Some New Results on Balanced Arrays of Strength Five

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Abstract

In this paper we derive some necessary existence conditions for a bi-level balanced arrays (B-arrays) with strength $t = 5$. We then describe how these existence conditions can be used to obtain an upper bound on the number of constraints of these arrays, and give some illustrative examples to this effect.

1 Introduction and Preliminaries

First of all, for ease of reference, we list here some basic definitions and concepts.

Definition 1.1 A matrix T of size $(m \times N)$ and with two elements (say, 0 and 1) is called a balanced array (B-array) with m rows (constraints), N columns (runs, treatment-combinations), with two levels (0 and 1) and having strength t $(t \leq m)$ if in every $(t \times N)$ submatrix T^* of T (clearly there are $\binom{m}{t}$ such submatrices), every $(t \times 1)$ vector $\underline{\alpha}$ of weight i $(0 \leq i \leq t$; the weight of $\underline{\alpha}$ means the number of non-zero elements in it) occurs with the same frequency (say) μ_i. The vector $\underline{\mu}' = (\mu_0, \mu_1, \ldots, \mu_t)$ is called the index set of the array T. It is quite obvious that $N = \sum_{i=0}^{t} \binom{t}{i} \mu_i$.

Note: The above definition of a B-array with two symbols can be easily extended to B-arrays with s symbols.

In this paper we restrict ourselves to arrays with $t = 5$. For these arrays, $N = \mu_0 + 5\mu_1 + 10\mu_2 + 10\mu_3 + 5\mu_4 + \mu_5$.

Definition 1.2 If $\mu_i = \mu$ for each i, then the B-array is reduced to an orthogonal array (O-array) with index set μ. For this case then $N = \mu 2^t$.

Thus O-arrays form a subset of B-arrays. B-arrays, a generalization of O-arrays, are also related to other combinatorial structures such as balanced incomplete block (BIB) designs, group divisible designs, nested BIB designs, rectangular designs, etc. These combinatorial arrays have been extensively used to construct fractional factorial designs in statistical design of experiments. B-arrays with different values of the strength t give rise to, under certain conditions, factorial designs of different resolutions. For example, a B-array with $t = 5$ will provide us with a balanced factorial design of resolution VI which would allow us to estimate all the effects up to and including two-factor interactions in the presence of three factor interactions under the assumption that higher order interactions are negligible. O-arrays, a subset of B-arrays, have been widely used in coding and information theory, in quality control in industry, in medicine, etc. To gain further insight into the importance of these arrays in combinatorics and statistical design of experiments, the interested reader may consult the list of references (by no means an exhaustive one) at the end of this paper, and also further references cited therein.

Thus, the existence and construction of these combinatorial arrays is very important from the point of view of applications and to study other combinatorial structures. It is quite obvious that to construct B-arrays for an arbitrary set of parameters is a very different problem. In particular we address here the problem of obtaining the maximum value of m for a given $\underline{\mu}'$ with $t = 5$ which is a nontrivial problem. Such problems for B-arrays and O-arrays have been discussed, among others, by Bose and Bush [1], Chopra and/or Bsharat [5, 6], Dios and Chopra [7], Hedayat et al [8], Rafter and Seiden [12], Rao [13], Seiden and Zemach [14], etc.

First of all, we derive some inequalities involving the parameters m and $\underline{\mu}'$ for arrays with $t = 5$. For a given $\underline{\mu}'$ (i.e. given N) these are inequalities involving only m. If any one of these inequalities is contradicted for a certain value of m ($> 5; m = m^*$), then the maximum value of m is $(m^* - 1)$. On the other hand, the B-array may or may not exist even if all the inequalities are satisfied for a given m (> 5) and $\underline{\mu}'$. These inequalities consequently would allow us to find an upper bound on the number of constraints m for a given $\underline{\mu}'$.

2 Main Results On Balanced Arrays with $t = 5$

The following results are easy to establish.

