Distance Two Vertex-Magic Graphs

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Abstract

Given an abelian group A, a graph $G = (V, E)$ is said to have a distance two magic labeling in A if there exists a labeling $l : E(G) \to A - \{0\}$ such that the induced vertex labeling $l^* : V(G) \to A$ defined by

$$l^*(v) = \sum_{e \in E(v)} l(e)$$

is a constant map, where $E(v) = \{e \in E(G) : d(v, e) < 2\}$. The set of all $h \in \mathbb{Z}_+$ for which G has a distance two magic labeling in $\mathbb{Z}_h$ is called the distance two magic spectrum of G and is denoted by $\Delta\mathrm{M}(G)$. In this paper, the distance two magic spectra of certain classes of graphs will be determined.

Key Words: Distance k magic; Distance k magic spectrum.

2000 Mathematics Subject Classification: 05C78

1 Introduction

In this paper all graphs $G = (V, E)$ are connected, finite, simple, and undirected. For graph theory notations and terminology not described in this paper, the readers are referred to [1]. Given a nontrivial abelian group A, written additively, a *standard magic* labeling of a graph G in A is any mapping $l : E(G) \to A - \{0\}$ with the property that the induced vertex labeling $l^+ : V(G) \to A$ defined by

$$l^+(v) = \sum_{uv \in E} l(uv), \tag{1.1}$$

is a constant map. Here, $l^+(v)$ is the sum of the labels of all edges incident with v. The *integer-magic spectrum* of G, denoted by $\mathrm{IM}(G)$, is defined to be the set of all positive integers $h \in \mathbb{N}$ such that G has a magic labeling in $\mathbb{Z}_h$. Variations of graph labeling, especially magic labeling, have been studied by many authors and the integer-magic spectra of different classes of graphs have been determined. Interested readers are directed to [3, 5].

The standard definition of magic graphs can be modified so that the local requirement that "the sum of the labels of all edges incident with any vertex be constant" is replaced by a more global requirement. To achieve this, we restate the requirement (1.1) in terms of distance; namely, $l^+(v) = \sum_{e \in E_1(v)} l(e)$, where $E_1(v) = \{e \in E(G) : d(v,e) < 1\}$ is the set of all edges incident with v. Defining the standard vertex-magic graphs in this fashion suggests the extension of this concept. Given an abelian group A and a fixed integer $k \geq 0$, a graph G is said to have a *distance k magic labeling* in A if there is an edge labeling $l : E(G) \to A - \{0\}$ such that its induced vertex labeling $l_k^* : V(G) \to A$ defined by

$$l_k^*(v) = \sum_{e \in E_k(v)} l(e) \tag{1.2}$$

is constant. Here, $E_k(v) = \{e \in E(G) : d(v,e) < k\}$. The *distance k spectrum* of G, denoted by $\Delta^k M(G)$, is defined to be the set of all positive integers $h \in \mathbb{N}$ such that G has a distance k magic labeling in $\mathbb{Z}_h$. Note that when $k = 1$, these definitions are equivalent to those of the standard vertex-magic labeling of G.

2 Distance Two Magic Graphs

In this paper we study the distance two magic graphs and when there is no ambiguity, we drop the index 2; that is, we use $l^*(v)$, $E(v)$ and $\Delta \mathrm{M}(G)$ instead of $l_2^*(v)$, $E_2(v)$ and $\Delta^2 M(G)$, respectively. Given an abelian group A, a graph G is said to have a *distance two magic* labeling in A if there is an edge labeling $l : E(G) \to A - \{0\}$ such that its induced vertex labeling $l^* : V(G) \to A$ defined by $l^*(v) = \sum_{e \in E(v)} l(e)$ is constant, where $E(v) = \{e \in E(G) : d(v,e) < 2\}$. We say G is *dmagic* in A (or *A-dmagic*) if G admits such a distance two magic labeling in A. In general, a graph G may admit more than one dmagic labeling; for example, if $|A| > 2$ and $l : E(G) \to A - \{0\}$ is a dmagic labeling of G with sum S, then $\lambda : E(G) \to A - \{0\}$, the *inverse labeling* of l, defined by $\lambda(uv) = -l(uv)$ provides another dmagic labeling of G with sum $-S$. An interesting observation is that a labeling that makes a graph A-magic does not necessarily make it A-dmagic and vice versa. This fact is illustrated by an example in Figure 1. The left hand side shows a magic labeling of the graph in $\mathbb{Z}$ with $l^+ \equiv 2$, which is not a dmagic labeling. Because, $l^*(u) = 2$ and $l^*(v) = 5$. The one on the right has a dmagic labeling in $\mathbb{Z}$ with $l^* \equiv 3$, which is not a magic labeling; here, $l^+(u) = 2$ and $l^+(v) = 1$.

