum of all of the expressions involving x_i is $(4)(5)\sum_{i=1}^{15} x_i$. We will show for the general case involving $r\left(P_{2^k}^{2^{k-1}}\right)$ the sum of all of the expressions involving x_i is $(2^{k-2}(2^{k-2}+1))\sum_{i=1}^{2^k-1} x_i$. The general expression corresponding to the first set of rows in Table 1 is

$\sum_{t=0}^{2^{k-1}-1}(\sum_{i=1}^{2^{k-2}} ix_{i+t} + (2^{k-2}+1-i)x_{i+2^{k-2}+t})$.

The expression corresponding to the second set of rows is

$$\sum_{j=1}^{2^{k-2}}(\sum_{i=1}^{j} i(x_i + x_{2^k-i}) + (j + 1 - i)(x_{i+j} + x_{2^k-i-j})).$$

Finally, the third expression equals

$$2\sum_{i=2}^{2^{k-2}}(\sum_{j=1}^{i-1}(i - j)x_j) + 2\sum_{i=2}^{2^{k-2}}(\sum_{j=1}^{i-1}(i - j)x_{2^k-j}).$$

The sum of all of the above expressions equals

$$(2^{k-2}(2^{k-2} + 1))\sum_{i=1}^{2^k-1} x_i.$$

We state this in our next theorem.

Lemma 7 *We have* $\sum_{t=0}^{2^{k-1}-1}(\sum_{i=1}^{2^{k-2}} ix_{i+t} + (2^{k-2} + 1 - i)x_{i+2^{k-2}+t})$

$$+ \sum_{j=1}^{2^{k-2}}(\sum_{i=1}^{j} i(x_i + x_{2^k-i}) + (j + 1 - i)(x_{i+j} + x_{2^k-i-j}))$$
$$+ 2\sum_{i=2}^{2^{k-2}}(\sum_{j=1}^{i-1}(i - j)x_j) + 2\sum_{i=2}^{2^{k-2}}(\sum_{j=1}^{i-1}(i - j)x_{2^k-j})$$
$$= (2^{k-2}(2^{k-2} + 1))\sum_{i=1}^{2^k-1} x_i.$$

Proof. Consider the coefficient of $x_{2^{k-1}}$. Examination of the first expression reveals that the first coefficient is the sum of the coefficients of the terms

$$x_{2^{k-1}} + 2x_{2^{k-1}} + \cdots + 2^{k-2}x_{2^{k-1}} + 2^{k-2}x_{2^{k-1}} + \cdots + x_{2^{k-1}},$$

and hence the coefficient of $x_{2^{k-1}}$ equals $\left(2^{k-2}\right)\left(2^{k-2}+1\right)$. By symmetry we have $x_j = x_{2^k-j}$ for all $j = 1, 2, ..., 2^{k-1} - 1$ in the sum of the three expressions. Without loss of generality assume $1 \le j \le 2^{k-1} - 1$. We consider two subcases, where $1 \le j \le 2^{k-2} - 1$ and $2^{k-2} \le j \le 2^{k-1} - 1$. Assume $1 \le j \le 2^{k-2} - 1$. Then the coefficient of x_j is

$$2\left(\sum_{i=1}^{2^{k-2}-j} i\right) + 2\left(\sum_{i=1}^{j} i\right) + 2\left(2^{k-2} - j\right)j$$
$$= \left(2^{k-2}\right)^2 + \left(2^{k-2}\right).$$

Finally we consider the case where $2^{k-2} \le j \le 2^{k-1} - 1$ and that the contributions to the coefficient of x_j come only from the first two expressions. The coefficient of $x_j = 1 + 2 + 3 + \cdots + 2^{k-2} + 2^{k-2} + \cdots + 2 + 1 = (2^{k-2}(2^{k-2} + 1))$. ■

In the next section, we will use this equality to obtain lower bounds for $r\left(P_{2^k}^{2^{k-1}}\right)$. For completeness we provide all of the details. Application of Lemma 4 yields lower bounds for the first two sets of inequalities from Lemma 7:

$\sum_{t=0}^{2^{k-1}-1}(\sum_{i=1}^{2^{k-2}} ix_{i+t} + (2^{k-2}+1-i)x_{i+2^{k-2}+t})$

and $\sum_{j=1}^{2^{k-2}} \sum_{i=1}^{j}(i(x_i + x_{2^k-i}) + (j+1-i)(x_{i+j} + x_{2^k-i-j}))$,

and we now seek a lower bound for the expressions

$2\sum_{i=2}^{2^{k-2}} \left(\sum_{j=1}^{i-1}(i-j)x_j\right)$

and

$2\sum_{i=2}^{2^{k-2}} \left(\sum_{j=1}^{i-1}(i-j)x_{2^k-j}\right)$.

