On Friendly Index Sets of Generalized Books

Harris Kwong
Department of Mathematical Sciences
State University of New York at Fredonia
Fredonia, NY 14063, USA

Sin-Min Lee
Department of Computer Science
San Jose State University
San Jose, CA 95192, USA

Abstract

Let $G = (V, E)$ be a graph with a vertex labeling $f : V \to \mathbb{Z}_2$ that induces an edge labeling $f^* : E \to \mathbb{Z}_2$ defined by $f^*(xy) = f(x) + f(y)$. For each $i \in \mathbb{Z}_2$, let $v_f(i) = \text{card}\{v \in V : f(v) = i\}$ and $e_f(i) = \text{card}\{e \in E : f^*(e) = i\}$. A labeling f of a graph G is said to be friendly if $|v_f(0) - v_f(1)| \leq 1$. The friendly index set of G is defined as $\{|e_f(1) - e_f(0)| : \text{the vertex labeling } f \text{ is friendly}\}$. In this paper, we determine the friendly index sets of generalized books.

1 Introduction

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$, and let A be an abelian group. A vertex labeling $f : V(G) \to A$ induces an edge labeling $f^* : E(G) \to A$ defined by $f^*(xy) = f(x) + f(y)$. For each $i \in A$, let $v_f(i) = \text{card}\{v \in V(G) : f(v) = i\}$ and $e_f(i) = \text{card}\{e \in E(G) : f^*(e) = i\}$. A vertex labeling f of a graph G is said to be A-friendly if $|v_f(i) - v_f(j)| \leq 1$ for all $(i, j) \in A \times A$. If $|e_f(i) - e_f(j)| \leq 1$ for all $(i, j) \in A \times A$, we say f is A-cordial. The notion of A-cordial labeling was first introduced by Hovey [4] to generalize the concept of cordial graphs [1] of Cahit, who considered $A = \mathbb{Z}_2$. In this paper, we will exclusively focus on $A = \mathbb{Z}_2$, and omit the reference to the group. When the context is clear, we will also drop the subscript f. In [3], the following concept was introduced.

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Definition. The ***friendly index set*** FI(G) of a graph G is defined as $\{|e_f(1) - e_f(0)| : f \text{ is friendly}\}$.

Note that G is cordial if 0 or 1 is in FI(G). Hence friendly index sets could be viewed as a generalization of cordiality. Cairnie and Edwards [2] proved that deciding whether a graph admits a cordial labeling is NP-complete. Even the restricted problem of deciding whether a connected graph of diameter 2 has a cordial labeling is still NP-complete. Thus, in general, it is difficult to find friendly index sets. In [5, 6, 7, 8] the friendly index sets of a few classes of graphs, including complete bipartite graphs and cycles, are determined. The following result was established.

Theorem 1.1 *For any graph with q edges,*

$$FI(G) \subseteq \begin{cases} \{0,2,4,\ldots,q\} & \textit{if } q \textit{ is even}, \\ \{1,3,5,\ldots,q\} & \textit{if } q \textit{ is odd}. \end{cases}$$

Example 1. The labelings in Figure 1 illustrates FI($K_{3,3}$) = $\{1, 9\}$ and FI($C_3 \times K_2$) = $\{1, 3, 5\}$. □

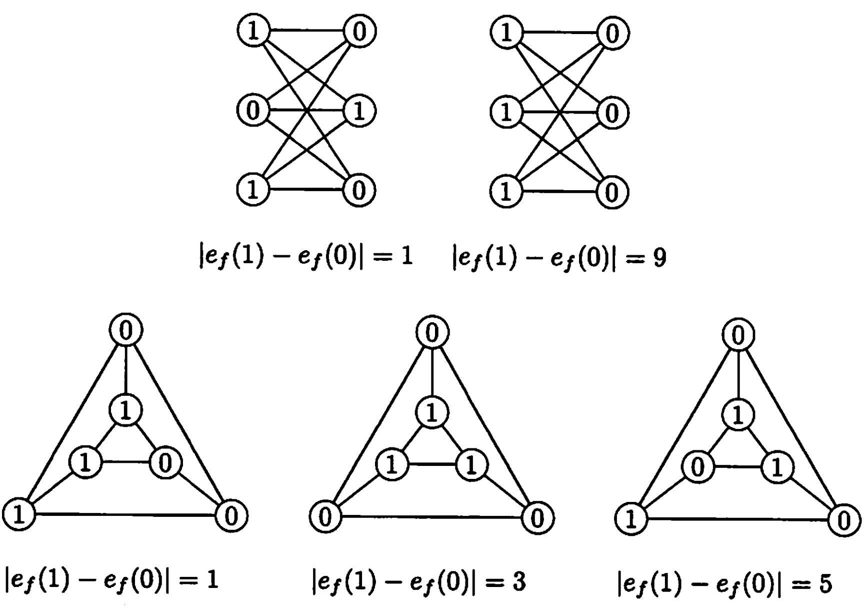

Figure 1: Friendly Labelings of $K_{3,3}$ and $C_3 \times K_2$.

In [6], we found

Theorem 1.2 *The friendly index set of a cycle is*

$$FI(C_n) = \begin{cases} \{0,4,8,\dots,n\} & \text{if } n \equiv 0 \pmod 4, \\ \{2,4,6,\dots,n\} & \text{if } n \equiv 2 \pmod 4, \\ \{1,3,5,\dots,n-2\} & \text{if } n \text{ is odd.} \end{cases}$$

Conjecture. The numbers in FI(T) form an arithmetic progression for any tree T.

In [7], we showed that for a cycle with an arbitrary nonempty set of parallel chords, the values in its friendly index set always form an arithmetic progression with common difference 2. However, this is not true if the chords are not parallel. Interestingly, the friendly index set of a union of disjoint cycles may not consist of an arithmetic progression either [5].

The ***book*** B_k is the graph consisting of k triangles sharing a common edge known as its "base." For example, the book B_4 is displayed below in Figure 2. A ***generalized book*** $B(n_1, n_2, \dots, n_k)$ consists of k cycles of length $n_1, n_2, \dots, n_k$ that share a common edge.

