Those Amazing Desargues Configurations

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Abstract

Given 2 triangles in a plane over a field F which are in perspective from a vertex V, the resulting Desargues line or axis l may or may not be on V. To avoid degenerate cases we assume that the union of the vertices of the 2 triangles is a set of six points with no three collinear. Our work then provides a detailed analysis of situations when V is on l for any F, finite or infinite.

1 Background, introduction

In a projective plane π, two triangles $P_1P_3P_5$ and $P_2P_4P_6$ are said to be in *perspective* from a point (= center) V if the lines P_1P_2, P_3P_4 and P_5P_6 all pass through V. If π is the plane over a field F — or even a division ring — then the 3 intersections of corresponding sides of the triangles must lie on a line, called the *Desargues line* or *axis*. This is the content of the celebrated theorem of Desargues. The resulting set of 10 points and 10 lines is a Desargues configuration. See Figure 1. (It is the case that the two triangles need not be in the same plane. In fact, the study of Desargues configurations first arose in the study of perspective painting in medieval times. Here the 2 triangles were not in the same plane. The vertex was the eye of the painter. The Desargues line was the intersection of the scene plane and the "plane of the canvas").

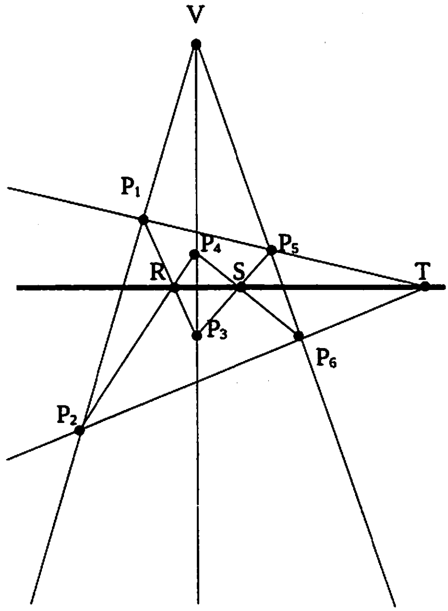

Figure 1: Desargues configuration

Throughout, we assume that F is in fact a field so that we may exploit also the theory of conics in π.

We will be interested in the situation where the vertex is on the axis. It is easy to construct such examples, for any field F, as follows. Let $P_1P_3P_5$ be any triangle in the Euclidean plane over $\mathbb{R}$. Take any translation T mapping $P_1P_3P_5$ to $P_2P_4P_6$ say. The translation T has a certain direction which can be identified with the point V at infinity corresponding to where the line P_1P_2 say meets the line at infinity denoted by l. Then the 2 triangles give a Desargues configuration with the vertex V lying on the axis l. This example works, not just for $\mathbb{R}$, but for any field (or division ring) F.

With this background, we proceed more formally, as follows. Consider any projective plane. Given two triangles $P_1P_3P_5$ and $P_2P_4P_6$ in perspective from a point V, $V = P_1P_2 \cap P_3P_4 \cap P_5P_6$ say, the points $R = P_1P_3 \cap P_2P_4$, $S = P_4P_6 \cap P_3P_5$ and $T = P_1P_5 \cap P_2P_6$ are collinear if

Desargues' Theorem holds. In that case, the line through R, S and T is the *Desargues line*, and the points $P_1, \ldots, P_6, V, R, S, T$ and the lines

$$P_1P_2, P_3P_4, P_5P_6, P_1P_3, P_2P_4, P_3P_5, P_4P_6, P_1P_5, P_2P_6, RS$$

form a *Desargues configuration.* What can we say when the Desargues line passes through the point V? We study this question here for projective planes over various fields. To avoid any degenerate examples, we only consider the case where the set consisting of the vertices of the two triangles in question, $\{P_1, \ldots, P_6\}$, is a 6-arc. Here, a *k-arc* in a projective plane is a set of k points, no three collinear.