A graph $G = (V, E)$ is called *fully dmagic* if it admits a dmagic labeling in every non-trivial abelian group A. The following theorem identifies a number of fully dmagic graphs.

Theorem 2.1. *If* $\mathrm{diam}(G) \leq 2$, *then G is fully dmagic.*

Proof. Since $\mathrm{diam}(G) \leq 2$, then $E(v) = E(G)$ holds for every $v \in V(G)$. Therefore,

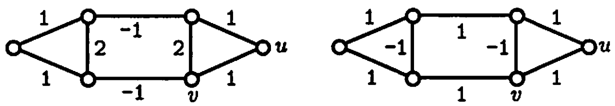

Figure 1: A magic labeling that is not a dmagic labeling and vice versa.

$l^*(v) = \sum_{e \in E(G)} l(e)$ is constant. □

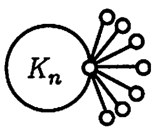

Figure 2: Amalgamation of K_n and star.

As a result of this theorem all complete graphs K_n, complete k-partite graphs $K(n_1, \cdots, n_k)$, and wheels $W_n = C_n + v$ are fully dmagic. In particular, the stars $ST(n) = K(1, n)$ are fully dmagic.

Corollary 2.2. *Given a complete graph K_n and a star $ST(m)$, if we identify the center of star with one of the vertices of K_n, the resulting graph is fully dmagic.*

A graph G is called *non-dmagic* if for every abelian group A, the graph does not have a dmagic labeling in A. The most obvious class of non-dmagic graphs is P_n ($n \geq 4$), the path of order n. Consider a typical labeling of P_4 that is illustrated in Figure 3.

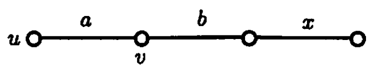

Figure 3: A typical labeling of P_4.

Here, $l^*(u) = a + b$ and $l^*(v) = a + b + x$. The requirement of $l^*(u) = l^*(v)$ would result in $x = 0$, which is not an acceptable dmagic label.

Observation 2.3. *Any graph with a pendant path of length $n \geq 3$ is non-dmagic.*

As another example of a non-dmagic graph, consider the graph H Figure 4. Given any labeling $l : E(G) \to A - \{0\}$ of this graph in A, let c be the sum of all labels of the edges.

Then $l^*(u) = c$ while $l^*(v) = c - x$. The requirement $l^*(u) = l^*(v)$ would result in $x = 0$, which is not acceptable. Thus, H does not have a dmagic labeling in A. On

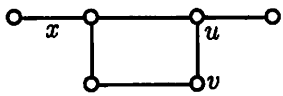

Figure 4: A non-dmagic graph H.

the other hand, if $|A| > 2$, then the graph H is A-magic, as illustrated in Figure 5

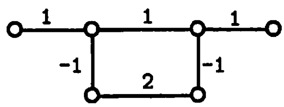

Figure 5: $IM(H) = \mathbb{N} - \{2\}$.

Observation 2.4. *Given an abelian group A, a graph that is A-magic is not necessarily A-dmagic and vice versa.*

Theorem 2.5. *Any complete graph K_n $(n \geq 2)$ with exactly two pendant edges is non-dmagic.*

Proof. The graph K_2 with two pendant edges is P_4, which is non-dmagic (see Figure 3). Suppose $n \geq 3$ and let $c = \sum_{e \in E(G)} l(e)$. Then $l^*(u) = c + x + y$ while $l^*(v) = c + x$. The requirement $l^*(u) = l^*(v)$ implies that $y = 0$, which is not an acceptable dmagic label. □

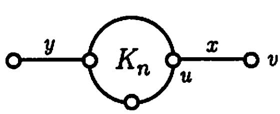

Figure 6: K_n with two pendant edges.