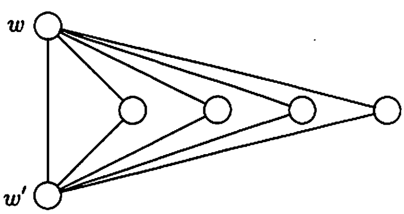

Figure 2: The book B_4 consists of four triangles sharing one common edge.

When $n_1 = n_2 = \cdots = n_k = n$, we denote the book $B(n^{[k]})$. We study the special cases of $n = 3, 4$ in Section 2. Although the solution method is difficult to extend and is different from the approach we use to attack the general problem, the results provide valuable data for verifying the general solution. The central idea behind the general solution is demonstrated in Section 3, in which we focus on $B(n_1, n_2)$. The complete solution is presented in Section 4. In Section 5, we look at its applications in several special cases.

2 Friendly Index Sets of $B(3^{[k]})$ and $B(4^{[k]})$

Let the two vertices in the base of $B(n^{[k]})$ be w and w', and write each n-cycle as $ww_2w_3 \dots w_{n-1}w'$. Observe that, in any friendly labeling, replacing each vertex label with its two's complement yields another friendly labeling with the same value in $|e(1)-e(0)|$. Hence, it suffices to consider two cases, depending on whether w and w' are labeled 0-0 or 0-1, respectively. In this section, we study the cases of $n = 3, 4$.

Theorem 2.1 *For $k \geq 2$,*

$$FI(B(3^{[k]}) = \begin{cases} \{1,3\} & \text{if } k \text{ is even,} \\ \{1,5\} & \text{if } k \text{ is odd.} \end{cases}$$

Proof. The graph $G = B(3^{[k]})$ has $k+2$ vertices and $2k+1$ edges. Let m_1 and m_2 be the number of 0-vertices and 1-vertices, respectively, amongst the k copies of w_2's. Since $m_1 + m_2 = k$, we obtain the following solutions:

$k = 2t + 1:$

w	w'	m_1	m_2	$e(0)$	$e(1)$	$\|e(1) - e(0)\|$
0	0	t	$t+1$	$1+2t$	$2(t+1)$	1
0	0	$t-1$	$t+2$	$1+2(t-1)$	$2(t+2)$	5
0	1	$t+1$	t	$(t+1)+t$	$1+t+(t+1)$	1
0	1	t	$t+1$	$t+(t+1)$	$1+t+(t+1)$	1

$k = 2t:$

w	w'	m_1	m_2	$e(0)$	$e(1)$	$\|e(1) - e(0)\|$
0	0	$t-1$	$t+1$	$1+2(t-1)$	$2(t+1)$	3
0	1	t	t	$2(t+1)$	$1+2(t+1)$	1

from which the result follows immediately. □

Theorem 2.2 *For $k \geq 2$,*

$$FI(B(4^{[k]})) = \begin{cases} \{1,3,5,\ldots,k+1\} & \\ \quad \cup \{k+5, k+9, \ldots, 3k+1\} & \text{if } k \text{ is even,} \\ \{0,4,8,\ldots,3k+1\} & \text{if } k \equiv 1 \pmod 4, \\ \{2,6,10,\ldots,3k+1\} & \text{if } k \equiv 3 \pmod 4. \end{cases}$$

Proof. The vertices w_2 and w_3 can be labeled as 0-0, 0-1, 1-0 or 1-1. Let m_1, m_2, m_3 and m_4, respectively, be the number of 4-cycles with such labelings. We need $0 \leq m_1, m_2, m_3, m_4 \leq k$, and $m_1 + m_2 + m_3 + m_4 = k$.

If w and w' are both labeled 0, we find

$$\begin{aligned} v(0) &= 2 + 2m_1 + m_2 + m_3, \\ v(1) &= m_2 + m_3 + 2m_4; \end{aligned}$$

Since $B(4^{[k]})$ has an even number of vertices, we need $v(0) = v(1)$; hence $1 + m_1 = m_4$. We also find

$$\begin{aligned} e(0) &= 1 + 3m_1 + m_2 + m_3 + m_4, \\ e(1) &= 2m_2 + 2m_3 + 2m_4. \end{aligned}$$

Thus

$$e(0) - e(1) = 1 + 3m_1 - m_2 - m_3 - m_4 = -(k-1) + 4m_1.$$

We could set $m_2 = m_3 = 0$. Since $1 + m_1 = m_4$, we obtain $k = 1 + 2m_1$. It follows that $0 \le m_1 \le (k-1)/2$.

If w and w' are labeled 0 and 1, similar argument leads to $m_1 = m_4$, and

$$e(0) - e(1) = -(k-1) + 4m_3.$$

By setting $m_1 = m_4 = 0$, we find $m_2 + m_3 = k$. Thus $0 \le m_3 \le k$. Combining the two cases, we determine that

$$\mathrm{FI}(B(4^{[k]})) = \{|-(k-1) + 4i| : 0 \le i \le k\}.$$

A careful examination of the values yields the sets stated above. □

Example 2. From $\{-5 + 4i : 0 \le i \le 6\} = \{-5, -1, 3, 7, 11, 15, 19\}$ we conclude that

$$\mathrm{FI}(B(4^{[6]})) = \{1, 3, 5, 7, 11, 15, 19\} = \{1, 3, 5, 7\} \cup \{11, 15, 19\}.$$

This example illustrates why, when k is even, $\mathrm{FI}(B(4^{[k]}))$ consists of two disjoint arithmetic progressions. □

This approach of distinguishing the possible labelings of the k disjoint copies of P_{n-2} resulting from the removal of w and w' becomes impractical as k increases. For example, finding $\mathrm{FI}(B(5^{[k]}))$ requires solving equations with eight variables $m_1, m_2, \ldots, m_8$. We need to develop a more effective way of computing $|e(1) - e(0)|$.