As a preliminary step we present lower bounds for $\sum_{i=1}^{w-1}(w-i)x_i$ in Table 2.

w	LB on $\sum_{i=1}^{w-1}(w-i)x_i$
2	1
3	4
4	8
5	15
6	23
7	30
8	44

Table 2

Then $\sum_{w=9}^{16}(\sum_{i=1}^{w-1}(w-i)x_i)$

$\geq (\sum_{i=9}^{16}\binom{i}{2}) + 4(\sum_{w=5}^{8}(\sum_{i=1}^{w-1}(w-i)x_i)) + \sum_{i=1}^{4}(w-i)x_i - \sum_{i=1}^{8}(w - i)x_i$

$= \binom{17}{3} - \binom{9}{3} + 4(112) + 8 - 44 = 1020.$

As mentioned earlier we develop lower bounds for a collection of expressions which have the form $\sum_{j=1}^{i-1}(i-j)x_j$ and the values of i form a set of consecutive integers. We use a bound for $\sum_{i=2^{k-3}+1}^{2^{k-2}}(\sum_{j=1}^{i-1}(i-j)x_j)$ to help determine a precise values for $r\left(P_{2^k}^{2^{k-1}}\right)$.

Lemma 8 *Let $k \geq 6$. Then* $\sum_{i=2^{k-3}+1}^{2^{k-2}}(\sum_{j=1}^{i-1}(i-j)x_j)$

$= \sum_{i=2^{k-3}+1}^{2^{k-2}}(\sum_{j=1}^{\lceil\frac{i-1}{2}\rceil} jx_j + \sum_{j=\lceil\frac{i-1}{2}\rceil+1}^{i-1}(i-j)x_j)$

$+ 4\sum_{i=2^{k-4}+1}^{2^{k-3}}(\sum_{j=1}^{i-1}(i-j)x_j) - (\sum_{j=1}^{2^{k-3}}(2^{k-3}-j)x_j)$

$+ (\sum_{j=1}^{2^{k-4}}(2^{k-4}-j)x_j)$.

Proof. The proof follows by induction and application of Lemma 5. ∎

The combination of Lemma 4,

$\sum_{i=2^{k-3}+1}^{2^{k-2}}(\sum_{j=1}^{\lceil\frac{i-1}{2}\rceil} jx_j + \sum_{j=\lceil\frac{i-1}{2}\rceil+1}^{i-1}(i - j)x_j) \geq \sum_{i=2^{k-3}+1}^{2^{k-2}} \binom{i}{2}$,

and bounds from [2] imply

$(\sum_{j=1}^{2^{k-3}}(2^{k-3} - j)x_j) \geq (2^{2k-4} - 2^{k-2} - 2^{k-3}(k-2))$
and
$(\sum_{j=1}^{2^{k-4}}(2^{k-4} - j)x_j) \geq (2^{2k-6} - 2^{k-3} - 2^{k-4}(k-3))$.

We establish a lower bound for $\sum_{i=2^{k-3}+1}^{2^{k-2}}(\sum_{j=1}^{i-1}(i - j)x_j)$, which is presented in our next lemma.

Lemma 9 *Let* $k \geq 6$. *Then* $\sum_{i=2^{k-3}+1}^{2^{k-2}}(\sum_{j=1}^{i-1}(i - j)x_j) \geq \frac{5}{64}4^{k-1} + \frac{7}{192}8^{k-1} - \frac{7}{48}2^{k-1} - \frac{1}{16}2^{k-1}(k-1) - \frac{3}{64}4^{k-1}(k-1)$.

Proof. For the base case ($k = 6$) note that

$\sum_{i=9}^{16}(\sum_{j=1}^{i-1}(i - j)x_j) \geq 1020 \geq \frac{5}{64}4^5 + \frac{7}{192}8^5 - \frac{7}{48}2^5 - \frac{1}{16}2^5(5) - \frac{3}{64}4^5(5)$.