3 Dmagic Spectra of Graphs

In this section we focus on group $\mathbb{Z}_h$, integers modulo $h \in \mathbb{N}$. For convenience, the notation 1-dmagic will be used to indicate $\mathbb{Z}$-dmagic and $\mathbb{Z}_h$-dmagic graphs will be referred to as h-dmagic graphs. Clearly, if a graph is h-dmagic, it is not necessarily k-dmagic ($h \neq k$). In fact, as we will see in Corollary 3.5, for any two distinct positive integers $h < k$, there is a graph G that is k-dmagic but not h-dmagic. However, it is often useful to observe that if G has a dmagic labeling

$l : E(G) \rightarrow \mathbb{Z}$, then it is k-dmagic for all $k > \max\{l(e) : e \in E(G)\}$. Moreover, G would be k-dmagic as long as k does not divide $l(e)$, $\forall e \in E(G)$.

Definition 3.1. *Given a graph G, the set of all positive integers h for which G is h-dmagic is called the dmagic spectrum of G and is denoted by $\Delta\mathrm{M}(G)$.*

Any fully dmagic graph is h-dmagic for all positive integers h; therefore, $\Delta\mathrm{M}(G) = \mathbb{N}$. On the other hand, the graph H, Figure 4, is non-dmagic, hence $\Delta\mathrm{M}(H) = \emptyset$.

Observation 3.2. *A graph G is 2-dmagic if and only if the numbers $|E(v)|$ have the same parity.*

Frucht and Harary [2] introduced the corona of two graphs G and H, denoted by $G©H$, to be the graph with base G such that each vertex $v \in V(G)$ is joined to all vertices of a separate copy of H.

Theorem 3.3. *Given a complete graph K_n $(n \geq 3)$, let $p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$ be the prime factorization of $n-1$. Then the dmagic spectrum of the corona $K_n©K_1$ is $\bigcup_{i=1}^{k} p_i\mathbb{N}$.*

Proof. Consider any dmagic labeling of $K_n©K_1$ in $\mathbb{Z}_h$ and let c be the sum of labels of the edges of K_n. For any two distinct terminal vertices u, v of $K_n©K_1$, the requirement $l^*(u) = l^*(v)$ implies that the label of the terminal edges must be the same nonzero element $x \in \mathbb{Z}_h$. Let w be any vertex of K_n. The requirement $l^*(w) = l^*(u)$ implies $nx + c \equiv x + c \pmod{h}$ or

$$(n-1)x \equiv 0 \pmod{h} \tag{3.1}$$

and this equation has a nonzero solution for x if and only if $\gcd(n-1, h) > 1$; that is, $h \in \bigcup_{i=1}^{k} p_i\mathbb{N}$. Also, we observe that the equation (3.1) does not have a nonzero solution in $\mathbb{Z}$, hence $K_n©K_1$ is not $\mathbb{Z}$-dmagic. □

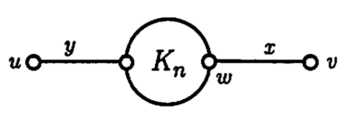

Figure 7: $K_n©K_1$.

Consider two complete graphs K_m and K_n and assume that

$V(K_m) = \{u_1, u_2, \cdots, u_m\}$ and $V(K_n) = \{v_1, v_2, \cdots, v_n\}$. Using these two graphs we Construct the new graph $G_{m,n}$ by 1) joining the vertices u_1 and v_1 with the edge u_1v_1; 2) adding a pendent edge to all other vertices, as illustrated in Figure 8. The following theorem determines the dmagic spectrum of $G_{m,n}$.

Theorem 3.4. *For any two positive integers m, n $(m > n \geq 3)$, let $m - n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$, and $m-2 = p_1^{\beta_1}p_2^{\beta_2}\cdots p_k^{\beta_k}$ be the prime factorizations of $m-n$ and*

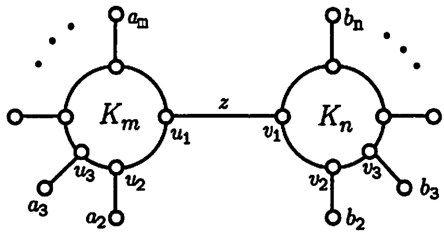

Figure 8: The graph $G_{m,n}$.