Lemma 2.1 *A B-array T with $m = t = 5$ always exists for any index set $\underline{\mu}'$.*

Lemma 2.2 *A B-array T with $t = 5$ is also of strength k where $0 \leq k \leq 5$.*

Remark: Let $A(j, k)$ be the j-th element ($0 \leq j \leq k$) of the parameter vector of T when it is of strength k. Clearly $A(j, 5) = \mu_j$, $A(j, 0) = N$ and $A(j, k)$ is a linear combination of μ_i's and is given by

$$A(j,k) = \sum_{i=0}^{5-k} \binom{5-k}{i} \mu_{i+j} \tag{1}$$

Definition 2.1. Two columns of a B-array T having m rows are said to have j coincidences ($0 \le j \le m$) if they have exactly the same symbols in j of the rows.

Lemma 2.3 *Consider a B-array T with m rows and having a certain column (say, the first one) of weight l ($0 \le l \le m$). Let X_j be the number of columns in T (other than the first one) having exactly j coincidences with the first one, then the following results hold:*

$$B_0 = \sum_{j=0}^{m} X_j = N - 1, \text{ where } B_k = \sum j^k X_j \tag{2}$$

$$B_k = \sum_{j=1}^{k-1} b(j,k) B_j + k! \sum_{i=0}^{k} \binom{l}{i} \binom{m-1}{k-i} (A(i,k) - 1) \qquad k = 1, 2, \ldots, 5$$

where $A(i,k) = \mu_i$ for $k = 5$ ($0 \le i \le k$).

Remark: One can easily obtain (2) by counting the number of coincidences in two ways – through rows and columns. The constants $b(j,k)$ for various values of k are known, and these appear in the process of deriving (2). For computational ease we provide next the values of $b(j,k)$, $0 \le j \le k$ for $k = 1, 2, 3, 4$ and 5. These values are $0, 1, (-2, 3), (6, -11, 6)$ and $(-24, 50, -35, 10)$ respectively.

Theorem 2.1 *For a B-array $T(m, N, 2, 5)$ with index set $\underline{\mu}'$ to exist, the following must hold:*

$$N^2 B_5 - 2N B_2 B_3 + B_1 B_2^2 \ge 0 \tag{3}$$

where B_k's are defined as above.

Proof: Consider $\sum_{j=0}^{m} j(j^2 - a)^2 X_j$ where $a = \frac{B_2}{N}$. Clearly $\sum_{j=0}^{m} j(j^2 - a)^2 X_j \ge 0$ for each $j \ge 0$. We obtain (3) if we expand the L.H.S.

Theorem 2.2 *Consider a B-array T with m rows and $t = 5$. For T to exist, the following must be satisfied:*

$$N^4 B_5 - 2N^2 B_3 B_1^2 + B_1^5 \ge 0. \tag{4}$$

Proof: Let $a = \frac{B_1}{N}$. Consider $\sum_{j=0}^{m} j(j^2 - a^2)^2 X_j$ which is clearly non-negative. We obtain (4) by using this inequality.

Theorem 2.3 *A necessary condition for $BA(m, N, 2, 5)$ with the index set $(\mu_0, \mu_1, \ldots, \mu_5)$ to exist, the following must be satisfied:*

$$\left(B_k + 3B_{\frac{k+2l}{3}} + 3B_{\frac{2k+l}{3}} + B_l\right)^{\frac{1}{3}} \le (B_l)^{\frac{1}{3}} + (B_k)^{\frac{1}{3}} \tag{5}$$

where $k, l \le t$.

Proof: First of all we state here the Minkowski's inequality for use to derive the above result. If $X_i, Y_i \geq 0$ and $p > 1$, then

$$\left[\sum_{i=1}^{n}(X_i + Y_i)^p\right]^{\frac{1}{p}} \leq \left[\sum_{i=1}^{n} X_i^p\right]^{\frac{1}{p}} + \left[\sum_{i=1}^{n} Y_i^p\right]^{\frac{1}{p}}$$

which can be extended to (for $Z_i \geq 0$):

$$\left[\sum_{i=1}^{n}(X_i + Y_i + Z_i)^p\right]^{\frac{1}{p}} \leq \left[\sum_{i=1}^{n} X_i^p\right]^{\frac{1}{p}} + \left[\sum_{i=1}^{n} Y_i^p\right]^{\frac{1}{p}} + \left[\sum_{i=1}^{n} Z_i^p\right]^{\frac{1}{p}}.$$

We set $p = 3$, replace X_i by $j^{\frac{l}{3}} X_j^{\frac{1}{3}}$ and Y_i by $j^{\frac{k}{3}} X_j^{\frac{1}{3}}$ and i from 1 to n by j from 1 to m,

$$\left[\sum_{j=1}^{m}\left(j^{\frac{l}{3}} + j^{\frac{k}{3}}\right)^3 X_j\right]^{\frac{1}{3}} \leq \left[\sum_{j=1}^{m} j^l X_j\right]^{\frac{1}{3}} + \left[\sum_{j=1}^{m} j^k X_j\right]^{\frac{1}{3}}.$$

Expanding L.H.S. and using $B_k = \sum j^k X_j$, we obtain the result.

