A lower bound of the l-edge-connectivity and optimal graphs

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Abstract

For an integer $l > 1$, the l-edge-connectivity of a graph G with $|V(G)| \geq l$ denoted by $\lambda_l(G)$, is the smallest number of edges whose removal results in a graph with l components. In this paper, we study lower bounds of $\lambda_l(G)$ and optimal graphs that reach the lower bounds. Former results by Boesch and Chen are extended.

We also present in this paper an optimal model of interconnection network G with a given $\lambda_l(G)$ such that $\lambda_2(G)$ is maximized while $|E(G)|$ is minimized.

Key workds: edge-connectivity, generalized edge-connectivity, circulant graphs

1 Introduction

Graphs in this paper are finite and loopless. Undefined terms and notations can be found in [3]. For a graph G and for an edge subset X which have ends in $V(G)$ and which are not in $E(G)$, $G + X$ denotes the graph with $V(G+X) = V(G)$ and $E(G+X) = E(G) \bigcup X$.

For an integer $l \geq 2$, Boesch and Chen [1] defined the l-**edge-connectivity** $\lambda_l(G)$ of a connected graph G to be the minimum number of edges that are

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required to be deleted from G to produce a graph with at least l components if $|V(G)| \geq l$, or to be $|E(G)|$, if $|V(G)| < l$. In particular, $\lambda_2(G)$ is the **edge-connectivity** of G. The parameter $\lambda_l(G)$ has been studied by many researchers. For overviews of the related literature, see [8], [9], and [10], among others.

For disjoint non empty subsets $A, B \subset V(G)$, the set $[A, B]$ denotes all edges in G with one vertex in A, and the other in B. We assume that G_1 and G_2 are two graphs with disjoint vertex sets. We use the notation that the degree of vertex v_i is deg v_i. We also use the notations $\lceil x \rceil$ to denote the smallest integer greater than or equal to x, and $\lfloor x \rfloor$ for the largest integer less than or equal to x. Let G be a graph and let $X \subseteq E(G)$ be an edge subset. The **contraction** G/X is the graph obtained from G by identifying the two ends of each edge in X and by deleting the resulting loops. Thus G/X is loopless and may have multiple edges, even when G is simple. If H is a subgraph of G, then G/H denotes $G/E(H)$. Note that each vertex v in G/X is the contraction image of a connected subgraph H_v of G. Thus H_v is called the **preimage** of v. A vertex v in the contraction G/X is **nontrivial** if $|V(H_v)| > 1$.

In Section 2, some former results on lower bounds of $\lambda_l(G)$ and a new best possible lower bound of $\lambda_l(G)$ in terms of $\lambda_2(G)$ are given. We also investigate in Section 2 when equality holds in our new lower bound. Section 3 is a brief introduction to circulant graphs and generalized circulant graphs, which will be used in Section 4 to determine the minimum size of optimal graphs.

2 Lower Bound of λ_l

We start with some former results concerning l-edge-connectivity.

Theorem 2.1 (Boesch and Chen [1]) Let G be a connected graph with $n = |V(G)|$ vertices. For each i with $1 \leq i < l - 1 < n$,

$$\lambda_l(G) \geq \frac{(l-1)(l-i+1)}{(l+1)(l-i-1)}\lambda_{l-i}(G).$$

Theorem 2.2 (Boesch and Chen [1]) Let $n \geq l > 1$ be two integers, and let G be a graph with n vertices and minimum degree $\delta(G)$. If $\delta(G) \geq \lfloor \frac{n}{l} \rfloor$, then $\lambda_l(G) \geq \delta(G)$.

Since $\delta(G)$, the minimum degree of a graph G, satisfies $\delta(G) \geq \lambda_2(G)$ for any graph G, Theorem 2.2 has an immediate corollary.

Corollary 2.3 (Boesch and Chen [1]) Let $n \geq l > 1$ be two integers, and let G be a graph with n vertices and minimum degree $\delta(G)$. If $\lambda_2(G) \geq \lfloor \frac{n}{l} \rfloor$, then $\lambda_l(G) \geq \delta(G)$.

Theorem 2.4 (Harary [6]) Among all graphs G with $|V(G)| = n$, and $|E(G)| = m$ the maximum value of $\lambda_2(G)$ is zero when $m < n-1$ and is $\lceil \frac{2m}{n} \rceil$ when $m \geq n-1$.