$m-2$, *respectively. Then* $\Delta M(G_{m,n}) = \bigcup_{i=1}^{k} A_i$, *where*

$$A_i = \begin{cases} \emptyset & \text{if } \beta_i \geq \alpha_i; \\ p_i^{\beta_i+1}\mathbb{N} & \text{otherwise,} \end{cases}$$

Proof. Given an arbitrary labeling ℓ of $G_{m,n}$, let $c = \sum_{e\in E(K_m)} \ell(e)$ and $d = \sum_{e\in E(K_n)} \ell(e)$. The condition $\ell^*(a_2) = \ell^*(a_i)$ implies $\ell(a_i u_i) = x$ for all $2 \leq i \leq m$. Also, the condition $\ell^*(u_1) = \ell^*(u_2)$ implies that $(m-1)x+c+z+d = (m-1)x+c+z$ or $d = 0$. Similarly, $c = 0$. The condition $\ell^*(a_2) = \ell^*(b_2)$ implies that $\ell(b_2v_2) = \ell(a_2u_2)$; that is, all the terminal edges of $G_{m,n}$ are labeled the same element x. The condition $\ell^*(u_1) = \ell^*(v_1)$ implies $(m-1)x + z = (n-1)x + z$ or

$$(m-n)x \equiv 0 \pmod{h}. \tag{3.2}$$

Finally, the condition $\ell^*(a_2) = \ell^*(u_2)$ gives us $x = (m-1)x + z$ or

$$(m-2)x + z \equiv 0 \pmod{h}. \tag{3.3}$$

The question of finding the dmagic spectrum of $G_{m,n}$ is reduced to determining those numbers $h \in \mathbb{N}$ for which the equations (3.2) and (3.3) have nonzero solutions for x and z. If $m-n=1$ or $(m-n)|(n-2)$, then the equations (3.2) and (3.3) lead us to $x \equiv 0$, or $z \equiv 0$, respectively, which are not acceptable dmagic labels. Therefore, in these cases, $DM(G_{m,n}) = \emptyset$. Assume that $m-n > 1$ and that $m-n$ does not divide $m-2$. Note that the equation (3.2) has nonzero solution for x if and only if $\gcd(m-n, h) > 1$; that is, $h \in \cup_{i=1}^{k} p_i\mathbb{N}$. To see if this nonzero solution for x provides a nonzero solution for z we consider the following two cases:

Case I. Suppose $\beta_i \geq \alpha_i$ and let $\gcd(m-n, r) = 1$. Then the system

$$\begin{cases} (m-n)x \equiv 0 & \pmod{rp_i^j}; \\ (m-2)x + z \equiv 0 & \pmod{rp_i^j}, \end{cases}$$

does not have nonzero solution for z. Because, if $rp_i^j|(m-n)x$, then $rp_i^j|(m-2)x$, which results in $z \equiv 0$.

Case II. Suppose $\beta_i < \alpha_i$. If $\gcd(m-n, r) = 1$ and $j \le \beta_i$, then as we saw in case I, the system

$$\begin{cases} (m-n)x \equiv 0 & \pmod{rp_i^j}; \\ (m-2)x + z \equiv 0 & \pmod{rp_i^j}, \end{cases}$$

does not have nonzero solution for z. However, the system

$$\begin{cases} (m-n)x \equiv 0 & \pmod{rp_i^{\beta_i+1}}; \\ (m-2)x + z \equiv 0 & \pmod{rp_i^{\beta_i+1}}, \end{cases}$$

has nonzero solutions for x and z; if we choose $x = r$, then $y \equiv -(m-2)r \not\equiv 0 \pmod{rp_i^{\beta_i+1}}$. Therefore, $p_i^{\beta_i+1}\mathbb{N} \subset \Delta\mathrm{M}(G_{m,n})$.