3 Friendly Index Sets of $B(n_1, n_2)$

To analyze the general problem, we find it helpful to study the labeling of each cycle as a stand-alone cycle. To demonstrate the idea, we will derive the friendly index set of $B(n_1, n_2)$ in this section. Due to symmetry, we may assume $n_1 \le n_2$. Given a graph G with any vertex labeling (which needs not be friendly), define its ***friendly index*** [9] as $i(G) = e(1) - e(0)$. In $B(n_1, n_2)$, the edge ww' is counted twice in the sum $i(C_{n_1}) + i(C_{n_2})$. Hence

$$i(B(n_1, n_2)) = i(C_{n_1}) + i(C_{n_2}) \pm 1,$$

depending on whether the edge ww' is labeled 0 or 1, respectively.

Consider any vertex labeling of C_n. Group the vertices into $2b$ blocks, where the $(2j-1)$th block consists of x_j consecutive 0-vertices and the

(2j)th block of y_j consecutive 1-vertices. In the event that all vertices in C_n are labeled the same, define $b = 0$. Then $e(1) = 2b$ and $e(0) = n - 2b$; hence $i(C_n) = e(1) - e(0) = -n + 4b$. It is easy to find a friendly labeling of C_n with $2b$ blocks for any b satisfying $1 \le b \le \lfloor n/2 \rfloor$. Thus

$$\begin{aligned} \mathrm{FI}(C_n) &= \{|-n+4b| : 1 \le b \le \lfloor n/2 \rfloor\} \\ &= \begin{cases} \{n-4k, n-4k+4, \ldots, n-4, n\} & \text{if } n = 4k \text{ or } 4k+2, \\ \{1, 3, 5, \ldots, n-2\} & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

This is exactly what Theorem 1.2 asserts.

For $B(n_1, n_2)$, we have $i(B(n_1, n_2)) = -(n_1+n_2)+4(b_1+b_2) \pm 1$, where $2b_i$ is the number of blocks in C_{n_i}. This friendly index can be written in a form similar to that of a cycle, namely,

$$i(B(n_1, n_2)) = -n \pm 1 + 4b,$$

where $n = n_1 + n_2$. To find $\mathrm{FI}(B(n_1, n_2))$, we first determine the bounds for b. Next, for each value for b, we find a friendly labeling with $b_1 + b_2 = b$. The friendly index set is obtained by gathering $|-n \pm 1 + 4b|$ into a set.

Since $i(C_{n_i})$ is independent of the individual block sizes, we may label the vertices in such a way that, with the exception of the last two blocks, and sometimes (see below) the first 0-block, most blocks are of size 1. The trick is to pick x_{ib} and y_{ib} in such a way that the overall vertex labeling is friendly.

Theorem 3.1 *Let $G = B(n_1, n_2)$, where $3 \le n_1 \le n_2$. Then*

$$FI(G) = \begin{cases} \{0, 2, 4, \ldots, n_1+n_2-3\} & \textit{if } n_1+n_2 \textit{ is odd,} \\ \{1, 3, 5, \ldots, n_1+n_2-3\} & \textit{if } n_1, n_2 \textit{ are odd,} \\ \{1, 3, 5, \ldots, n_1+n_2-5\} \cup \{n_1+n_2-1\} & \textit{if } n_1, n_2 \textit{ are even.} \end{cases}$$

Proof. If w and w' are labeled 0 and 1, then $1 \le b_i \le \lfloor n_i/2 \rfloor$. We shall label the vertices in the order of $ww_2w_3 \ldots w_{n-1}w'$. For any b_1 and b_2 within their respective ranges, setting $x_{ij} = y_{ij} = 1$ if $1 \le j < b_i$, and picking

$$\begin{aligned} x_{1b_1} &= \lfloor (n_1 - 2b_1 + 2)/2 \rfloor, & y_{1b_1} &= \lceil (n_1 - 2b_1 + 2)/2 \rceil, \\ x_{2b_2} &= \lceil (n_2 - 2b_2 + 2)/2 \rceil, & y_{2b_2} &= \lfloor (n_2 - 2b_2 + 2)/2 \rfloor, \end{aligned}$$

yields a friendly labeling. Pictorially, the two cycles are labeled as follows:

$$C_{n_1}: \quad \underbrace{f(w) = 0101\ldots01}_{2(b_1-1)}\ \underbrace{00 \quad \ldots \quad 0}_{\lfloor (n_1-2b_1+2)/2 \rfloor}\ \underbrace{11\ldots1 = f(w')}_{\lceil (n_1-2b_1+2)/2 \rceil},$$

$$C_{n_2}: \quad \underbrace{f(w) = 0101\ldots01}_{2(b_2-1)}\ \underbrace{00 \quad \ldots \quad 0}_{\lceil (n_2-2b_2+2)/2 \rceil}\ \underbrace{11\ldots1 = f(w')}_{\lfloor (n_2-2b_2+2)/2 \rfloor}.$$

We find $i(G) = -(n_1+n_2)-1+4b$, where $2 \leq b = b_1+b_2 \leq \lfloor n_1/2 \rfloor + \lfloor n_2/2 \rfloor$. More specifically, if $[s,t]_d$ denotes an arithmetic progression with common difference d that starts with s and ends with t, then

$$i(G) \in \begin{cases} [-(n_1+n_2-7), n_1+n_2-5]_4 & \text{if } n_1, n_2 \text{ are odd,} \\ [-(n_1+n_2-7), n_1+n_2-3]_4 & \text{if } n_1+n_2 \text{ is odd,} \\ [-(n_1+n_2-7), n_1+n_2-1]_4 & \text{if } n_1, n_2 \text{ are even.} \end{cases}$$

If w and w' are both 0-vertices, then $0 \leq b_i \leq \lfloor (n_i-1)/2 \rfloor$, and $i(G) = -(n_1+n_2)+1+4b$, where $b \leq \lfloor (n_1-1)/2 \rfloor + \lfloor (n_2-1)/2 \rfloor$. The lower bound of b is a bit trickier. We cannot have $b = 0$, because it would require $b_1 = b_2 = 0$, which in turn implies that all vertices are labeled 0. We now describe the vertex labels in the order of $w'ww_2w_3 \ldots w_{n-1}$.

