Global Domination in Planar Graphs

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Abstract

For any graph $G = (V, E)$, $D \subseteq V$ is a global dominating set if D dominates both G and its complement $\overline{G}$. The global domination number $\gamma_g(G)$ of a graph G is the fewest number of vertices required of a global dominating set. In general, $\max\{\gamma(G), \gamma(\overline{G})\} \leq \gamma_g(G) \leq \gamma(G)+\gamma(\overline{G})$, where $\gamma(G)$ and $\gamma(\overline{G})$ are the respective domination numbers of G and $\overline{G}$. We show, when G is a planar graph, that $\gamma_g(G) \leq \max\{\gamma(G)+1, 4\}$.

Keywords: Domination, global domination, planar graphs

1. Introduction

In a graph $G = (V, E)$, $D \subseteq V$ is said to dominate G when every vertex in V–D is adjacent to (a neighbor of) a vertex in D. A global dominating set is a set of vertices that dominates both G and the complement graph $\overline{G}$. The number of vertices in a smallest dominating set of G is $\gamma(G)$, and the number in a smallest global dominating set is denoted by $\gamma_g(G)$. A minimum (global) dominating set of vertices is referred to as a γ–set (γ_g–set). Global domination was introduced by Sampathkumar [6] and, independently, by Brigham and Dutton [2] as a special case of factor domination of a graph G. Further results on factor domination appear in Dankelman and Laskar [3]. A survey of global domination, as of 1998, was given by Brigham and Carrington [1]. Additional global domination results are given by Dutton [4] and by Dutton and Brigham [5].

The global domination number for several families of graphs is known or at least restricted to within a fairly limited range [1]; for example, when G (or $\overline{G}$) is disconnected, $\gamma_g(G) = \max\{\gamma(G), \gamma(\overline{G})\}$; for triangle–free graphs, $\gamma(G) \leq \gamma_g(G) \leq \gamma(G)+1$ [1, Corollary 1]; and the special case for trees has been completely characterized [1, Theorem 14]. The work presented here examines the global domination number for planar graphs.

Additional notation used includes: the *order* of a graph G is $n = |V|$; diam(G) is the *diameter* and r(G) is the *radius*; and $\gamma_c(G)$ is the *connected domination number*. For any vertex $v \in V$, $N_G(v)$ is the *open neighborhood* of v in G and is the set of vertices adjacent to v. The *closed neighborhood* of v is $N_G[v] = N_G(v) \cup \{v\}$. The subscript in the neighborhood notation will be omitted unless referring specifically to a graph other than G. For example,

$N_{\overline{G}}(v) = V - N[v]$ is the open neighborhood of v in the complement graph $\overline{G}$. For $W \subseteq V$, N(W) and N[W] are the unions of the open and closed neighborhoods, respectively, for every $v \in W$, and $\langle W \rangle$ is the subgraph of G induced by W. A full explanation of these and other graph theory terms can be found in [7].

2. Preliminary Results

In this section, results are presented that hold for all graphs, and are useful in the next section on planar graphs. The first is obvious and appears in [1].

Lemma 1. For any graph G, $\max\{\gamma(G), \gamma(\overline{G})\} \leq \gamma_g(G) \leq \gamma(G)+\gamma(\overline{G})$.

It was also stated in [1] that for any integers m, n, and k such that $2 \leq m \leq n \leq k \leq m+n$, there exists a graph G for which $\gamma(G) = m$, $\gamma(\overline{G}) = n$, and $\gamma_g(G) = k$. Next, Lemma 2 shows that $\gamma_g(G) \leq \gamma(G)+2$ for graphs with sufficiently large diameter.

Lemma 2. For any graph G, $\text{diam}(G) \geq 3$ if and only if $\gamma(\overline{G}) \leq \gamma_c(\overline{G}) \leq 2$ and $G \neq K_1$.

Proof: Suppose G has two vertices x and y that are distance 3. Then G is not a K_1 and, in $\overline{G}$, x and y are a connected dominating set. When $\text{diam}(G) \leq 2$, every pair of non adjacent vertices has a common neighbor. Thus, in $\overline{G}$, no pair of adjacent vertices can be a dominating set. ■

Lemma 3. For any graph G, if $r(G) \geq 3$, then every dominating set of G is a dominating set of $\overline{G}$.

Proof: Let D be an arbitrary dominating set of G and suppose D does not dominate $\overline{G}$. Then there exists a vertex $v \in V-D$ for which $D \subseteq N(v)$. Therefore, the distance in G between v and any other vertex is at most 2, that is $r(G) \leq 2$. ■

An immediate consequence of Lemma's 2 and 3, since $\text{diam}(G) \geq r(G)$, is that when $r(G) \geq 3$, then $\gamma(\overline{G}) \leq \gamma_c(\overline{G}) \leq 2 \leq \gamma(G) = \gamma_g(G)$.