This shows that $\cup_{i=1}^{k} A_i \subset \Delta\mathrm{M}(G_{m,n})$. Now suppose $G_{m,n}$ is h-dmagic. Then $h|(m-2)x$ but $h \nmid (m-2)x$. Therefore, in the prime factorization of h there is a prime factor p^c with the property that $p^c|(m-n)x$ but $p^c \nmid (m-2)x$. Let $\beta \ge 0$ be the number such that $b^\beta|(m-2)$ but $p^{\beta+1} \nmid (m-2)$. Then, by case II, $G_{m,n}$ is $p^{\beta+1}$-dmagic and $h \in p^{\beta+1}\mathbb{N}$. The proof would be complete if one can show that the edges of K_m and K_n can be labeled so that $c = \sum_{e\in E(K_m)} \ell(e) = 0$ and $d = \sum_{e\in E(K_n)} \ell(e) = 0$. This fact has been established in [4]. □

Corollary 3.5. *Given any two distinct positive integers h and k with $h < k$, there is a graph G that is k-dmagic but it is not h-dmagic.*

Proof. Assume $h < k$ and consider $G_{m,n}$ with $m = k + h + 2$ and $n = h + 2$. This graph is k-dmagic, but is not h-dmagic. □

Corollary 3.6. *Using the notation of Theorem 3.4, if $m-n = 1$ or $(m-n)|(m-2)$, then the graph $G_{m,n}$ is non-dmagic.*

4 Trees with Diameter at most four

Trees of diameter two are stars $ST(n) = K(1, n)$ and are fully dmagic. Therefore, $\Delta\mathrm{M}(ST(n)) = \mathbb{N}$. Trees of diameter three, also known as *double-stars*, have two central vertices c and d plus leaves. We use $DS(m, n)$ to denote the double-star whose two central vertices have degrees m and n, respectively. Note that if $m = 1$ or $n = 1$, then $DS(m, n)$ would be a star.

Theorem 4.1. *Let $m, n \ge 2$. Then*

$$\Delta\mathrm{M}(DS(m, n)) = \begin{cases} \emptyset & \text{if } m \text{ or } n = 2; \\ \mathbb{N} & \text{if } m, n \text{ are odd}; \\ \mathbb{N} - \{2\} & \text{otherwise}. \end{cases}$$

Proof. We observe that if $m = 2$ or $n = 2$, then $DS(m, n)$ would be non-dmagic. Because, if $\deg c = 2$, then $l^*(v_1) = l^*(d)$ implies that $l(cu_1) = 0$, which is not an acceptable dmagic label. Suppose $m, n \ge 3$, $\deg(c) = m$ and $\deg(d) = n$. First, we observe that $|E(u_i)| = m$, $|E(v_j)| = n$, and $|E(c)| = m + n - 1$. Therefore,

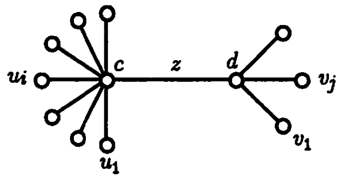

Figure 9: $DS(8,4)$

by observation 3.2, $DS(m,n)$ is 2-dmagic if and only if m,n are both odd. Now, let l be any labeling of $DS(m, n)$ by elements of $\mathbb{Z}_h$. Then the conditions $l^*(c) = l^*(u_i) = l^*(v_j)$ provide $\sum l(cu_i) = \sum l(dv_j) \equiv 0 \pmod h$. So, it is enough to show that such a dmagic labeling is possible.

If m is odd, then let $l(cu_i) = (-1)^i$ $(1 \leq i \leq m-1)$. If m is even, then let $l(cu_1) = 2$, $l(cu_2) = l(cu_3) = -1$, and $l(cu_i) = (-1)^i$ $(4 \leq i \leq m-1)$. This guarantees that $\sum l(cu_i) = 0$ and $l^* \equiv z$ is a constant, where z is the label of the edge cd. □

Definition 4.2. A tree of diameter four, denoted by $T_4(a_1, a_2, \cdots, a_n)$, consists of n stars $ST(a_1)$, $ST(a_2), \cdots$, $ST(a_n)$, one of their edges is incident with a common vertex and $a_i \geq 2$ for at least two values of i. The common vertex is the center of tree and is denoted by c. Equivalently, $T_4(a_1, a_2, \cdots, a_n)$ is a tree with center-vertex c, in which n edges $\{cd_1, cd_2, \cdots, cd_n\}$ are emanated from c, and $\deg(d_i) = a_i$ for $i = 1, 2, \cdots, n$, as illustrated in the Figure 10.