If $b_1, b_2 \geq 2$, letting $x_{11} = x_{21} = 2$ and $y_{11} = y_{12} = 1$ produces a partially completed labeling which is friendly. The remaining vertices can be labeled in the same manner described above; that is, $x_{ij} = y_{ij} = 1$ if $j < b_i$, and

$$x_{1b_1} = \lfloor (n_1 - 2b_1 + 1)/2 \rfloor, \qquad y_{1b_1} = \lceil (n_1 - 2b_1 + 1)/2 \rceil,$$
$$x_{2b_2} = \lceil (n_2 - 2b_2 + 1)/2 \rceil, \qquad y_{2b_2} = \lfloor (n_2 - 2b_2 + 1)/2 \rceil.$$

It remains to analyze $1 \leq b \leq 3$.

For $b_1 = 1$, we need $x_{11} = \lceil n_1/2 \rceil$ to ensure $x_{11} \geq 2$, hence we would set $y_{11} = \lfloor n_1/2 \rfloor$. For $b_2 = 2$, we could pick $x_{21} = 2$ and $y_{21} = 1$. Thus far, the partially completed labeling has

$$v(1) - v(0) = \begin{cases} 0 & \text{if } n_1 \text{ is odd,} \\ 1 & \text{if } n_1 \text{ is even.} \end{cases}$$

Choosing $x_{22} = \lceil (n_2-3)/2 \rceil$ and $y_{22} = \lfloor (n_2-3)/2 \rfloor$ settles the case of $b = 3$. For $b_2 = 1$, letting $x_{21} = \lceil n_2/2 \rceil$ and $y_{21} = \lfloor n_2/2 \rfloor$ if n_1 or n_2 is odd, but $x_{21} = n_2/2+1$ and $y_{21} = n_2/2-1$ if both n_1 and n_2 are even, yields a friendly labeling with $b = 2$.

The last case is $b = 1$. We need $b_1 = 0$ and $b_2 = 1$. Friendliness also requires $n_1 < n_2$. Since all n_1 vertices in C_{n_1} are labeled 0, at least $n_1 - 1$ vertices in C_{n_2} must be labeled 1, leaving $n_2 - (n_1+1)$ vertices that need to be evenly labeled 0 and 1. We could select $x_{21} = 2 + \lfloor (n_2-n_1-1)/2 \rfloor$ and $y_{21} = n_2 - 2 - \lfloor (n_2 - n_1 - 1))/2 \rfloor$.

To summarize the event in which both w and w' are labeled 0, we have $i(G) = -(n_1+n_2)+1+4b$, where $b \leq \lfloor (n_1-1)/2 \rfloor + \lfloor (n_2-1)/2 \rfloor$. If $n_1 < n_2$, then $b \geq 1$, in which case,

$$i(G) \in \begin{cases} [-(n_1+n_2-5), n_1+n_2-3]_4 & \text{if } n_1 < n_2 \text{ are both odd,} \\ [-(n_1+n_2-5), n_1+n_2-5]_4 & \text{if } n_1 \text{ or } n_2 \text{ is odd,} \\ [-(n_1+n_2-5), n_1+n_2-7]_4 & \text{if } n_1 < n_2 \text{ are both even.} \end{cases}$$

If $n_1 = n_2$, then $b \geq 2$, in which case,

$$i(G) \in \begin{cases} [-(n_1+n_2-9), n_1+n_2-3]_4 & \text{if } n_1 = n_2 \text{ are both odd,} \\ [-(n_1+n_2-9), n_1+n_2-7]_4 & \text{if } n_1 = n_2 \text{ are both even.} \end{cases}$$

Finally, we need to combine these friendly indices with those from the case in which w and w' are labeled 0 and 1.

- **Case 1.** When $n_1 + n_2$ is odd,

  $$i(B(n_1, n_2)) \in [-(n_1+n_2-5), n_1+n_2-3]_4.$$

  Consequently, $\mathrm{FI}(G) = [0, n_1+n_2-3]_2$.

- **Case 2.** When n_1 and n_2 are odd,

  $$i(B(n_1, n_2)) \in \begin{cases} [-(n_1+n_2-5), n_1+n_2-3]_4 & \text{if } n_1 < n_2, \\ [-(n_1+n_2-7), n_1+n_2-3]_4 & \text{if } n_1 = n_2. \end{cases}$$

  In either case, we find $\mathrm{FI}(G) = [1, n_1+n_2-3]_2$.

- **Case 3.** When n_1 and n_2 are even,

  $$i(B(n_1, n_2)) \in \begin{cases} [-(n_1+n_2-5), n_1+n_2-1]_4 & \text{if } n_1 < n_2, \\ [-(n_1+n_2-7), n_1+n_2-1]_4 & \text{if } n_1 = n_2. \end{cases}$$

  Hence $\mathrm{FI}(G) = [1, n_1+n_2-5]_2 \cup \{n_1+n_2-1\}$.