Lemma 4. For any graph G, every set of $\gamma(\overline{G})-1$ vertices has a common neighbor, and every maximal complete subgraph of G has at least $\gamma(\overline{G})$ vertices.

Proof: No set of $\gamma(\overline{G})-1$ vertices can dominate $\overline{G}$. Thus, in G, every such set of vertices has a common neighbor. Suppose G has a maximal complete subgraph W on $k < \gamma(\overline{G})$ vertices. Since $k < \gamma(\overline{G})$, the vertices of W must have a common neighbor, contradicting that W is maximal complete subgraph. ■

3. Planar Graphs

For any graph G, the graph formed by either deleting an edge or identifying two adjacent vertices (an "edge contraction") is a "minor" of G. From Wagner's Theorem, a graph G is planar if and only if no subgraph of G has a minor isomorphic to K_5 or $K_{3,3}$.

Lemma 5. If G is a planar graph, $\gamma(\overline{G}) \leq 4$.

Proof: If $\gamma(\overline{G}) > 4$, from Lemma 4, G must have a subgraph isomorphic to a K_5, a contradiction for planar graphs. ■

From Lemmas 1 and 5 it is immediate that $\gamma_g(G) \leq \gamma(G)+4$ for any planar graph G. We show in the remainder of this section that equality is not possible and, in fact, that $\gamma_g(G)$ is rarely larger than $\gamma(G)+1$.

Theorem 6. If G is planar graph and $\gamma(G) \geq 3$, then $\gamma_g(G) \leq \gamma(G)+1$.

Proof: Let D be an arbitrary γ–set of G and $X = \{v \mid N(v) \supseteq D\}$. If $|X| = 0$, then D is a global dominating set and $\gamma_g(G) = \gamma(G)$. If $|X| = 1$, then $D \cup X$ is a global dominating set of $\gamma(G)+1$ vertices. If $|X| \geq 3$, then $\langle D \cup X \rangle$ has a $K_{3,3}$ minor, a contradiction for planar graphs. Thus, we must have that $|X| = 2$. If there exists a vertex $w \in V-D$ for which $N(w) \cap X$ is empty, then $D \cup \{w\}$ is a global dominating set with $\gamma(G)+1$ vertices. Otherwise, X is a dominating set of G, a contradiction since $|X| = 2 < \gamma(G)$. ■

The next result is not surprising, although its proof is not straightforward.

Theorem 7. If G is a planar graph, $\gamma_g(G) \leq \max\{\gamma(G)+1, 4\}$.

Proof: By way of contradiction, assume G is planar and $\gamma_g(G) > \max\{\gamma(G)+1, 4\}$. We present a series of claims resulting from this assumption. A culminating contradiction will establish that $\gamma_g(G) \leq \max\{\gamma(G)+1, 4\}$.

Claim 1. G and $\overline{G}$ are connected and $\gamma(\overline{G}) > \gamma(G) = r(G) = r(\overline{G}) = \text{diam}(G) = 2$.

Proof: From Lemma 5, $\max\{\gamma(G)+1, 4\} \geq \max\{\gamma(G), \gamma(\overline{G})\}$. Thus, $\gamma_g(G) > \max\{\gamma(G), \gamma(\overline{G})\}$ and, hence, no γ–set of G can be a dominating set of $\overline{G}$. Then from Lemma 3, $r(G) \leq 2$ and $r(\overline{G}) \leq 2$. Therefore, both G and $\overline{G}$ are connected. This in turn implies $r(G) > 1$ and $r(\overline{G}) > 1$. Thus, $r(G) = r(\overline{G}) = 2$ and $\gamma(G) \geq 2$. From Theorem 6, $\gamma(G) \leq 2$ implying $\gamma(G) = 2$. From Lemma 1, $\gamma_g(G) \leq \gamma(G)+\gamma(\overline{G})$. Since we assume $\gamma_g(G) > 4$, we must have that $5 \leq \gamma_g(G) \leq \gamma(\overline{G})+2$. Hence, $\gamma(\overline{G}) \geq 3$. Finally, from Lemma 4, every pair of vertices in G has a common neighbor and, therefore, $\text{diam}(G) = 2$.

The following definitions are similar to those made in the proof of Theorem 6. Let $D = \{a, b\}$ be a γ–set of G and $X = \{v \mid N(v) \supseteq D\}$. Also, let $X_a = \{v \mid v \notin$

$D \cup X$ and v is adjacent to $a \in D\}$ and $X_b = \{v \mid v \notin D \cup X$ and v is adjacent to $b \in D\}$. Then X_a, X_b, X, and D form a partition of V.

Claim 2. Every vertex in V–D has exactly two neighbors in X.