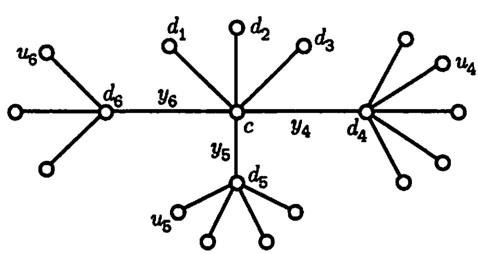

Figure 10: $T_4(1^3, 4, 5, 6)$; An example of a tree of diameter 4.

In order to have a tree of diameter four, one needs $n \geq 2$ and $a_i \geq 2$ for at least two values of i. Let $b_1, \cdots, b_n$ be any permutation of $a_1, \cdots, a_n$. Then the tree $T_4(a_1, \cdots, a_n)$ is isomorphic with $T_4(b_1, \cdots, b_n)$. Therefore, when considering a tree of diameter four, without loss of generality, we may assume that $a_1 \leq a_2 \leq \cdots \leq a_n$. Also, we use the notation $T_4(a^n)$ when $a_i = a$ $(1 \leq i \leq n)$.

Theorem 4.3. *Given a tree of diameter four $T_4 = T_4(a_1, \cdots, a_n)$, let $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be the prime factorization of $n-1$. Then*

$$\Delta M(T_4) = \begin{cases} \emptyset & \text{if } a_1 = 1 \text{ and } a_i = 2 \text{ for some } i \geq 2; \\ \bigcup_{i=1}^{k} p_i\mathbb{N} & \text{if } n \text{ is odd and} \\ & a_i\ (1 \leq i \leq n) \textit{ have the same parity;} \\ \bigcup_{i=1}^{k} p_i\mathbb{N} - \{2\} & \text{otherwise.} \end{cases}$$

Proof. Let u_{ij} $(1 \leq i \leq n)$ be the terminal vertices adjacent to d_i. Note that $|E(u_{ij})| = a_i$, $|E(d_i)| = a_i + n - 1$, and $|E(c)| = \sum_{i=1}^{n} a_i$. By 3.2, T_4 is 2-dmagic if and only if these numbers have the same parity. That is, n is odd and the numbers a_i have the same parity $(1 \leq i \leq n)$.

Assume $h > 2$ is a positive integer and consider the labeling $l : E(T_4) \to \mathbb{Z}_h$ and let $l(cd_i) = y_i$. First a couple of observations:

(A) The conditions $l^*(d_1) = l^*(d_i)$ imply that the sum of labels of the terminal edges incident with d_i must be the same $(1 \leq i \leq n)$. We denote this common value by x. Note that if $a_1 = 1$, then $x = 0$.

(B) The condition $l^*(u_{i1}) = l^*(u_{j1})$ and observation **(A)** imply that $y_i = y_j$. Therefore, all the edges cd_i $(1 \leq i \leq n)$ have the same label, we denote this common label by y.

Now we prove the theorem by considering three cases.

Case 1. If $a_1 = 1$ and $a_i = 2$ for some $i > 1$, then by (A), $x = 0$ is the label of the only terminal edge incident with d_i, which is not an acceptable dmagic label. Thus, T_4 is non-dmagic and its dmagic spectrum is $\emptyset$.

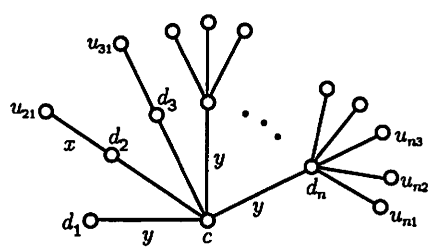

Figure 11: $a_1 = 1$ and $a_2 = 2$ imply $x = 0$.

Case 2. Suppose $a_1 = 1$ and $a_i \geq 3$ $(i \geq 2)$. Then by (A) the sum of labels of the terminal edges incident with d_i must be 0 $(2 \leq i \leq n)$, and $l^*(c) = l^*(u_{n1})$ or $(n-1)y \equiv 0 \pmod{h}$. This equation has a nonzero solution for $y \in \mathbb{Z}_h$ if and only if $\gcd(n-1, h) > 1$; That is, $h \in \bigcup_{i=1}^{k} p_i\mathbb{N}$. It only remains to show that we can label the terminal edges that are incident with d_i with nonzero elements of $\mathbb{Z}_h$ so that they add up to 0, and this is similar to the labeling that was presented in 4.1.