The proof is now complete. □

Example 3. Besides Theorems 2.1 and 2.2, it is an easy exercise to deduce $\mathrm{FI}(B(5^{[2]})) = \{1, 3, 5, 7\}$, and

$$\mathrm{FI}(B(5,n)) = \begin{cases} \{1,3,5,\ldots,n+2\} & \text{if } n \text{ is odd,} \\ \{0,2,4,\ldots,n+2\} & \text{if } n \text{ is even,} \end{cases}$$

$$\mathrm{FI}(B(6,n)) = \begin{cases} \{0,2,4,\ldots,n+3\} & \text{if } n \text{ is odd,} \\ \{1,3,5,\ldots,n+1\} \cup \{n+5\} & \text{if } n \text{ is even,} \end{cases}$$

from Theorem 3.1. □

4 Friendly Index Sets of $B(n_1, n_2, \ldots, n_k)$

In this section, we discuss how to find $\mathrm{FI}(B(n_1, n_2, \ldots, n_k))$, where $k \geq 2$ and $n_i \geq 3$ for each i. Following [9], we define the ***full friendly index set*** of a graph G to be

$$\mathrm{FFI}(G) = \{i_f(G) : f \text{ is a friendly labeling of } G\};$$

hence $\mathrm{FI}(G) = \{|x| : x \in \mathrm{FFI}(G)\}$. For $G = B(n_1, n_2, \dots, n_k)$, define

$$\begin{aligned} S_1 &= \{i_f(G) : f(w) = 0 \text{ and } f(w') = 1\}, \\ S_2 &= \{i_f(G) : f(w) = 0 \text{ and } f(w') = 0\}, \end{aligned}$$

so that $\mathrm{FFI}(B(n_1, n_2, \dots, n_k)) = S_1 \cup S_2$.

Set $n = n_1 + n_2 + \cdots + n_k$, and let ℓ denote the number of odd numbers among $n_1, n_2, \dots, n_k$. If w and w' are labeled 0 and 1, respectively, then

$$i(B(n_1, n_2, \dots, n_k)) = -n - (k-1) + 4b,$$

where $b = b_1 + b_2 + \cdots + b_k$. Since $1 \le b_i \le \lfloor n_i/2 \rfloor$ for each i, we find $k \le b \le \sum_{i=1}^{k} \lfloor n_i/2 \rfloor = \lfloor n/2 \rfloor - \lfloor \ell/2 \rfloor$. Rename and rearrange the n_i's as m_i's such that $m_1 \le m_2 \le \cdots \le m_\ell$ are odd, and $m_{\ell+1} \le m_{\ell+2} \le \cdots \le m_k$ are even. We can now apply the same labeling method we used in Section 3 to obtain every possible value of b. We determine that $S_1 = \{-n-k+1+4b : k \le b \le \lfloor n/2 \rfloor - \lfloor \ell/2 \rfloor\}$.

The situation is much more complicated if w and w' are both 0-vertices. In this case, we have $i(B(n_1, n_2, \dots, n_k)) = -n + (k-1) + 4b$. Although $0 \le b_i \le \lfloor (n_i - 1)/2 \rfloor$ for each i, we cannot have $\sum_{i=1}^{k} b_i = 0$, because it would have implied that all vertices are 0-vertices. Let m be the maximum number of b_i's that could equal 0, so that $k - m$ is the lower bound for b. Since $n_1 \le n_2 \le \cdots \le n_k$, we may assume $n_1 = n_2 = \cdots = n_m = 0$. For a vertex labeling to be friendly, we need $2 + \sum_{i \le m}(n_i - 2) \le \sum_{i>m}(n_i - 2) + 1$. This implies

$$2 - 2m + \sum_{i \le m} n_i \le n - \sum_{i \le m} n_i - 2(k-m) + 1.$$

Thus $\sum_{i \le m} n_i \le (n-1)/2 + 2m - k$.

The upper bound for b is also problematic. Supposedly, the upper bound for $\sum_{i=1}^{k} b_i$ is

$$\sum_{i=1}^{k} \left\lfloor \frac{n_i - 1}{2} \right\rfloor = \left\lfloor \frac{n-k}{2} \right\rfloor - \left\lfloor \frac{k-\ell}{2} \right\rfloor.$$

However, this upper bound may not be attainable. If n_i is even, we can label the vertices between w and w' alternately with 1 and 0 to obtain $b_i \le \lfloor (n_i - 1)/2 \rfloor$. More importantly, this gives an equal number of 0- and 1-vertices between w and w', thereby preserving the friendliness of the labeling.

If n_i is odd, to obtain the maximum value in b_i, the vertices between w and w' must be labeled as $1010\dots101$. Therefore each odd-sized cycle contains an extra unmatched 1-vertex. Since w and w' are 0-vertices, the overall labeling is still friendly, provided $\ell \le 3$. If $\ell \ge 4$, the vertices between

w and w' on half of these odd-sized cycles beyond the first three must be labeled 1010...100 so as to maintain friendliness. In other words, $\lfloor \ell/2 \rfloor - 1$ of the odd-sized cycles can only have $b_i \leq \lfloor (n_i - 1)/2 \rfloor - 1$. Consequently, the upper bound for b must be reduced by $\lfloor \ell/2 \rfloor - 1$ if $\ell \geq 4$.

It remains to show that any b within the lower and upper bounds is attainable. First consider $b \geq k$. We need to pick b_i's such that $b = \sum_{i=1}^{k} b_i$. We could use an greedy algorithm to select b_i. Start with b_k, and work backward: choose the largest possible b_k, then the largest possible b_{k-1}, and so forth, until $b = \sum_{i=1}^{k} b_i$, where $b_i \geq 1$ for each i. When we pick the largest possible value of b_i when n_i is odd, take into account the remark above pertaining to the overall value of b, and make appropriate adjustment whenever necessary.

We now describe a labeling method that would produce a friendly labeling. If $b_i < \lfloor (n_i - 1)/2 \rfloor$ for each i, we first use the same labeling method in Section 3 to label C_{n_1} and C_{n_2}. If there are any cycles with b_i's reaching the maximum values allowed, that is, with $b_i = \lfloor (n_i - 1)/2 \rfloor$, they must be labeled first, because they do not allow much flexibility in their vertex labeling. Compute $v(1)$ and $v(0)$ in this partially completed labeling. If $v(1) > v(0)$, we need to select $v(1) - v(0) - 1$ vertices from the unlabeled cycles and label them 0 to maintain friendliness. Likewise, if $v(1) < v(0)$, we need to select $v(0) - v(1) - 1$ vertices from the unlabeled cycles and label them 1. Again, we use a greedy algorithm to select these vertices. From the next unlabeled cycle of the largest size, pick as many vertices as possible until the remaining vertices, when they are eventually labeled alternately with 0 and 1, together with the selected and subsequently labeled vertices, would yield the desired b_i for that cycle. Repeat the process until all required vertices are selected.