Next, assume the identity holds for k and show it holds for $k + 1$. By Lemma 8,

$\sum_{i=2^{k-2}+1}^{2^{k-1}}(\sum_{j=1}^{i-1}(i - j)x_j)$

$\geq \sum_{i=2^{k-2}+1}^{2^{k-1}} \binom{i}{2} + 4\sum_{i=2^{k-3}+1}^{2^{k-2}}(\sum_{j=1}^{i-1}(i - j)x_j) - (2^{2k-4} - 2^{k-2} - 2^{k-3}(k-2))$
$+ (2^{2k-6} - 2^{k-3} - 2^{k-4}(k-3))$.

Application of the induction hypothesis yields

$\sum_{i=2^{k-2}+1}^{2^{k-1}}(\sum_{j=1}^{i-1}(i - j)x_j)$

$= \sum_{i=2^{k-2}+1}^{2^{k-1}} \binom{i}{2} + 4(\frac{5}{64}4^{k-1} + \frac{7}{192}8^{k-1} - \frac{7}{48}2^{k-1} - \frac{1}{16}2^{k-1}(k-1) - \frac{3}{64}4^{k-1}(k-1))$
$- (2^{2k-4} - 2^{k-2} - 2^{k-3}(k-2)) + (2^{2k-6} - 2^{k-3} - 2^{k-4}(k-3))$

$= \frac{5}{64}4^k + \frac{7}{192}8^k - \frac{7}{48}2^k - \frac{1}{16}2^k k - \frac{3}{64}4^k k$. ∎

Lemma 10 *We have the inequality* $\sum_{i=1}^{2^{k-2}}(\sum_{j=1}^{i-1}(i-j)x_j) \geq \frac{3}{64}4^k + \frac{1}{192}8^k - \frac{1}{48}2^k - \frac{1}{16}2^k k - \frac{1}{64}4^k k$.

Proof. The proof will follow by induction on k. The key observation is that

$\sum_{i=1}^{2^{k-1}}(\sum_{j=1}^{i-1}(i-j)x_j) \geq \sum_{i=1}^{2^{k-2}}(\sum_{j=1}^{i-1}(i-j)x_j) + \sum_{i=2^{k-2}+1}^{2^{k-1}}(\sum_{j=1}^{i-1}(i-j)x_j)$.

Application of Lemma 9 yields

$(\frac{3}{64}4^k + \frac{1}{192}8^k - \frac{1}{48}2^k - \frac{1}{16}2^k k - \frac{1}{64}4^k k) + (\frac{5}{64}4^k + \frac{7}{192}8^k - \frac{7}{48}2^k - \frac{1}{16}2^k k - \frac{3}{64}4^k k)$

$= \frac{1}{8}4^k + \frac{1}{24}8^k - \frac{1}{6}2^k - \frac{1}{8}2^k k - \frac{1}{16}4^k k$

$= \frac{3}{64}4^{k+1} + \frac{1}{192}8^{k+1} - \frac{1}{48}2^{k+1} - \frac{1}{16}2^{k+1}(k+1) - \frac{1}{64}4^{k+1}(k+1)$. ■

Finally we obtain a lower bound for $2\sum_{i=2}^{2^{k-2}}(\sum_{j=1}^{i-1}(i-j)x_j) + 2\sum_{i=2}^{2^{k-2}}(\sum_{j=1}^{i-1}(i-j)x_{2^k-j})$. We use symmetry to obtain

$2\sum_{i=2}^{2^{k-2}}(\sum_{j=1}^{i-1}(i-j)x_j) + 2\sum_{i=2}^{2^{k-2}}(\sum_{j=1}^{i-1}(i-j)x_j) = 4\sum_{i=1}^{2^{k-2}}(\sum_{j=1}^{i-1}(i-j)x_j)$. Hence

$\sum_{i=1}^{2^{k-2}}(\sum_{j=1}^{i-1}(i-j)x_j) \geq \frac{3}{64}4^k + \frac{1}{192}8^k - \frac{1}{48}2^k - \frac{1}{16}2^k k - \frac{1}{64}4^k k$.

The above bound is used to establish the following result.