Case 3. Suppose $a_1 \geq 2$. The condition $l^*(d_1) = l^*(u_{n1})$ gives us $x + ny \equiv x + y$ or $(n-1)y \equiv 0 \pmod h$. Also, $l^*(c) = l^*(d_i)$ gives us $nx + ny \equiv x + ny$ or $(n-1)x \equiv 0 \pmod h$. And these two equations have nonzero solutions in $\mathbb{Z}_h$ if and only if $h \in \bigcup_{i=1}^k p_i\mathbb{N}$. Note that in this case one can choose $x = y$. □

Corollary 4.4. *Let $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be the prime factorization of $n-1$. Then*

$$\Delta\mathrm{M}(T_4(2^n)) = \bigcup_{i=1}^{k} p_i\mathbb{N}.$$

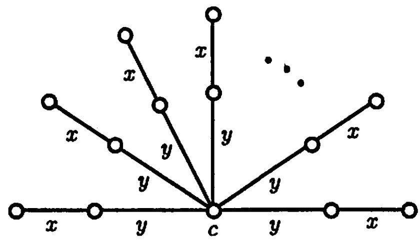

Figure 12: $T_4(2^n)$ and its dmagic labeling.

5 Caterpillars

Caterpillar is a tree having the property that the removal of its end-vertices results in a path (the spine). We use $CR(a_1, a_2, \cdots, a_n)$ to denote the caterpillar with a P_n-spine, where d_i, the ith vertex of P_n has degree a_i. Since $CR(1, a_1, \cdots, a_n, 1) = CR(a_1, \cdots, a_n)$ and $a_i \neq 1$ $(2 \leq i \leq n-1)$, we assume that $a_i \geq 2$.

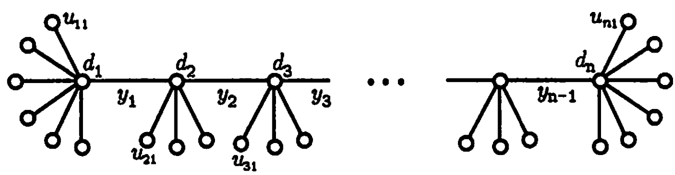

Figure 13: A Caterpillar of diameter $n+1$ (P_n-spine).

Theorem 5.1. *If $a_1, a_n \geq 3$ and $a_i \geq 2$ for $1 < i < n$, then the caterpillar $G = CR(a_1, a_2, \ldots, a_n)$ is $\mathbb{Z}$-dmagic.*

Proof. Let $l : E(G) \to \mathbb{Z}$ be a labeling of $G = CR(a_1, \cdots a_n)$, y_i be the label of $u_i u_{i+1}$ $(1 \leq i \leq n-1)$, and u_{i1} be one of the terminal vertices adjacent to d_i, as illustrated in Figure 13. Also, let σ_i be the sum of the labels of all terminal edges incident with d_i. For $l : E(G) \to \mathbb{Z}$ to be a dmagic labeling of G, we require $l^*(u_{i1}) = l^*(d_i)$ for $1 \le i \le n$ and $l^*(u_{i1}) = l^*(u_{(i+1)\,1})$ for $1 \le i \le n-1$.

The condition $l^*(u_{11}) = l^*(d_1)$ implies that

$$y_2 + \sigma_2 \equiv 0 \pmod{h}, \tag{5.1}$$

and the condition $l^*(u_{11}) = l^*(u_{21})$ together with equation (5.1) imply that $\sigma_1 \equiv 0 \pmod{h}$. Now we proceed by induction to show that $\sigma_i = -y_1$ and $y_{i-1} = y_1$ $(2 \le i \le n-1)$. Note that $l^*(u_{21}) = l^*(d_2)$, which is the same as $l^*(u_{21}) = l^*(u_{21}) + y_3 + \sigma_3$ implies that

$$y_3 + \sigma_3 \equiv 0 \pmod{h}, \tag{5.2}$$

and $l^*(u_{21}) = l^*(u_{31})$ together with equation (5.2) imply that $\boxed{y_2 = y_1.}$