Thus far, in both cases, since w and w' have been counted, and the partially constructed labeling is friendly, the remaining unlabeled vertices can be labeled using the same method described above, with the understanding that, if necessary, we may need to use ceiling functions before floor functions to ensure friendliness.

Example 4. Consider $B(4, 5, 7, 7, 8, 10)$ with w and w' both labeled 0. The maximum value for the b_i's are recorded as $(1, 2, 3, 3, 3, 4)$, hence $b \leq 16$. To obtain $b = 10$, we could use $(b_1, b_2, b_3, b_4, b_5, b_6) = (1, 1, 1, 1, 2, 4)$, in which b_1 and b_6 reach their maximum values. We need to start the labeling with the first and the last cycles:

w	w'	C_4	C_5	C_7	C_7	C_8	C_{10}
0	0	0110	xxx0	0xxxxx0	0xxxxx0	0xxxxxx0	0101010100

At this stage, $v(0) = 6$ and $v(1) = 6$, so the partially finished labeling is friendly. Since C_8 has an even number of unlabeled vertices, we will label it next.

w	w'	C_4	C_5	C_7	C_7	C_8	C_{10}
0	0	0110	0xxx0	0xxxxx0	0xxxxx0	00100110	0101010100

Each unfinished cycle has an odd number of unlabeled vertices, label them with the usual approach via floor and ceiling functions:

w	w'	C_4	C_5	C_7	C_7	C_8	C_{10}
0	0	0110	00110	0000110	0001110	00100110	0101010100

The result is a friendly labeling with $v(0) = 15$, $v(1) = 16$, and $b = 10$. □

Example 5. Consider $B(4,5,7,7,7,9)$ with w and w' both labeled 0. The maximum value for each b_i is recorded as $(1,2,3,3,3,4)$, hence $b \leq 16$. To obtain $b = 11$, we could use $(b_1, b_2, b_3, b_4, b_5, b_6) = (1,1,1,1,3,4)$, in which b_1, b_5 and b_6 reach their maximum values. We need to start the labeling with their respective cycles:

w	w'	C_4	C_5	C_7	C_7	C_7	C_9
0	0	0110	0xxx0	0xxxxx0	0xxxxx0	0101010	010101010

At this stage, $v(0) = 7$ and $v(1) = 9$, so we need an extra 0-vertex. Put it in the last unfinished C_7.

w	w'	C_4	C_5	C_7	C_7	C_7	C_9
0	0	0110	0xxx0	0xxxxx0	00xxxx0	0101010	010101010

The last unfinished C_7 has an even number of unlabeled vertices, so we will label it next:

w	w'	C_4	C_5	C_7	C_7	C_7	C_9
0	0	0110	0xxx0	0xxxxx0	0000110	0101010	010101010

Finally, the two remaining cycles will be filled.

w	w'	C_4	C_5	C_7	C_7	C_7	C_9
0	0	0110	00010	0001110	0000110	0101010	010101010

The result is a friendly labeling with $v(0) = 14$, $v(1) = 15$, and $b = 11$. □

Example 6. For $B(3,3,3,3,3,7,10)$, we have $n = 32$, $k = 7$, $\ell = 6$, and $M = 2$. The maximum value for each b_i is recorded as $(1,1,1,1,1,3,4)$, hence $b \leq 10$. To obtain $b = 8$ with $b_i \geq 1$ for each i, and w and w' labeled 0, we could use $(b_1, b_2, b_3, b_4, b_5, b_6, b_7) = (1,1,1,1,1,1,2)$, in which b_i reaches its maximum value for each $i \leq 5$. We could start with the following labeling

w	w'	C_3	C_3	C_3	C_3	C_3	C_7	C_{10}
0	0	010	010	010	010	010	0xxxxx0	0xxxxxxxx0

Thus far, $v(0) = 2$ and $v(1) = 5$, so we need to add two 0-vertices in C_{10}:

w	w'	C_3	C_3	C_3	C_3	C_3	C_7	C_{10}
0	0	010	010	010	010	010	0xxxxx0	000xxxxxx0

Here is the final labeling:

w	w'	C_3	C_3	C_3	C_3	C_3	C_7	C_{10}
0	0	010	010	010	010	010	0000110	0000100110

This is a friendly labeling with $v(0) = 10$, $v(1) = 10$, and $b = 8$. □

If $k - m \leq b < k$, all vertices in the first $k - b$ cycles must be labeled 0, and we want $b_i = 1$ for each $i > k - b$. First, select $1 + \sum_{i \leq k-b}(n_i - 2)$ vertices between w and w' from the last b cycles, and select them as evenly as possible from each cycle. Label them with 1. The partially completed labeling is friendly. We still need to label the remaining unlabeled vertices.

If the number of unlabeled vertices in a cycle is even, label half of them with 0 and the other half with 1. Among the remaining cycles with odd number of unlabeled vertices, use the same old strategy (of applying floor and ceiling functions alternately) to label them with 0's and 1's. The result is a friendly labeling with $b_i = 0$ if $i \leq k - b$ and $b_i = 1$ if $i > k - b$.

Example 7. For $B(4, 5, 7, 7, 8, 10)$, we have $n = 41$ and $k = 6$. We find $m = 3$, hence $b \geq 3$. To obtain a friendly labeling f with $f(w) = f(w') = 0$ and $b = 3$, all the vertices in the first three cycles must be labeled 0.

w	w'	C_4	C_5	C_7	C_7	C_8	C_{10}
0	0	0000	00000	0000000	0xxxxx0	0xxxxxx0	0xxxxxxxx0

We now have twelve 0-vertices, so we need eleven 1-vertices to maintain friendliness. Distribute them as evenly as possible among the last three cycles. We could use four in C_7 and C_8 and three in C_{10}:

w	w'	C_4	C_5	C_7	C_7	C_8	C_{10}
0	0	0000	00000	0000000	0x11110	0xx11110	0xxxxx1110

Only C_8 has an even number of unlabeled vertices, half of which we will label with 0, and the other half with 1:

w	w'	C_4	C_5	C_7	C_7	C_8	C_{10}
0	0	0000	00000	0000000	0x11110	00111110	0xxxxx1110

The remaining vertices are labeled as follows:

w	w'	C_4	C_5	C_7	C_7	C_8	C_{10}
0	0	0000	00000	0000000	0111110	00111110	0000111110

This gives $v(0) = 16$, $v(1) = 15$, and $b = 3$.