Proof: Consider any vertex $w \in V-D$ and let $Z = N(w) \cap X$. If $|Z| \leq 1$, then $D \cup \{w\} \cup Z$ is a global dominating set of at most 4 vertices, contradicting the assumption that $4 < \gamma_g(G)$. If $|Z| \geq 3$, then $\langle D \cup Z \cup \{w\}\rangle$ has a $K_{3,3}$ minor, where one partite set is {a, b, w} and the other is any three vertices in Z, a contradiction for planar graphs.

Claim 3. D is an independent set and $\langle X \rangle$ is a cycle.

Proof: From Claim 2, $\langle X \rangle$ is a collection of one or more cycles. In any of the cycles, contract edges until the cycle has exactly three vertices that we denote by {x, y, z}. If ab is an edge, then D and {x, y, z} form a K_5, a contradiction for planar graphs. Thus, D must be an independent set. If $\langle X \rangle$ contains another cycle, let w be any vertex in the second cycle and identify w with $a \in D$, thereby creating a K_5 minor, a contradiction showing $\langle X \rangle$ is a single cycle.

Claim 4. $|X_a| \geq 2$, $|X_b| \geq 2$, and $|X| \geq 4$.

Proof: If X_a is empty, $b \in D$ and any vertex in X is a connected γ–set, and if $X_a = \{w\}$, then $b \in D$ and any neighbor of w in X is a connected γ–set. Both instances contradict Claim 3. Therefore, $|X_a| \geq 2$. A similar argument shows $|X_b| \geq 2$. From Claim 3, $\langle X \rangle$ is a cycle, hence, $|X| \geq 3$. If $|X| = 3$, any two vertices of X, by Claim 2, is a connected dominating set of G, again contradicting Claim 3.

Claim 5. Every $w \in X_a \cup X_b$ has two adjacent neighbors in X.

Proof: From Claim 2, w has exactly two neighbors in X. If they are not adjacent, w can be identified with either neighbor in X, resulting in a vertex w' having three neighbors in X. As in the proof of Claim 2, this graph, and hence G, has a $K_{3,3}$ minor, a contradiction.

Claim 6. If v and w are two vertices in $X_a \cup X_b$ that have two common neighbors in X, then $|\{v, w\} \cap X_a| = |\{v, w\} \cap X_b| = 1$.

Proof: Suppose both of v and w are, without loss of generality, in X_a and have common neighbors x and y in X. Then xy is an edge by Claim 5. Select any $z \in X-\{x, y\}$ and identify z and $a \in D$, creating the vertex a' and edge a'b. Then $K_{3,3}$ is a minor of $\langle\{v, w, b, a', y, x\}\rangle$ where{v, w, b} is a partite set, a contradiction.

Claim 7. $X_a \cup X_b$ is an independent set.

Proof: If v and w are adjacent vertices in $X_a \cup X_b$, they must have the same two neighbors in X. If not, contracting the edge vw produces a vertex with three neighbors in X. Thus, similar to the proof of Claim 2, G has a $K_{3,3}$ minor, a contradiction. Therefore, if v and w have common neighbors x and y (adjacent, by Claim 5) in X, they must have different neighbors in D, by Claim 6. Therefore, without loss of generality, $v \in X_a$ and $w \in X_b$. Then identifying $a \in D$ with any vertex $z \in X-\{x, y\}$ creates the vertex a' with edge a'b. A subgraph results that has a $K_{3,3}$ minor, where the partite sets are {a', x, w} and {b, v, y}, a contradiction.

Claim 8. If $v \in X_a$ and $w \in X_b$, then v and w must have a common neighbor in X.

Proof: The set $X_a \cup X_b$ is independent, by Claim 7. Since diam(G) = 2, v and w must have a common neighbor that can only be in X.

Finally, consider distinct vertices guaranteed by Claim 4, $\{a_1, a_2\} \subseteq X_a$ and $\{b_1, b_2\} \subseteq X_b$. By Claim 5, a_1 and a_2 each have two adjacent neighbors in X. Let the two neighbors for a_1 be $\{x_1, x_2\}$ and those of a_2 be $\{x_3, x_4\}$. Then by Claim 6, $\{x_1, x_2\} \neq \{x_3, x_4\}$. Therefore, either the two sets have one vertex in common, or none. We examine each case.