Theorem 2.5 Let $n \geq l > 1$ be two integers, and let G be a connected graph with n vertices. Then

$$\lambda_l(G) \geq \frac{l\lambda_2(G)}{2}.$$

Proof. Let G be a connected graph and Y be a set of $\lambda_l(G)$ edges of G, such that $G-Y$ has l components $C_1, C_2, \ldots, C_l$ of $G-Y$. By the definition of edge-connectivity we have,

$$|[V(C_i), V(G-C_i)]| \geq \lambda_2(G) \text{ for each } i \text{ with } 1 \leq i \leq l.$$

Take the sum from $i = 1$ to l to get,

$$\sum_{i=1}^{l} |[V(C_i), V(G-C_i)]| \geq l\lambda_2(G).$$

It follows that

$$\lambda_l(G) = \frac{\sum_{i=1}^{l} |[V(C_i), V(G-C_i)]|}{2} \geq \frac{l}{2}\lambda_2(G).\square$$

Note that if $l = n$ then $\lambda_n = m$. Thus Theorem 2.5 implies Theorem 2.4. When $i = l-2$, Theorem 2.1 asserts that $\lambda_l(G) \geq \frac{3(l-1)}{l+1}\lambda_2(G)$ for any connected graphs with n vertices such that $n \geq l > 2$. Simple algebraic manipulation yields

$$\frac{l}{2} > \frac{3(l-1)}{l+1} \Longleftrightarrow (l-2)(l-3) > 0.$$

Therefore when $l = 3$ and $i = l - 2$, both Theorems 2.1 and 2.5 give the same bound and when $l > 3$ and $i = l - 2$,, Theorem 2.5 gives a better bound than Theorem 2.1.

Theorem 2.5 also extends Corollary 2.3 when $|V(G)| \geq 2\delta(G)$.

<u>**Corollary 2.6**</u> Let G be a connected graph. If $\lambda_2(G) \geq \lceil \frac{2\delta}{l} \rceil$ then $\lambda_l(G) \geq \delta(G)$.

By Theorem 2.5, when $\lambda_l(G)$ is given, the maximum $\lambda_2(G)$ can reach is to have the equality

$$\lambda_l(G) = \frac{l\lambda_2(G)}{2}. \tag{1}$$

To investigate graphs satisfying (1), we first note that $\lambda_l(G)$ is an integer. Thus if (1) holds for a graph, then $l\lambda_2(G)$ must be an even integer.

<u>**Lemma 2.7.**</u> Let G be a graph satisfying (1). Let Y be a set of $\lambda_l(G)$ edges of G such that $G - Y$ has l components $C_1, C_2, \ldots, C_l$. Then

$$|[V(C_i), V(G - C_i)]| = \lambda_2(G) \text{ for all } 1 \leq i \leq l.$$

<u>**Proof**</u>: By the definition of $\lambda_2(G)$,

$$|[V(C_i), V(G - C_i)]| \geq \lambda_2(G) \text{ for all } 1 \leq i \leq l. \tag{2}$$

By (1) and by the definition of Y,

$$\frac{1}{2}\sum_{i=1}^{l} |[V(C_i), V(G - C_i)]| = |Y| = \lambda_l(G) = \frac{l}{2}\lambda_2(G)$$

and so we have,

$$\sum_{i=1}^{l} |[V(C_i), V(G - C_i)]| = l\lambda_2(G). \tag{3}$$

It follows by (2) and (3) that $|[V(C_i), V(G - C_i)]| = \lambda_2(G)$.□

<u>**Lemma 2.8**</u> Let G be a graph satisfying (1). Let Y be a set of $\lambda_l(G)$ edges of G such that $G - Y$ has l components $C_1, C_2, \ldots, C_l$. If a component

C_i has at least two vertices, then the number of vertices in C_i is at least $\lambda_2(G)$, for each i with $1 \leq i \leq l$.

<u>**Proof:**</u> Fix an i with $1 \leq i \leq l$ and let $n_i = |V(C_i)|$. By Lemma 2.7,

$$|[V(C_i), V(G - C_i)]| = \lambda_2(G).$$

Thus $n_i(n_i - 1) + \lambda_2(G) \geq$ number of incidences with vertices in $V(C_i) \geq n_i\lambda_2(G)$, and so $(n_i - \lambda_2(G))(n_i - 1) \geq 0$. Lemma 2.8 now follows by $n_i > 1$. □

<u>**Theorem 2.9**</u> Assume that $l \geq 3$ is an integer. Let G be a simple graph with $\lambda_2(G) = s$ and $\lambda_l(G) = t$. Then G satisfies (1) if and only if each of the following holds:
(i) G can be contracted to an s-regular graph G' with $|V(G')| = l$ and $|E(G')| = t$;
(ii) the preimage of each nontrivial vertex in G' has at least s vertices; and
(iii) there is at most one edge joining two trivial vertices in G'.