What if we want $b = 4$? We may proceed as follows:

w	w'	C_4	C_5	C_7	C_7	C_8	C_{10}
0	0	0000	00000	0xxxxx0	0xxxxx0	0xxxxxx0	0xxxxxxxx0
0	0	0000	00000	0xxx110	0xxx110	0xxxxx10	0xxxxxxx10
0	0	0000	00000	0001110	0011110	00001110	0000111110

This gives $v(0) = 16$, $v(1) = 15$, and $b = 4$. □

We summarize the result in the next theorem.

Theorem 4.1 *Let m be the largest nonnegative integer such that $\sum_{i=1}^{m} n_i \leq (n-1)/2 + 2m - k$. Let M be 0 if $\ell \leq 3$, but set $M = \lfloor \ell/2 \rfloor - 1$ if $\ell \geq 4$. Define*

$$\begin{aligned} S_1 &= \left\{ -n - k + 1 + 4b : k \leq b \leq \left\lfloor \frac{n}{2} \right\rfloor - \left\lfloor \frac{\ell}{2} \right\rfloor \right\}, \\ S_2 &= \left\{ -n + k - 1 + 4b : k - m \leq b \leq \left\lfloor \frac{n-k}{2} \right\rfloor - \left\lfloor \frac{k-\ell}{2} \right\rfloor - M \right\}. \end{aligned}$$

Then $FI(B(n_1, n_2, \ldots, n_k)) = \{|z| : z \in S_1 \cup S_2\}$.

5 Some Special Cases

It is straightforward to verify that Theorem 4.1 reduces to Theorem 3.1 when $k = 2$. If the n_i's are constant, we obtain the following result.

Theorem 5.1 *Let $r \geq 3$ and $k \geq 2$. Set $M = 0$ if r is even or if $k \leq 3$, otherwise let $M = \lfloor k/2 \rfloor - 1$. Define*

$$\begin{aligned} S_1 &= \left\{ -(r+1)k + 1 + 4b : k \leq b \leq \left\lceil \tfrac{r-1}{2} \right\rceil k \right\}, \\ S_2 &= \left\{ -(r-1)k - 1 + 4b : \left\lceil \frac{k+1}{2} \right\rceil \leq b \leq \frac{(r-2)k}{2} - M \right\}. \end{aligned}$$

Then $FI(B(r^{[k]})) = \{|z| : z \in S_1 \cup S_2\}$.

Proof. We want to find the largest nonnegative integer m such that $rm \leq (rk-1)/2 + 2m - k$, which is equivalent to

$$m \leq \frac{(r-2)k - 1}{2(r-2)}.$$

Therefore $m = \lfloor (k-1)/2 \rfloor$. This gives $k - m = \lceil (k+1)/2 \rceil$. The remaining bounds are obtained from direct computation. □

It is an easy exercise to deduce Theorems 2.1 and 2.2 from Theorem 5.1. We also obtain the following new result.

Theorem 5.2 *For $k \geq 2$,*

$$FI(B(5^{[k]})) = \begin{cases} \{1,3,5,\ldots,2k+3\} & \text{if } k \text{ is even,} \\ \{1,3,5,\ldots,2k+1\} \cup \{2k+5\} & \text{if } k \text{ is odd.} \end{cases}$$

Proof. For $B(5^{[k]})$, we have

$$\begin{aligned} S_1 &= \{-6k+1+4b : k \leq b \leq 2k\} = [-(2k-1), 2k+1]_4, \\ S_2 &= \{-4k-1+4b : \lceil (k+1)/2 \rceil \leq b \leq 2k - M\}. \end{aligned}$$

Direct computation gives

$$-4k - 1 + 4\lceil (k+1)/2 \rceil = \begin{cases} -(2k+3) & \text{if } k \text{ is even,} \\ -(2k-1) & \text{if } k \text{ is odd,} \end{cases}$$

and

$$-4k - 1 + 4(2k - M) = \begin{cases} 4k - 1 & \text{if } k \leq 3, \\ 4k - \lfloor k/2 \rfloor + 3 & \text{if } k \geq 4. \end{cases}$$

Notice that

$$4k - \lfloor k/2 \rfloor + 3 = \begin{cases} 2k+3 & \text{if } k \text{ is even,} \\ 2k+5 & \text{if } k \text{ is odd;} \end{cases}$$

and

$$4k - 1 = \begin{cases} 8 = 2k+3 & \text{if } k = 2, \\ 11 = 2k+5 & \text{if } k = 3. \end{cases}$$

Therefore $S_2 - \{\pm(2k+3)\} \subseteq S_1$ if k is even, but $S_2 - \{2k+5\} \subseteq S_1$ if k is odd. We conclude that

$$\mathrm{FI}(B(5^{[k]})) = \begin{cases} T_1 \cup \{2k+3\} & \text{if } k \text{ is even,} \\ T_1 \cup \{2k+5\} & \text{if } k \text{ is odd,} \end{cases}$$

where $T_1 = \{|z| : z \in S_1\} = \{1,3,5,\ldots,2k+1\}$. □

The last problem we study is $B(3^{[t]}, n)$, where $t \geq 2$ and $n \geq 4$, in which we may regard C_n as the $(t+1)$st cycle. We find an interesting result.