We now introduce the concepts of Special-Desargues configurations and Very Special Desargues configurations or arcs.

Definition 1.1. *Let $P_1P_3P_5$ and $P_2P_4P_6$ be triangles in perspective from a point V with $V = P_1P_2 \cap P_3P_4 \cap P_5P_6$, say, and denote by R, S and T the points $R = P_1P_3 \cap P_2P_4$, $S = P_4P_6 \cap P_3P_5$ and $T = P_1P_5 \cap P_2P_6$. Then the points $P_1, \ldots, P_6, V, R, S, T$ and the lines*

$$P_1P_2, P_3P_4, P_5P_6, P_1P_3, P_2P_4, P_3P_5, P_4P_6, P_1P_5, P_2P_6, RS$$

form a Special-Desargues configuration *if*

(i) they form a Desargues configuration,

(ii) V is on the Desargues line of that Desargues configuration, and

(iii) $\{P_1, \ldots, P_6\}$ is a 6-arc Γ.

See Figure 2.

Remark 1.1. *As pointed out in the introduction these configurations exist, over any field. In fact, what we have called a Special-Desargues configuration is usually called the Little Desargues configuration (see [4], [8] p. 719) except that here we impose the extra condition (iii) above.*

We now specialize even more to Very Special Desarguesian configurations denoted also by VS-Desarguesian configurations.

Definition 1.2. *Let $\Gamma = \{P_1, \ldots, P_6\}$ be a 6-arc in a projective plane. Then Γ is a* Very Special Desarguesian arc, *abbreviated* VS-Desarguesian arc, *if there exists a point $V \notin \Gamma$ such that the following two conditions hold:*

(i) There exist three distinct lines through V each of which meets Γ in two points. (Relabelling then gives $V = P_1P_2 = P_3P_4 = P_5P_6$.)

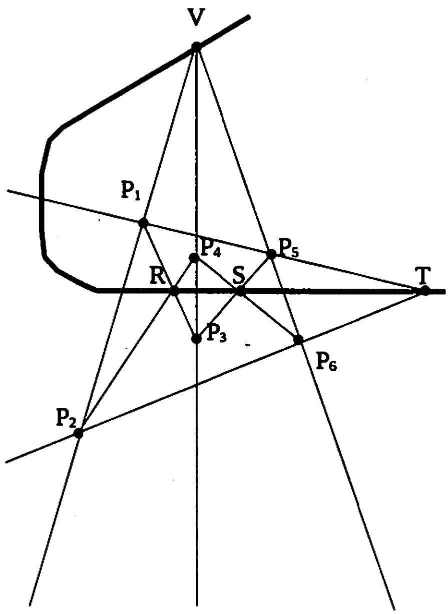

Figure 2: Special-Desargues configuration

(ii) Of the four ways to partition $P_1, \ldots, P_6$ into pairs of disjoint triangles that are in perspective from V, at least two of these pairs, together with V, yield Special-Desargues configurations.
(The four ways to partition $P_1, \ldots, P_6$ into pairs of disjoint triangles that are in perspective from V are $P_1P_3P_5$, $P_2P_4P_6$; $P_1P_3P_6$, $P_2P_4P_5$; $P_1P_4P_5$, $P_2P_3P_6$; and $P_1P_4P_6$, $P_2P_3P_5$.)

It will transpire that in the detailed study of such configurations the characteristic of the field F will play a crucial role.

Given a quadrangle (a 4-arc) $\{P_1, ..., P_4\}$ in a projective plane over a field of characteristic 2, the *Fano line* of this quadrangle is the line containing the points $R = P_1P_2 \cap P_3P_4$, $S = P_1P_3 \cap P_2P_4$ and $T = P_1P_4 \cap P_2P_3$. The seven points

$$P_1, P_2, P_3, P_4, R, S, T$$

and seven lines

$$P_1P_2, P_1P_3, P_1P_4, P_2P_3, P_2P_4, P_3P_4, RS$$

form a *Fano configuration* (a projective plane of order 2). Such Fano configurations do not exist in projective planes over fields of characteristic different from 2. Please see [5] and [7] Section 1.7 for more information on projective planes, and conics and their nuclei, and [6] for more on conics and fano configurations, for example.