<u>**Proof:**</u> Suppose first (1) holds. Then G has $Y \subseteq E(G)$ such that $G - Y$ has l components $C_1, C_2, \ldots, C_l$. Let $X = \cup_{i=1}^{l} E(C_i)$ and $G' = G/X$. Then the l components of $G - Y$ are vertices of G' and the edges in Y are the edges of G'. By Lemma 2.7, $|[V(C_i), V(G - C_i)]| = \lambda_2(G) = s$ for all $1 \leq i \leq l$ and so G' is an s-regular graph. Note that $|V(G')| = l$ and $|E(G')| = s|V(G')|/2 = sl/2 = t$. This proves (i). Theorem 2.9 (ii) and (iii) follows by Lemma 2.8 and the simpleness of G respectively.

Conversely, by (i) G' is an s-regular graph with $|V(G')| = l$ and $|E(G')| = t$. It is well known that for an s-regular graph G', $|E(G')| = s|V(G')|/2$. Thus $ls = 2t$. □

<u>**Corollary 2.10**</u> Let G satisfy (1) and G' be the graph defined in Theorem 2.9. Let b denote the number of nontrivial vertices in G'. Then $|V(G)| \geq (l - b) + b\lambda_2$.

3 Circulant Component Graphs

Let $|V(G)| = n$ be a positive integer. Assume that the vertices of a graph are labeled $0, 1, 2, \cdots, n-1$, and we refer to vertex i instead of saying the vertex labeled with i. The **circulant graph** $C_n[a_1, a_2, \cdots, a_k]$ or briefly $C_n[a_i]$, where $0 < a_1 < a_2 < \cdots < a_k < \frac{n+1}{2}$, has $i \pm a_1, i \pm a_2, \cdots, i \pm a_k (mod\ n)$ adjacent to each vertex i. The sequence $\langle a_i \rangle$ is called the **jump sequence** and the a_i's are called the **jumps**. Notice that our definition precludes jumps a of size greater than $\frac{n}{2}$ as such jumps would produce the same result as a jump of size $(n - a)$, as $n - a < \frac{n}{2}$. Also note that if $a_k \neq \frac{n}{2}$ then the circulant is always regular of degree $2k$. When n is even we have allowed $a_k = \frac{n}{2}$ (called a diagonal jump), and when $a_k = \frac{n}{2}$ the circulant has degree $2k - 1$.

Now we extend the definition of circulant graphs to define circulant component graphs. Let G be a graph and T be a set of edges of G such that $G - T$ has l components, $C_0, C_1, \ldots, C_{l-1}$. If a component has only one vertex then it is called a **trivial** component. Two components C_i and C_j are said to be **adjacent** in G if there is a vertex x in C_i and vertex y in C_j such that the edge $xy \in T$. The **circulant component graph** $CC_l[a_1(b_1), a_2(b_2), \ldots, a_k(b_k)]$ or briefly $CC_l[a_i(b_i)]$, where $0 < a_1 < a_2 < \cdots < a_k < \lceil \frac{l}{2} \rceil$, represents a family of graphs. A graph G is in $CC_l[a_1(b_1), a_2(b_2), \cdots, a_k(b_k)]$ if and only if G has an edge set T such that $G - T$ has l components $C_1, C_2, \cdots, C_l$, such that for each $i = 1, 2, \cdots, l$, C_i is adjacent to $C_{i \pm a_1 (mod\ l)}, C_{i \pm a_2 (mod\ l)}, \ldots C_{i \pm a_k (mod\ l)}$ with $b_1, b_2, \cdots, b_k$ edges, respectively. Figure 3.1 gives $CC_5[1(2), 2(1)]$. For notational convention, we also use $CC_l[a_1(b_1), a_2(b_2), \ldots, a_k(b_k)]$ or $CC_l[a_i(b_i)]$ to denote a member in it.