Corollary 2.1 *For a B-array with $t = 5$ to exist, we must have the following result:*

$$(B_5 + 3B_3 + 3B_4 + B_2)^{\frac{1}{3}} \leq (B_5)^{\frac{1}{3}} + (B_2)^{\frac{1}{3}}. \tag{6}$$

Proof: It can be easily obtained by setting $k = 5$ and $l = 2$ in equation (5).

Theorem 2.4 *For a B-array with m rows and of strength $t = 5$ the following must be true:*

$$\left[B_k + B_l + B_f + 3\left(B_{\frac{2k+l}{3}} + B_{\frac{2k+f}{3}} + B_{\frac{2l+k}{3}} + B_{\frac{2l+f}{3}} + B_{\frac{2f+k}{3}} + B_{\frac{2f+l}{3}}\right) + 6B_{\frac{k+l+f}{3}}\right]^{\frac{1}{3}} \leq (B_k)^{\frac{1}{3}} + (B_l)^{\frac{1}{3}} + (B_f)^{\frac{1}{3}}$$

where each of k, l and $f \leq 5$.

Proof: In order to establish this result, we use the extended Minkowski's inequality with $p = 3$, replacing each of X_i, Y_i and Z_i by $j^{\frac{k}{3}} X_j^{\frac{1}{3}}, j^{\frac{l}{3}} X_j^{\frac{1}{3}}$ and $j^{\frac{f}{3}} X_j^{\frac{1}{3}}$ respectively and changing i (from 1 to n) to j (from 1 to m). Then expanding the L.H.S., we obtain the desired result.

The following result is a special case of Theorem 2.4.

Result: Take $(k, l, f) = (2, 2, 5)$, then we obtain

$$(8B_2 + 12B_3 + 6B_4 + B_5)^{\frac{1}{3}} \leq 2\sqrt[3]{B_2} + \sqrt[3]{B_5}. \tag{7}$$

Theorem 2.5 *A necessary condition for the existence of a B-array T of strength 5 with m rows is*

$$\sqrt[3]{B_5} \leq \sqrt[3]{B_2} + \sqrt[3]{B_5 - 3B_4 + 3B_3 - B_2} \tag{8}$$

Proof outline: Here we replace X_i by $\left(j^{\frac{k}{3}} - j^{\frac{r}{3}}\right) X_j^{\frac{1}{3}}$ and replace Y_j by $j^{\frac{r}{3}} X_j^{\frac{1}{3}}$ where $k \geq r$ are positive integers not exceeding $t = 5$. Using these with $p = 3$ in Minkowski's inequality, we obtain $\sqrt[3]{B_k} \leq \sqrt[3]{B_r} + \sqrt[3]{B_k - 3B_{\frac{2k+r}{3}} + 3B_{\frac{k+2r}{3}} - B_r}$. We obtain (8) by setting $k = 5$ and $r = 2$ in the above.

Remark: All the above inequalities are merely functions of m for a given $\underline{\mu}'$. A computer program was prepared to check if the obtained inequalities are satisfied for a given $\underline{\mu}'$ starting with $m = 6$. If any of these inequalities is contradicted for the first time for $m = k+1$ (say), then $m = k$ is an upper bound for the number of constraints of T. Next, we give some illustrative examples.

Examples: We took three arrays with index sets $(1,1,1,1,1,1)$, $(1,1,1,1,1,2)$ and $(1,1,1,1,2,1)$. We observed that, with $l = 0$ condition (6) is contradicted for $m = 8$. Hence the maximum value of m for each array is 7. It means none of these arrays can exist for $m = 8$ having a column of weight 0. However, we do not claim that any of these arrays exists for $m = 7$ since all these are necessary conditions.

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