Suppose $y_k = y_1$ and $y_k + \sigma_k \equiv 0 \pmod{h}$ for all k $(2 \le k < n-1)$. Then the condition $l^*(u_{k1}) = l^*(d_k)$ and the induction hypothesis imply that $y_{k+1} + \sigma_{k+1} \equiv 0 \pmod{h}$. Furthermore, the condition $l^*(u_{k1}) = l^*(u_{(k+1)\,1})$ provides $y_{k+1} = y_k = y_1$, which completes the induction. Finally, it is easy to see that $\sigma_n \equiv 0 \pmod{h}$. Therefore, the caterpillar G is $\mathbb{Z}$-dmagic if and only if

$$\sigma_1, \sigma_n \equiv 0 \pmod{h},$$

all the edges of the spine be labeled the same $y_i = y$ $(1 < i < n)$, and $\sigma_i = -y$ $(1 < i < n)$, which would provide $l^* \equiv y$.

The above dmagic labeling can be done by just using at most for integers $\pm 1, \pm 2 \in \mathbb{Z}$. Label all the edges of spine by 1. If $a_1 = \deg(d_1)$ is odd, label all the terminal edges incident with d_1 alternatively by 1 and -1. If a_1 is even, label one of the terminal edges by 2, the next two by -1, and the rest by 1 and -1, alternatively. This guarantees that $\sigma_1 = 0$. Similar scheme applies to $\sigma_n = 0$ and $\sigma_i = -1$ $(2 < i < n)$. □

Corollary 5.2. *Using the notations of theorem 5.1,* $G = CR(a_1, a_2, \ldots, a_n)$ *is non-dmagic if and only if* $a_1 = 2$ *or* $a_n = 2$.

Proof. Suppose $a_1 = 2$. Using the labeling of Figure 13, we notice that $l^*(d_1) = l^*(u_{21})$ implies that $l(d_1u_{11}) + l^*(u_{21}) = l^*(u_{21})$, or $l(d_1u_{11}) = 0$, which is not an acceptable dmagic label. □

Corollary 5.3. *For the dmagic spectrum of caterpillar* $CR(a_1, a_2, \cdots, \alpha_n)$ *we have*

$$\Delta\text{M}(G) = \begin{cases} \emptyset & \text{if } a_1 = 2 \text{ or } a_n = 2; \\ \mathbb{N} & \text{if } a_i \text{ is odd } 1 \le i \le n; \\ \mathbb{N} - \{2\} & \text{otherwise.} \end{cases}$$

Examples 5.4.

(a) The dmagic spectrum of the graph in Figure 10 is $5\mathbb{N}$. Because, $n - 1 = 5$.

(b) $\Delta M(T_4(1,2,6,7^4)) = \emptyset$. Here, $a_1 = 1$ and $a_2 = 2$.

(c) $\Delta M(T_4(1^3,4^2,5,15)) = 2\mathbb{N} \cup 3\mathbb{N} - \{2\}$. Here, $n-1=6$ and the numbers a_i do not have the same parity, which means the graph is not 2-dmagic.

(d) $\Delta M(CR(13,9,6)) = \mathbb{N} - \{2\}$.

(e) $\Delta M(CR(17,10,2)) = \emptyset$. Here, $a_n = 2$.

(f) $\Delta M(CR(5,11,15,81)) = \mathbb{N}$. Here, $a_1, \cdots, a_n$ all are odd.

6 Some Open Problems

The observation 3.2 characterizes all the graphs that have distance two magic labeling in $\mathbb{Z}_2$. It would be desirable to find some kind of characterization for other $\mathbb{Z}_h$ groups. This issue has not even been resolved for standard 3-magic graphs.

Problem 6.1. *Characterize all graphs that have distance two magic labeling in* $\mathbb{Z}_3$.

Figure 1 illustrates a graph that is both magic and dmagic. The given labelings are either magic but not dmagic, or vice versa. However, there is another labeling shown in Figure 14 that simultaneously provides both magic and dmagic labelings.

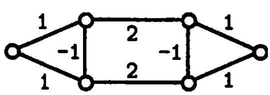

Figure 14: A magic labeling that is dmagic too.

Problem 6.2. *Characterize the graphs for which there exists a magic labeling that is dmagic as well.*

7 Acknowledgments

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