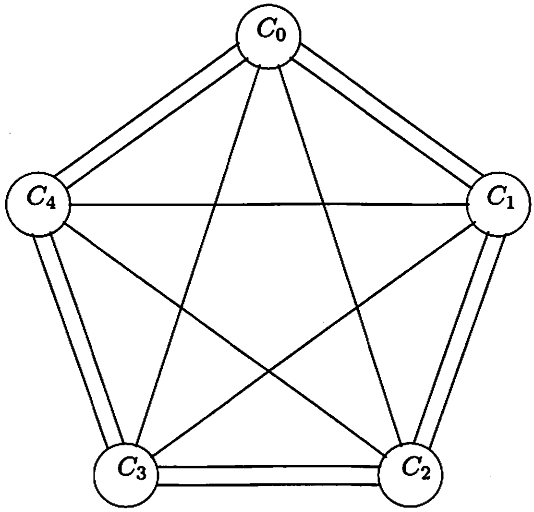

Figure 3.1 Circulant component graph $CC_5[1(2), 2(1)]$

Intuitively, a graph G in $CC_l[a_1(b_1), a_2(b_2), \ldots, a_k(b_k)]$ can be obtained from $C_l[a_1, a_2, \ldots, a_k]$ by replacing each edge in $C_l[a_1, a_2, \ldots, a_k]$ joining vertex i and vertex $i \pm a_1(mod\ l)$ by b_i edges, and by expanding each vertex i in $C_l[a_1, a_2, \ldots, a_k]$ by a (possibly trivial) connected graph C_i. We shall refer these $C_1, C_2, \cdots, C_l$ as **the components** of G. Note that the definition of circulant component graph does not say any thing about the structure of the components $C_1, C_2, \cdots, C_l$ of $G - T$. In Section 4 we use circulant component graphs in the construction of minimal graphs. Then all we have to do is to give the structure of each component of $G - T$.

Proposition 3.1 Let G be a circulant component graph, where T is an edge subset of G such that $G - T$ has l components $C_1, C_2, \cdots, C_l$. Then G can be contracted to an $2|T|/l$-regular graph G' with $|V(G')| = l$ and $|E(G')| = |T|$.

Proof: Let $E(G) - T = X$ and $G' = G/X$. Then the l components of $G - T$ are vertices of G' and the edges in T are the edges of G'. Thus $|V(G')| = l$ and $|E(G')| = |T|$. □

Let G be a circulant component graph and T be an edge subset of G such that $G - T$ has l components. Let C be a component of $G - T$. A vertex v of C is **internal** if v is not incident with any edge of T; otherwise, v will be **external.** If $e \in T$ then the edge e joins two external vertices of two different components of $G - T$. Furthermore e is called an **external edge** of the circulant component graph. Thus all the edges of T are external edges of the circulant component graph. Therefore the definition of the circulant component graph gives only the arrangement of the external edges.

4 Graphs reaching the lower bound with minimum number of edges

In this section, we present a best possible lower bound of the size of graphs satisfying (1).

Theorem 4.1 Let $n \geq l > 1$ be integers.
(i) Let G be a simple graph satisfying (1) with $|V(G)| = n$ vertices. Then $|E(G)| \geq \frac{1}{2}\lambda_2(G)|V(G)|$.
(ii) There exists a graph H satisfying (1) with $n = |V(H)|$ such that $|E(H)| = \frac{1}{2}\lambda_2(H)|V(H)|$.

Proof: (i). Let T be a set of $\lambda_l(G)$ edges of G such that $G - T$ has l components $C_1, C_2, \ldots, C_l$. Consider a component C_i of $G - T$. Let $v \in V(C_i)$. Then

$$\deg_G v \geq \lambda_2(G). \tag{4}$$

Let $|V(C_i)| = n_i$. By Lemma 2.7, $|[V(C_i), V(G - C_i)]| = \lambda_2(G)$. By (4) and by Lemma 2.8, $\lambda_2 n_i \leq \sum_{v \in V(C_i)} \deg\, v = 2|E(C_i)| + \lambda_2$ and so $\lambda_2(n_i - 1) \leq 2|E(C_i)|$. It follows that

$$|E(C_i)| \geq \frac{\lambda_2(G)}{2}(n_i - 1), 1 \leq i \leq l. \tag{5}$$

Note that $|E(C_i)|$ is an integer. If the equality of (5) holds for a graph G, then $\lambda_2(G)(n_i - 1)$ must be even. Thus,

$$\begin{aligned} |E(G)| &= \sum_{i=1}^{l} |E(C_i)| + \lambda_l(G) \geq \sum_{i=1}^{l} \frac{\lambda_2(G)(n_i - 1)}{2} + \frac{l}{2}\lambda_2(G) \\ &= \frac{\lambda_2(G)}{2}\left(\sum_{i=1}^{l} n_i - \sum_{i=1}^{l} 1\right) + \frac{l}{2}\lambda_2(G) \end{aligned}$$

$$= \frac{\lambda_2(G)}{2}|V(G)| - \frac{\lambda_2(G)}{2}l + \frac{l}{2}\lambda_2(G) = \frac{\lambda_2(G)|V(G)|}{2}.$$