Theorem 11 *Let $k \geq 2$. Then $r\left(P_{2^k}^{2^{k-1}}\right) = r(T_{2^k}) = 2^{k+1} - k - 2$.*

Proof. We need only show $r\left(P_{2^k}^{2^{k-1}}\right) \geq r(T_{2^k})$ since the converse will follow by Lemma 1. The cases where $2 \leq k \leq 5$ are shown in Lemma 6. If $k \geq 6$, earlier constructions yield

$$r\left(P_{2^k}^{2^{k-1}}\right) \geq$$

$$(1/((2^{k-2}(2^{k-2}+1))) \left(\begin{array}{c} 2^{k-1}\binom{2^{k-1}+1}{2} + 2\binom{2^{k-1}+1}{3} \\ +4(\frac{3}{64}4^k + \frac{1}{192}8^k - \frac{1}{48}2^k - \frac{1}{16}2^k k - \frac{1}{64}4^k k \end{array} \right)$$

$$= \left(\frac{1}{2^k+4}\right)(-2^k k - 4k + 2 \times 4^k + 5 \times 2^k - 4).$$

Then

$\left(\frac{1}{2^k+4}\right)(-2^k k - 4k + 2 \times 4^k + 5 \times 2^k - 4) - 2^{k+1} + k + 3$

$= \frac{8}{2^k+4}$.

Since $k \geq 6$, it follows that $0 < \frac{8}{2^k+4} < 1$ and hence $r\left(P_{2^k}^{2^{k-1}}\right) \geq 2^{k+1} - k - 2 = r(T_{2^k})$. Application of Lemma 2 yields the reverse inequality. ■

In our next corollary we note that the reversing number for T_{2^k} established in [4], actually applies to a larger family of digraphs.

Corollary 12 *Let D be an acyclic digraph on 2^k vertices containing $P_{2^k}^{2^{k-1}}$. Then $r(D) = r(T_{2^k}) = 2^{k+1} - k - 2$.*

Theorem 13 *We have $r\left(P_{2^k-2^{k-2}}^{2^{k-1}}\right) = r(T_{2^k-2^{k-2}}) = 2(2^k - 2^{k-2}) - (k-1) - 2$.*

Proof. We have

$$r\left(P_{2^k-2^{k-2}}^{2^{k-1}}\right) \geq$$

$$(1/(2^{k-2}(2^{k-2}+1)))\left[\begin{array}{c}\left(2^{k-2}\binom{2^{k-1}+1}{2} + 2\binom{2^{k-1}+1}{3}\right)\\ +4(\frac{3}{64}4^k + \frac{1}{192}8^k - \frac{1}{48}2^k - \frac{1}{16}2^k k - \frac{1}{64}4^k k)\end{array}\right].$$

Then

$$(1/((2^{k-2})(2^{k-2}+1)))\left[\begin{array}{c}\left(2^{k-2}\binom{2^{k-1}+1}{2} + 2\binom{2^{k-1}+1}{3}\right)\\ +4(\frac{3}{64}4^k + \frac{1}{192}8^k - \frac{1}{48}2^k - \frac{1}{16}2^k k - \frac{1}{64}4^k k)\end{array}\right]$$
$$- (2(2^k - 2^{k-2}) - (k-1) - 3)$$
$$= \frac{1}{2^{k-2}+1}.$$

Since $0 < \frac{1}{2^{k-2}+1} < 1$, it follows that

$$r\left(P_{2^k-2^{k-2}}^{2^{k-1}}\right) \geq 2^{k+1} - k - 2 = 2(2^k - 2^{k-2}) - (k-1) - 2.$$

Application of Theorem 2 yields the reverse inequality. ■

Example 14 *We will show that $r(P_{24}^{16}) = r(T_{24}) = 2(24)-(5-1)-2 = 42$.*

Theorem 15 *Let $0 \leq t < k-2$. Then $r(T_{2^k-2^t}) \geq r\left(P_{2^k-2^t}^{2^{k-1}}\right) \geq r(T_{2^k-2^t}) -1 = 2(2^k - 2^t) - (k-1) - 3$.*

Proof. We first note that the upper bound follows from Lemma 1 and Theorem 2. For the reverse inequality we note

$$r\left(P_{2^k-2^t}^{2^{k-1}}\right) \geq$$

$$(1/(2^{k-2}(2^{k-2}+1)))\left[\begin{array}{c}(2^{k-1}-2^t)\binom{2^{k-1}+1}{2}+2\binom{2^{k-1}+1}{3}\\ +4(\frac{3}{64}4^k+\frac{1}{192}8^k-\frac{1}{48}2^k-\frac{1}{16}2^k k-\frac{1}{64}4^k k)\end{array}\right]$$

$$= -\tfrac{1}{4(2^{k-2}+1)}(-2\cdot 4^k - 5\cdot 2^k + 2^{k+t+1} + 2^{t+2} + 4 + 4k + 2^k k).$$

Then $-\frac{1}{2^k+4}(-2\cdot 4^k - 5\cdot 2^k + 2^{k+t+1} + 2^{t+2} + 4 + 4k + 2^k k)$
$-\ (2(2^k - 2^t) - (k-1) - 4)$
$= 4\frac{2^t+2}{2^k+4}$.