In this paper, we will use the fact that given a non-degenerate conic C in $PG(2, F)$, where F is any field of characteristic 2 and order greater than 2, the Fano line of a quadrangle of points of C must contain the nucleus of C.

Let us now summarize our results.

We show that, if F has characteristic 2, then the six points of the arc Γ (belonging to a Special Desargues configuration) must lie on a non-degenerate conic whose nucleus must lie on the Desargues line. The converse is also true. This is Theorem 2.1. On the other hand, if F has characteristic different from 2 we show (see Lemma 2.1) that the six points of Γ cannot lie on a conic. This result also has implications for collineations fixing conics, which we will discuss in a future paper. In Lemma 2.4 we prove that VS-Desarguesian arcs do not exist in planes $PG(2, F)$ where F has characteristic unequal to two. However, in the case when F has characteristic 2, it is shown (see Theorem 2.3) that every Special-Desargues configuration gives rise to a VS-Desarguesian arc.

2 Some existence results

We will use (homogeneous) coordinates occasionally for investigating properties of Special-Desargues configurations.

Proposition 2.1. *Let $\pi = PG(2, F)$, where F is a field. Given a Special-Desargues configuration formed by the triangles $P_1P_3P_5$, $P_2P_4P_6$ in perspective from V with Desargues line through $V = P_1P_2 = P_3P_4 = P_5P_6$, $R = P_1P_3 \cap P_2P_4$, $S = P_4P_6 \cap P_3P_5$ and $T = P_1P_5 \cap P_2P_6$, π may be coordinatized so that $V = (1, 1, 0)$, $P_1 = (1, 0, 0)$, $P_2 = (0, 1, 0)$, $P_3 = (0, 0, 1)$, and $P_4 = (1, 1, 1)$. With this coordinatization, $P_5 = (u, v, 1)$ for some $u, v \in F$. Therefore, $P_6 = (1 + v, 1 + 2v - u, 1)$, $R = (1, 0, 1)$, $S = (u, v, u - v)$ and $T = (1 + v, v, 1)$.*

Proof. Let l_0 be the line through V, R, S, T. Coordinatize so that $P_1 = (1, 0, 0)$, $P_2 = (0, 1, 0)$, $P_3 = (0, 0, 1)$, and $P_4 = (1, 1, 1)$. Thus, $V = (1, 1, 0)$ and $R = P_1P_3 \cap P_2P_4 = (1, 0, 1)$. Now $P_5 = (u, v, 1)$, for some $u, v \in F$. Since we have a Special-Desargues configuration, l_0 is the line $Y = X - Z$. Because $T = P_1P_5 \cap P_2P_6$ is on l_0, $T = (1 + v, v, 1)$. Because

$S = P_4P_6 \cap P_3P_5$ is on l_0, $S = (u, v, u - v)$. Because P_6 is on P_2T and VP_5, $P_6 = (1 + v, 1 + 2v - u, 1)$. □

We opted to include the 6-arc restriction in our definitions of Special-Desargues configuration and VS-Desarguesian arc to avoid any degenerate cases. Because of this, some small projective planes do not contain any Special-Desargues configurations or VS-Desarguesian arcs. The smallest projective plane containing a 6-arc is $PG(2, 4)$, but it does not contain any Special-Desargues configurations.

2.1 Special-Desargues and conics

Given a 6-arc $\{P_1, \ldots, P_6\}$ with triangles $P_1P_3P_5$, $P_2P_4P_6$ in perspective from V, we study whether or not $P_1, \ldots, P_6$ lie on a conic. We first consider projective planes over fields of characteristic different from 2.