(ii). We shall construct a family of graphs satisfying (1) and $|E(H)| = \frac{1}{2}\lambda_2(H)|V(H)|$. We use terminology from Theorem 2.9 in the construction of graph H. Thus H can be contracted to a s-regular graph H' with $|V(H')| = l$ and $|E(H')| = t$. We shall prove that such constructed H satisfies $\lambda_2(H) = s$ and $\lambda_l(H) = t$. It is convenient to give construction separately for even and odd values of s. It is well known that

$$\frac{sl}{2} = \frac{s|V(H')|}{2} = |E(H')| = t. \qquad (6)$$

For even s let H' be the graph $C_l[1(s/2)]$, and so H is a $CC_l[1(s/2)]$. Figure 4.1 gives the graph $CC_5[1(3)] = CC_5[1(6/2)]$, that is $s = 6$ and $l = 5$. When s is odd l must be even. In this case we let H' be the graph $C_l[1(\frac{s-1}{2}), \frac{l}{2}(1)]$ and so H is in $CC_l[1(\frac{s-1}{2}), \frac{l}{2}(1)]$. Figure 4.2 gives the graph $CC_6[1(2), 3(1)] = CC_6[1(\frac{5-1}{2}), \frac{6}{2}(1)]$, that is $s = 5$ and $l = 6$. In both Figure 4.1 and Figure 4.2 the structure of the components were not given. Note that in both cases, we have

$$\lambda_2(H') = s = \delta(H') = \Delta(H'). \qquad (7)$$

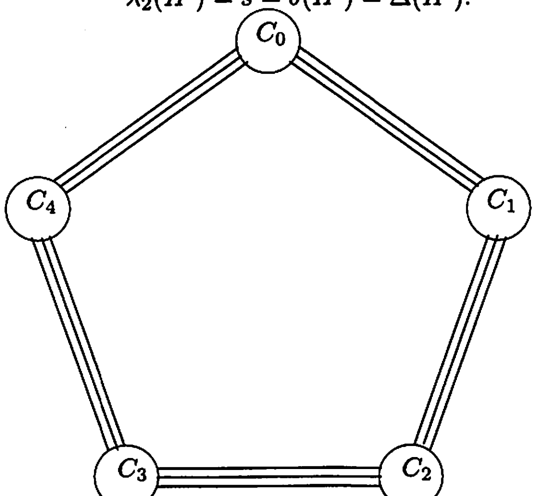

Figure 4.1 Circulant component graph $CC_5[1(3)]$

Note also that $E(H') = T$ and $H - T$ has l components. Below we shall define the structure of each component. The edges joining two components are in T, thus $H - T$ gives l components $C_0, C_1, \ldots, C_{l-1}$. Let C be a component of $H - T$. A vertex v of C is **internal** if v is not incident with any edge of T; otherwise, v will be **external**. Let C_i and C_j be two distinct components of $H - T$, for $0 \le i \ne j \le l - 1$. If there is an edge joining a vertex v of C_i and a vertex u of C_j then this edge e must be in T. Further more C_i and C_j are called **adjacent components**. The edge e is called an **external edge** of C_i and C_j.

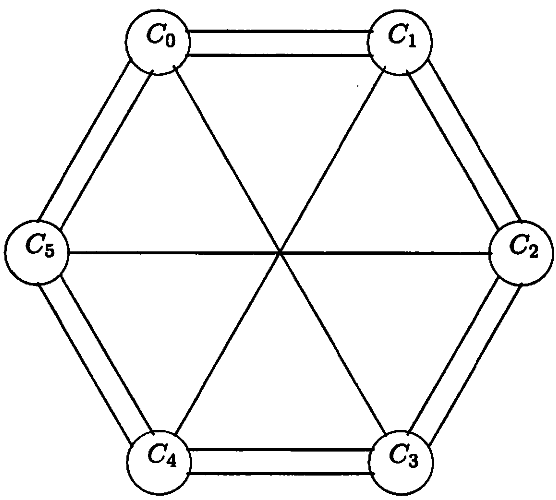

Figure 4.2 Circulant component graph $CC_6[1(2), 3(1)]$

Note that T is an edge subset of $E(H)$, such that $H - T$ has l components. Let $X = E(H) - T$, and $H' = H/X$. Thus the vertices of H' are components of $H - T$. Also note that the elements of T are edges of H'. Label the l components of $H - T$ by $C_0, C_1, \ldots, C_{l-1}$. Now we look at the structure of these components. By Lemma 2.8, if a component C_i has more than one vertex then the number of vertices in C_i is at least s. Recall that we want to construct graphs with edge-connectivity equals to s. Instead of constructing l components $C_0, C_1, \ldots, C_{l-1}$, we just construct one such component (say C) and give several different cases. The components of $H - T$ can be any combination of these components provided the components C_i and C_{i+1} both cannot be trivial components at the same time for $0 \le i \le l$, where component $C_l = C_0$. Let $|V(C)| = n'$. By Lemma 2.8, if $n' > 1$ then $n' \ge s$. Thus we break the construction of C into five cases depending on the values of n' and s. They are $n' = 1$, $n' = s$, $n' > s$ for even s, $n' > s$ for odd s and even n', and $n' > s$ for odd s and odd n'. For all the cases and sub cases label the n' vertices $v_1, v_2, \ldots, v_s, v_{s+1}, \ldots, v_{n'}$ so that s edges of T are incident with vertices $v_1, v_2, \ldots, v_s$, respectively, when $n' \geq s$ in C. Thus these s vertices are the external vertices of C. All the other vertices are internal.