Theorem 5.3 *$FI(C_n) \subseteq FI(B(3^{[t]}, n))$ for all $n \geq 4$.*

Proof. Consider any friendly labeling of C_n. It must contain a 0-vertex adjacent to an 1-vertex. Let them be w and w' respectively. Each w_2 in the t copies of C_3 can be labeled either 0 or 1 without altering the overall value of $e(1) - e(0)$, so we can easily distribute 0's and 1's among them and C_n to produce a friendly labeling of $B(3^{[t]}, n)$. This proves that every friendly labeling of C_n induces a friendly labeling of $B(3^{[t]}, n)$. □

It is clear that any friendly labeling of $B(3^{[t]}, n)$ with w and w' labeled 0 and 1 has a friendly index that can be found in $\mathrm{FI}(C_n)$. Consequently, we only need to analyze the vertex labelings that assign 0 to both w and w'. Note that it is possible for $b_{t+1} = 0$, and the upper bound for b_{t+1} could be lowered, depending on how many 0- and 1-vertices are there among the w_2's in the C_3's, which in turn could change the value of $e(1) - e(0)$. By carefully studying whether $e(1) - e(0)$ could attain new values not found in $\mathrm{FI}(C_n)$, we could derive the following results. Nonetheless, it is easier to prove them with Theorem 4.1.

Theorem 5.4 *For $n \geq 4$,*

$$FI(B(3^{[2]}, n)) = \begin{cases} \{0, 4, 8, \ldots, n\} & \text{if } n \equiv 0 \pmod 4, \\ \{2, 6, 10, \ldots, n\} & \text{if } n \equiv 2 \pmod 4, \\ \{1, 3, 5, \ldots, n+2\} & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Apply Theorem 4.1. We find $m = 1$ if $n = 4$, but $m = 2$ if $n \geq 5$. For $n = 2t$, where $t \geq 3$,

$$\begin{aligned} S_1 &= \{-n - 8 + 4b : 3 \leq b \leq t+2\} &= [-(n-4), n]_4, \\ S_2 &= \{-n - 4 + 4b : 1 \leq b \leq t+1\} &= [-n, n]_4. \end{aligned}$$

When $n = 4$, the only change is in S_2, in which $b \geq 2$; hence it starts with $-(n-4)$, which does not affect the friendly index set. We conclude that $\mathrm{FI}(B(3^{[2]}, n)) = \{0 \leq z \leq n : z \equiv n \pmod 4\}$.

For $n = 2t + 1$, where $t \geq 2$,

$$\begin{aligned} S_1 &= \{-n - 8 + 4b : 3 \leq b \leq t+2\} &= [-(n-4), n-2]_4, \\ S_2 &= \{-n - 4 + 4b : 1 \leq b \leq t+2\} &= [-n, n+2]_4. \end{aligned}$$

Since $S_1 \subseteq S_2$, we find $\mathrm{FI}(B(3^{[2]}, n)) = \{1, 3, 5, \ldots, n+2\}$. □

Theorem 5.5 *For $n \geq 4$,*

$$FI(B(3^{[3]}, n)) = \begin{cases} \{0, 2, 4, \ldots, n+2\} & \text{if } n \text{ is even,} \\ \{1, 3, 5\} & \text{if } n = 5, \\ \{1, 3, 5, \ldots, n+2\} & \text{if } n > 5 \text{ is odd.} \end{cases}$$

Proof. We find $m = 2$ if $n = 4, 5$; but $m = 3$ if $n \geq 6$. For $n = 2t$, where $t \geq 3$, we have

$$\begin{aligned} S_1 &= \{-n-12+4b : 4 \leq b \leq t+3\} &&= [-(n-4), n]_4, \\ S_2 &= \{-n-6+4b : 1 \leq b \leq t+2\} &&= [-(n+2), n+2]_4. \end{aligned}$$

Since $-(n+2) \notin S_2$ when $n = 4$, $\mathrm{FI}(B(3^{[3]}, n)) = \{1, 3, 5, \ldots, n+2\}$.

If $n = 2t+1$, where $t \geq 3$, we have

$$\begin{aligned} S_1 &= \{-n-12+4b : 4 \leq b \leq t+3\} &&= [-(n-4), n-2]_4, \\ S_2 &= \{-n-6+4b : 1 \leq b \leq t+2\} &&= [-(n+2), n]_4. \end{aligned}$$

Since $-(n+2) \notin S_2$ if $n = 5$, we find $S_1 = \{-3, 1, 5\}$ and $S_2 = \{-1, 3\}$. Thus $\mathrm{FI}(B(3^{[3]}, 5)) = \{1, 3, 5\}$, and $\mathrm{FI}(B(3^{[3]}, n)) = \{1, 3, 5, \ldots, n+2\}$ if $n > 5$ is odd. □

References

[1] I. Cahit, Cordial graphs: a weaker version of graceful and harmonious graphs, *Ars Combin.* **23** (1987), 201–207.

[2] N. Cairnie and K. Edwards, The computational complexity of cordial and equitable labelling, *Discrete Math.* **216** (2000), 29–34.

[3] G. Chartrand, S-M. Lee and P. Zhang, Uniformly cordial graphs, *Discrete Math.* **306** (2006), 726–737.

[4] M. Hovey, *A*-cordial graphs, *Discrete Math.* **93** (1991), 183–194.

[5] H. Kwong, S-M. Lee and H.K. Ng, On friendly index set of 2-regular graphs, *Discrete Math*, to appear (doi:10.1016/j.disc.2007.10.018).

[6] S-M. Lee and H.K. Ng, On friendly index sets of bipartite graphs, *Ars Combin.*, to appear.

[7] S-M. Lee and H.K. Ng, On friendly index sets of cycles with parallel chords, *Ars Combin.*, to appear.

[8] E. Salehi and S.M Lee, On friendly index sets of trees, *Congr. Numer.* **178** (2006), 173–183.

[9] W.C. Shiu and H. Kwong, Full friendly index sets of $P_2 \times P_n$, *Discrete Math*, to appear (doi:10.1016/j.disc.2007.07.002).