Lemma 2.1. *Let $\pi = PG(2, F)$, where F is a finite or infinite field of characteristic different from 2. Suppose that the triangles $P_1P_3P_5$ and $P_2P_4P_6$ in perspective from V form a Special-Desargues configuration in π. Then the points of the 6-arc $\{P_1, \ldots, P_6\}$ do not lie on a conic.*

Proof. Coordinatize as in Proposition 2.1. By way of contradiction, suppose that $C : aX^2 + bY^2 + cZ^2 + dXY + eXZ + fYZ = 0$ is a conic through $P_1, \ldots, P_6$. Now $a = b = c = 0$ since $P_1, \ldots, P_3$ are on C. Because $P_4, \ldots, P_6$ are on C, we have $d + e + f = 0$, $duv + eu + fv = 0$, and $d(1 + v)(1 + 2v - u) + e(1 + v) + f(1 + 2v - u) = 0$. Thinking of this as a system of three equations in d, e, f, the determinant of the coefficient matrix equals $2v(u - v - 1)(u - v)$. Note that this is never equal to 0 since $\{P_1, \ldots, P_6\}$ is a 6-arc. Thus, there is no conic through $P_1, \ldots, P_6$. □

Next, we consider projective planes over fields of characteristic 2.

Lemma 2.2. *Let $\pi = PG(2, F)$, where F is a finite or infinite field of characteristic 2. Suppose that the triangles $P_1P_3P_5$, $P_2P_4P_6$ in perspective from V form a Special-Desargues configuration in π. Then the points of the 6-arc $\{P_1, \ldots, P_6\}$ must lie on a (non-degenerate) conic. Moreover, the Desargues line must equal the line VN where N is the nucleus of the conic.*

Proof. As mentioned, there are no Special-Desargues configurations in $PG(2, 2)$ or $PG(2, 4)$.

To prove the result we can use Lemma 2.1. In particular, since we are now in characteristic 2, the determinant there is zero. This shows that the 6 points lie on a non-degenerate conic. Then, using the fact that the Fano line of any inscribed quadrangle of a conic passes through the nucleus, the result follows. □

This same argument concerning the Fano line also proves our next result which is a converse to Lemma 2.2. For completeness sake we state it here as follows.

Lemma 2.3. *Let $\pi = PG(2, F)$, where F is a finite or infinite field of characteristic 2. Let C be a (non-degenerate) conic with nucleus N. Let V be any point not on $C \cup \{N\}$, and let $A_1A_3A_5$, $A_2A_4A_6$ be triangles of points of C in perspective from V, $A_i \neq A_j$, $i \neq j$. Then the resulting Desargues configuration is a Special-Desargues configuration. The Desargues line is the line VN.*

Combining Lemmas 2.2 and 2.3, we have the following.

Theorem 2.1. *Let $\pi = PG(2, F)$, where F is a finite or infinite field of characteristic 2. Consider a pair of triangles in perspective from a point V, where the set containing the vertices of that pair of triangles is a 6-arc, Γ. The resulting Desargues configuration is a Special-Desargues configuration if and only if Γ lies on a unique (non-degenerate) conic. Moreover, the nucleus N of the conic is on the Desargues line which is therefore the line VN.*

Proof. This follows from the Lemma 2.2 and Lemma 2.3. The uniqueness of the conic follows from the fact that Γ is a 6-arc. □

2.2 VS-Desarguesian arcs

Lemma 2.4. *If a VS-Desarguesian arc exists in a projective plane over a finite or infinite field F, then F must have characteristic 2.*