(1) First, suppose $x_2 = x_3$, that is, $\{x_1, x_2, x_4\}$ induces a path. Since by Claim 5, $|X| \geq 4$, x_4 must have a neighbor in X other than x_2, and it can not be x_1 by Claim 3. Next, b_1 must have two adjacent neighbors in X and, from Claim 8, a neighbor in common with each of a_1 and a_2. We may assume these are x_1 and x_2. Furthermore, b_2 also must have two adjacent neighbors in X and a common neighbor with each of a_1 and a_2. These can not be x_1 and x_2, from Claim 6. Since x_4 is not adjacent to x_1, they also can not be x_1 and x_4. Hence, they must be x_2 and x_4. Suppose there is another vertex z in either X_a or X_b. Then z also must have $\{x_1, x_2\}$, or $\{x_2, x_4\}$, as neighbors, contradicting Claim 6. Therefore $|X_a| = |X_b| = 2$ and x_2 is a common neighbor of the four vertices in $X_a \cup X_b$. Since ax_2 is an edge, the set $\{a, x_2\}$ is a connected γ-set of G, contradicting Claim 3.

(2) Next, assume $\{x_1, x_2\}$ and $\{x_3, x_4\}$ are distinct sets of vertices. From Claim 8, b_1 must have a neighbor in each of these sets and we may assume they are x_2 and x_3 and, hence, that x_2 and x_3 must be adjacent by Claim 5. Similarly, b_2 also must have a neighbor in each set, but neither can be x_2 or x_3. Thus, the neighbors of b_2 must be x_1 and x_4 which implies x_1 and x_4 are adjacent, that is, $|X| = 4$. As above, there can be no other vertices in X_a or X_b. As constructed, $\{a_1, x_3, b_2\}$ forms a global dominating set, contradicting that $\gamma_g(G) \geq 5$.

Therefore, $\gamma_g(G) \leq \max\{\gamma(G)+1, 4\}$ and the proof of the theorem is complete. ■

Consider the class of graphs G = (V, E) having a vertex $v \in V$ for which (1) $\deg(v) = |V|-1$, and (2) $G-\{v\}$ is a collection of cycles and/or paths. From this class, let $H_3 = \{K_4\}$ and H_1 be the subset having $\delta(G) = 1$. Finally, let H_2 be the remaining graphs in the class. It is straightforward for any graph G in this class, that $\gamma(G) = 1$ and, for $1 \leq i \leq 3$, if $G \in H_i$, then $\gamma_g(G) = \gamma(\overline{G}) = i+1$. Notice that no graph in H_1, H_2, and H_3 possesses a subgraph isomorphic to $K_{2,3}$, but all other graphs with $\gamma(G) = 1$ do have $K_{2,3}$ as a subgraph.

Theorem 8. If G is planar and has no subgraph isomorphic to a $K_{2,3}$, then

(1) $\gamma_g(G) = \gamma(\overline{G}) = i+1$, when $G \in H_i$, for $1 \le i \le 3$, and

(2) $\gamma_g(G) \le \gamma(G)+1$, otherwise.

Proof: The comments in the preceding paragraph establish the validity of (1) and that we may assume $\gamma(G) \ge 2$. Suppose G is a counterexample to (2). Then $\gamma(G)+1 < \gamma_g(G)$ and, from Theorem 7, $\gamma(G) = 2$ and $\gamma_g(G) = 4$.

As in the proof of Theorem 6, let D be a γ–set of G and $X = \{v \mid N(v) \supseteq D\}$. Then $|D| = 2$. If $|X| \le 1$, then $D \cup X$ is a global dominating set with at most 3 vertices. That is, $\gamma_g(G) \le 3 = \gamma(G)+1$, a contradiction for a counterexample. If $|X| \ge 3$, then D with any three vertices in X has a subgraph isomorphic to a $K_{2,3}$. Therefore, we may assume $|X| = 2$. For any $w \in V-D$, let $Z = N(w) \cap X$. If $|Z| = 0$, then $D \cup \{w\}$ is a global dominating set with three vertices and, again, $\gamma_g(G) \le 3$. If $|Z| \ge 2$, then $w \notin X$ and the three vertices in $D \cup \{w\}$ and X has a subgraph isomorphic to $K_{2,3}$, a contradiction. Therefore, every vertex in V–D has exactly one neighbor in X. Thus, X is a connected γ–set of G and we may consider that D is connected. Then $D \cup X$ induces a K_4. Since $G \ne K_4$, there is a vertex w with one neighbor in X and one in D. These 5 vertices have a subgraph isomorphic to $K_{2,3}$, where one partite set is $N(w) \cap (D \cup X)$. This final contradiction establishes the theorem. ■

It is well known that a graph G is outerplanar if and only if no subgraph of G has a $K_{2,3}$ or K_4 minor. Specifically, an outerplanar graph has no subgraph isomorphic to a $K_{2,3}$. Thus, we have the following.

Corollary 9. If G is outerplanar then

(1) $\gamma_g(G) = \gamma(\overline{G}) = i+1$, when $G \in H_i$, for $1 \le i \le 3$, and

(2) $\gamma_g(G) \le \gamma(G)+1$, otherwise.

4. References

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