Since, $0 < 4\frac{2^t+2}{2^k+4} < 1$ whenever $2 \leq t < k-2$, it follows that

$$r\left(P_{2^k-2^t}^{2^{k-1}}\right) \geq 2^{k+1} - k - 2 = r\left(T_{2^k-2^{k-2}}\right) - 1. \ \blacksquare$$

Example 16 *We have* $r(T_{28}) - 1 \leq r\left(P_{28}^{16}\right) \leq r(T_{28}) = 50.$

Next we show that Theorems 11 and 13 cannot be extended by decreasing the path power while keeping the reversing number fixed. We will show for each of the theorems that if we decrease the path power by one, the reversing number must decrease by at least one. We will investigate upper bounds for $r(P_n^k)$. Until this point we have used known values for $r(T_n)$ as our upper bounds for $r(P_n^k)$. Now will use methods similar to those presented in [4] and [8]. Upper bounds for $r(P_n^k)$ will be obtained by explicitly describing a set of arc disjoint cycles in $T\left(\overrightarrow{x}, P_n^k\right)$ for a given $\overrightarrow{x}$. We will adopt a method from [4] for viewing a collection of arc disjoint 3-cycles, it maybe helpful to refer to Figure 1. Each triangle corresponds to an entry in an upper triangular portion of the array L_n where the entries $L_n(i,j)$ for $1 \leq i < j \leq n$ are distinct integral ordered pairs (a,b) where $a \geq i$ and $a < j$. The conditions $a \geq i$ and $a < j$ are necessary to construct a 3-cycle containing arc (v_i, v_j) and no other arcs of D. To insure that the 3-cycles are disjoint, entries with the same second coordinate may not appear in the same row or column. The entry (i,j) in L_n corresponds to vertex $u_{i,j}$ in $T\left(\overrightarrow{x}, P_n^k\right)$. The 3-cycle is represented by $(v_i, u_{i,j}, v_j)$. In the case that D is a tournament, the upper right corner of the array is filled. However when $D = P_n^k$ only array positions $\{(i,j), i < j$ and $j - i \leq k\}$ are filled. We will refer to these positions as the *first k diagonals* of L_n. We give an example showing a collection of five arc disjoint 3-cycles found in $T\left(\overrightarrow{x}, P_4^2\right)$. Entries in the first two diagonals of L_n are used to construct the following set of five arc disjoint 3-cycles $\{(v_1, u_{1,0}, v_2), (v_1, u_{2,1}, v_3), (v_2, u_{2,0}, v_3), (v_2, u_{2,1}, v_4), (v_3, u_{3,0}, v_4)\}$.

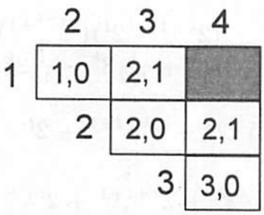

Figure 1. Five arc disjoint 3-cycles in $T(\vec{x}, P_n^k)$

Theorem 17 *We have* $r(P_n^k) \le n - 1 + \lceil\frac{n-2}{2}\rceil + \lceil\frac{n-4}{4}\rceil + \cdots + \lceil\frac{n-2^m}{2^m}\rceil$ *where* $m = \lfloor \log_2 k \rfloor$.