Proof. Suppose that $P_1, \ldots, P_6$ is a VS-Desarguesian arc such that triangles $P_1P_3P_5$ and $P_2P_4P_6$ in perspective from $V = P_1P_2 = P_3P_4 = P_5P_6$ form a Special-Desargues configuration. This gives us (at least) two Special-Desargues configurations. Relabel $P_1, \ldots, P_6$ if necessary so that one Special-Desargues configuration corresponds to triangles $P_1P_3P_5$ and $P_2P_4P_6$ in perspective from V so that the points V, $R = P_1P_3 \cap P_2P_4$, $S = P_4P_6 \cap P_3P_5$ and $T = P_1P_5 \cap P_2P_6$ are collinear. Let l_0 the the line through V, R, S, T. Coordinatize as in Proposition 2.1. Consider the other three ways to partition $P_1, \ldots, P_6$ into pairs of disjoint triangles that are in perspective from V, $P_1P_3P_6, P_2P_4P_5$, $P_1P_4P_5, P_2P_3P_6$ or $P_1P_4P_6, P_2P_3P_5$. A direct calculation as in Proposition 2.1 shows that, if any of these pairs, together with V, yield Special-Desargues configurations, then the characteristic of F must be 2. □

Lemma 2.5. *Let $\pi = PG(2, F)$, where F is a finite or infinite field of characteristic 2 and of order greater than 4. Let C be a (non-degenerate)* *conic with nucleus N. Let V be any point not on $C \cup \{N\}$, and l_1, l_2 and l_3 any three lines through V meeting C in two points each. Let P_1, P_2 be the two points of l_1 on C, P_3, P_4 the two points of l_2 on C, and P_5, P_6 the two points of l_3 on C. Then $\{P_1, \ldots, P_6\}$ is a VS-Desarguesian arc. Each of the four possible pairs of triangles in perspective from V yields a Special-Desargues configuration. In each case the Desargues line is the line VH.*

Proof. Note that Lemma 2.3 applies to any of the four possible pairs of triangles in perspective from V. The result follows from Lemma 2.3. □

Theorem 2.2. *Let $\pi = PG(2, F)$, where F is a finite or infinite field of order greater than 4. Then π contains a VS-Desarguesian arc if and only if F has characteristic 2.*

Proof. This follows from Lemma 2.4 and Lemma 2.5. □

Theorem 2.3. *Let $\pi = PG(2, F)$, where F is a finite or infinite field of characteristic 2. Then every Special-Desargues configuration yields a VS-Desarguesian arc.*

Proof. Consider a Special-Desargues configuration with triangles $P_1P_3P_5$, $P_2P_4P_6$ in perspective from V. Since the given triangles are part of a Special-Desargues configuration, $P_1, \ldots, P_6$ must lie on a unique conic by Theorem 2.1. Therefore, $\{P_1, \ldots, P_6\}$ is a VS-Desarguesian arc by Lemma 2.5. □

Finally, we give a constructive example of a 6-arc in a projective plane over a field of characteristic 2 which is not a VS-Desarguesian arc.

Theorem 2.4. *In $PG(2, F)$, where F is a finite or infinite field of characteristic 2 and order greater than 2, there exist 6-arcs which are not VS-Desarguesian arcs.*

Proof. Let C be a non-degenerate conic, and N its nucleus. Pick $\Gamma = \{P_1, \ldots, P_5, N\}$, where $P_1, \ldots, P_5$ are any five points of C. Then there is no non-degenerate conic through Γ. By Theorem 2.1, Γ is not a VS-Desarguesian arc. □

3 Conclusion

We have completely characterized the Special-Desargues configurations in projective planes over finite and infinite fields of characteristic 2. Given a pair of triangles in perspective from a point V, where the set containing the vertices of that pair of triangles is a 6-arc Γ, the resulting Desargues configuration is a Special-Desargues configuration if and only if Γ lies on some non-degenerate conic. Moreover, the nucleus of the conic is on the Desargues line. The 6-arc Γ is a VS-Desarguesian arc.

If the characteristic of the field is different from 2, then the points of Γ cannot be on a conic if the resulting Desargues configuration is a Special-Desargues configuration. There are no VS-Desarguesian arcs if the characteristic is different from 2.

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