Case (1): If $n' = 1$ then the component C is a single vertex. Thus s edges of T are all incident with this vertex. This is a trivial component of $H - T$.

Case (2): If $n' = s$ then let the component C be the complete graph K_s. In this case, each vertex in C is incident with exactly one edge in T. Thus all the s vertices are external with $\deg_H v_i = s$ for $1 \leq i \leq s$.
Case (3): If $n' > s$ and s is even. Let the component C be

$$C_{n'}[1, 2, \ldots, \frac{s}{2}] - \{v_1v_2, v_3v_4, \ldots, v_{s-1}v_s\}.$$

There are s external vertices and $n' - s$ internal vertices. $\deg_H v_i = s$ for $1 \leq i \leq n'$. The graph $C_7[1, 2, 3] - \{v_1v_2, v_3v_4, v_5v_6\}$ is shown in the Figure 4.3.

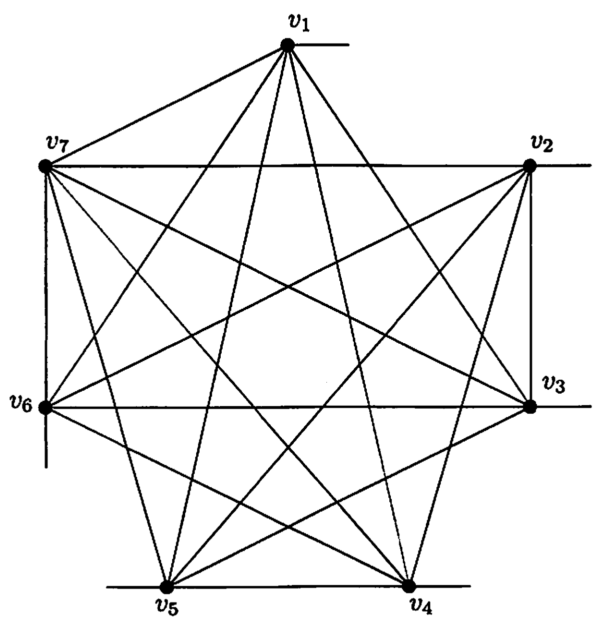

Figure 4.3 The graph $C_7[1, 2, 3] - \{v_1v_2, v_3v_4, v_5v_6\}$

Case (4): If $n' > s$, s is odd and n' is even. Let the component C be

$$C_{n'}[1, 2, \ldots, \frac{s-1}{2}, \frac{n'}{2}] - \{v_2v_3, v_4v_5, \ldots, v_{s-1}v_s\}.$$

There are s external vertices and $n' - s$ internal vertices, $\deg_H v_1 = s + 1$ and $\deg_H v_i = s$ for all $2 \leq i \leq n'$. The graph $C_6[1, 2, 3] - \{v_2v_3, v_4v_5\}$ is shown in the Figure 4.4.

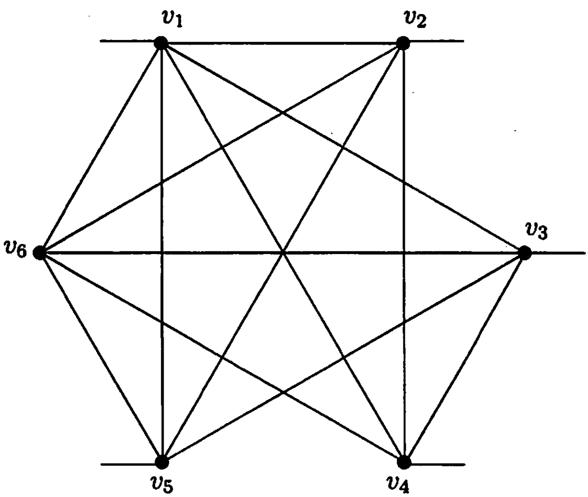

Figure 4.4 The graph $C_6[1, 2, 3] - \{v_2v_3, v_4v_5\}$

Case (5): If $n' > s$, and s and n' are both odd. This we break into two sub cases as $n' < 2s$ and $n' > 2s$.