Proof. Using the definition of $L_n(i,j)$ described above, let:

$$L(i,j) = \begin{cases} (\lfloor\frac{i+j}{2}\rfloor, 0) & \text{if } i+j \text{ is odd} \\ (2^k\lfloor\frac{i+j}{2^{k+1}}\rfloor, k) & \text{if } i+j \text{ is even, } 2^k \mid (j-i) \text{ and } 2^{k+1} \nmid (j-i) \end{cases}.$$

Then for a given k, the number of distinct entries present in the first k diagonals of the array L, is given by $n - 1 + \lceil\frac{n-2}{2}\rceil + \lceil\frac{n-4}{4}\rceil + \cdots + \lceil\frac{n-2^m}{2^m}\rceil$. These entries can be used to form a collection of $|A(D)|$ arc disjoint 3-cycles, yielding the desired upper bound for the reversing number. ■

Corollary 18 *We have* $r\left(P_{2^k}^{2^{k-1}-1}\right) < r\left(P_{2^k}^{2^{k-1}}\right) = 2^{k+1} - k - 2$.

Proof. Following the arrangement of the array elements described in Theorem 17. The number of distinct entries in the array is $2^k + \frac{1}{2}(2^k - 2) + \frac{1}{4}(2^k - 4) + \cdots + \frac{1}{2^{k-2}}(2^k - 2^{k-2}) = 2^k(1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{k-2}}) - (k - 1) = 2^{k+1} - k - 3$. ■

Corollary 19 *We have* $r\left(P_{2^k-2^{k-2}}^{2^{k-1}-1}\right) < r\left(P_{2^k-2^{k-2}}^{2^{k-1}}\right) = 2(2^k - 2^{k-2}) - k - 2$.

Proof. Following the arrangement of the array elements described in Theorem 17. The number of distinct entries in the array is $(2^k - 2^{k-2} - 1) + (2^{k-1} - 2^{k-3} - 1) + \cdots + (2^2 - 2^0 - 1) = 2^k + 2^{k-1} - (k-1) - 2^1 - 2^0 = 2(2^k - 2^{k-2}) - k - 2$. ■

The combination of Theorems 11, 13 and 15 yields a proof for Theorem 3.

4 Conclusion

Precise reversing numbers were presented in Theorems 11 and 13. A bound that is within one of the correct answer is given in Theorem 15. It is possible that the integer programming methods used in this paper may be improved, possibly by adding additional inequalities, to raise the lower bound enough to close the gap. It can be verified that $r\left(P_{28}^{16}\right) = 49$ (the lower bound) if all inequalities of the form given in Theorem 4 are considered. We conjecture that in general it is the lower bound which equals the reversing number. However it is not known for the general case if there even exists an integer linear program having these types of inequalities whose minimum value is $r\left(P_n^k\right)$. Another strategy would be to close the gap by improving the upper bounds. However since the existence of an appropriately sized collection of arc disjoint 3-cycles is a sufficient, but not necessary condition, this approach may not be suitable. Because of the potential difficulties that arise in either raising the lower bound or dropping the upper bound, the closing of this gap could conceivably be more difficult than it appears. However getting the bounds to match for particular families of path powers may be quite reasonable.

Acknowledgements

This research was partially supported by an RIT College of Science Dean's Summer Research Fellowship Grant. The author thanks an anonymous referee for closely reading this paper and providing many valuable suggestions and comments.

References

[1] J. L. Baldwin, W. C. Kronholm, and D. A. Narayan, Tournaments with a transitive subtournament as a feedback arc set, Congr. Numer. (2002) **158**, 43-51.

[2] J. -P. Barthélemy, O. Hudry, G. Isaak, F. S. Roberts and B. Tesman, The reversing number of a digraph, Discrete Appl. Math, **60** (1995), 39-76.

[3] G. Dufore, J. L. Baldwin, and D. A. Narayan, Constructions for tournaments with an acyclic tournament as a minimum feedback arc set, preprint.

[4] G. Isaak, Tournaments as Feedback Arc Sets, Elec. J. Combin. **R20** (1995) 19 pp.

[5] G. Isaak and D. A. Narayan, Complete classification of all tournaments having a disjoint union of directed paths as a minimum feedback arc set, J. Graph Theory **45** (2004), 28-47.

[6] G. Isaak and D. A. Narayan, A classification of all tournaments having an acyclic tournament as a minimum feedback arc set, Info. Proc. Let., **92** no. 3. (2004), 107-111.

[7] D. A. Narayan, On the reversing number of powers of directed Hamiltonian Paths, Utilitas Math. **73** (2007), 181-206.

[8] D. A. Narayan, Tournaments with feedback path powers, J. Comb. Math. Comb. Comput. **49** (2004) 85-86.

[9] D. A. Narayan, The reversing number of a digraph; A disjoint union of directed stars, Congr. Numer. (2000) **145**, 153-164.