Subcase(5a): $s < n' < 2s$ for odd s and n'. Let the component C be

$$\begin{aligned} C_{n'}[1, 2, \ldots, \frac{s-1}{2}] &+ \{v_{s+1}v_{r+1}, v_{s+2}v_{r+2}, \ldots, v_{n'}v_{r+n'-s}\} \\ &- \{v_{r+1}v_{r+2}, v_{r+3}v_{r+4}, \ldots, v_{r+n'-s-1}v_{r+n'-s}\}, \end{aligned}$$

where $r = (\frac{3s+1}{2}) \pmod{n'}$. There are s external vertices and $n' - s$ internal vertices and $\deg_H v_i = s$ for $1 \leq i \leq n'$. The graph $C_7[1, 2] + \{v_2v_6, v_3v_7\} - \{v_2v_3\}$ is shown in Figure 4.5.

Subcase(5b): $n' > 2s$ for odd s and n'. Let the component C be

$$C_{n'}[1, 2, \ldots, \frac{s-1}{2}] + \{v_{r+s+1}v_{s+1}, v_{r+s+2}v_{s+2}, \ldots, v_{n'}v_{r+s}\},$$

where $r = \frac{n'-s}{2}$. There are s external vertices and $n' - s$ internal vertices. $\deg_H v_i = s$ for $1 \le i \le n'$. The graph $C_{11}[1,2] + \{v_6v_9, v_7v_{10}\} - \{v_8v_{11}\}$ is shown in Figure 4.6.

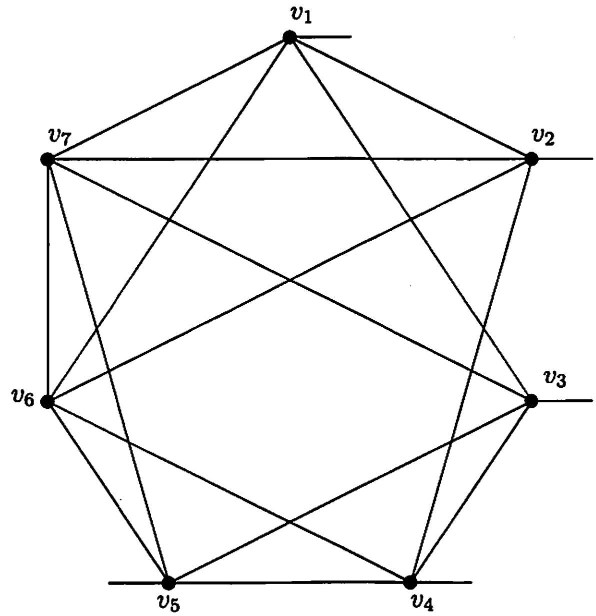

Figure 4.5 The graph $C_7[1,2] + \{v_2v_6, v_3v_7\} - \{v_2v_3\}$

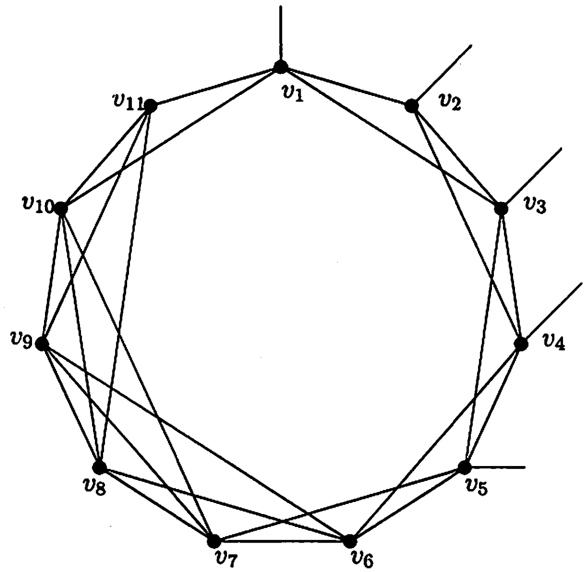

Figure 4.6 The graph $C_{11}[1,2] + \{v_6v_9, v_7v_{10}, v_8v_{11}\}$

For each component C of $H - T$. The component C has either one or s external vertices. When it has s external vertices, we assume that these vertices are labelled as $v_1, v_2, \cdots, v_s$, adjacent to external edges $e_1, e_2, \cdots, e_s$, respectively. Let v denote a vertex not in C and let $C \cup v$ denote the graph obtained from C by adding a new vertex v and edges $e_1, e_2, \cdots, e_s$ such that each e_i joins v to v_i in C.

Claim 4.2 For any component C in the above construction, $\lambda_2(C \cup v) = s$.

Proof of Claim 4.2 Let X be an edge-cut of $C \cup v$ such that $\lambda_2(C \cup v) = |X|$. If the edge cut X separate the vertex v and the component C then $|X| = s$. In following cases we assume that the edge cut X does not separate the vertex v and component C.

Case(1): $n' = 1$. Thus the graph $C \cup v$ has only two vertices and s edges between them. Thus the edge cut X separates the vertex v and the component C. Therefore $|X| = s$ and $\lambda_2(C \cup v) = s$.

Case(2): $n' = s$. Thus the graph $\lambda_2(C \cup v)$ is the complete graph K_{s+1}.

Therefore $|X| = s$ and $\lambda_2(C \cup v) = s$.

Case(3): $n' > s$. for even s. In this case, the component $C = C_{n'}[1, 2, \ldots, \frac{s}{2}] - \{v_1v_2, v_3v_4, \ldots, v_{s-1}v_s\}$. Note that $\lambda_2(C_{n'}[1, 2, \ldots, \frac{s}{2}]) = s$. As C is obtained by removing the edges $\{v_1v_2, v_3v_4, \ldots, v_{s-1}v_s\}$, and joining $v_1, v_2, \cdots, v_s$ to the new vertex v, we still have $\lambda_2(C \cup v) = s$.

Case(4): $n' > s$ for odd s and even n'. In this case, the component $C = C_{n'}[1, 2, \ldots, \frac{s-1}{2}, \frac{n'}{2}] - \{v_2v_3, v_4v_5, \ldots, v_{s-1}v_s\}$. It is routine to check that $\lambda_2(C_{n'}[1, 2, \ldots, \frac{s-1}{2}, \frac{n'}{2}]) = s$, and so $\lambda_2(C \cup v) = s$.

Subcase(5a): $s < n' < 2s$ for odd s and n'. In this case the component $C = C_{n'}[1, 2, \ldots, \frac{s-1}{2}] + \{v_{t+s+1}v_{s+1}, v_{t+s+2}v_{s+2}, \ldots, v_{n'}v_{t+s}\}$. It is routine to check that $\lambda_2(C_{n'}[1, 2, \ldots, \frac{s-1}{2}]) = s$. We again have $\lambda_2(C \cup v) = s$.

Subcase(5b): $n' > 2s$ for odd s and n'. In this case, the component $C = C_{n'}[1, 2, \ldots, \frac{s-1}{2}] + \{v_{t+s+1}v_{s+1}, v_{t+s+2}v_{s+2}, \ldots, v_{n'}v_{t+s}\}$, where $t = \frac{n'-s}{2}$. It is routine to check that $\lambda_2(C_{n'}[1, 2, \ldots, \frac{s-1}{2}]) = s$. Thus $\lambda_2(C \cup v) = s$. This proves Claim 4.2. □

To complete the proof for Theorem 4.1(ii), it remains to prove that $\lambda_2(H) = s$, $\lambda_l(H) = t$ and that

$$|E(H)| = \frac{1}{2}\lambda_2(H)|V(H)|. \tag{8}$$

Let X_i denote the set of all edges with exactly one end in a given component C_i for any $1 \le i \le l$, then $H - X_i$ has two components. By the construction, $|X_i| = s$, and so $\lambda_2(H) \le s$. On the other hand, we argue by contradiction and assume that there exists a minimal edge cut $E' \subseteq E(H)$ such that $H - E'$ has two components and $|E'| < s$. Suppose first that $E' \cap T = \emptyset$, and so we may assume that for some nontrivial component C_i of $H - T$, $E' \cap E(C_i) \neq \emptyset$. Since E' is minimal, $E' \cap E(C_i)$ must be an edge cut of C_i, and so E' contains an edge-cut of $C_i \cup v$. By Claim 4.2, $|E'| \ge s$, contrary to the assumption that $|E'| < s$. Hence we must have $E' \subseteq T$, and so E' is an edge-cut of H'. By (7), $|E'| \ge s$, contrary the the assumption that $|E'| < s$ again. Therefore, we must have $\lambda_2(H) = s$.

By Theorem 2.9, by $\lambda_2(H) = s$ and by (6), we have $\lambda_l(H) \ge \lceil \frac{ls}{2} \rceil = t$. Recall that $|T| = |E(H')| = t$ and that $H - T$ has l components. Therefore,

$\lambda_l(H) \leq |T| = t$. It follows that $\lambda_l(H) = t$.

We argue by induction on $f(H) = \sum_{i=1}^{l} |V(C_1)|$ to prove (8), where $C_1, C_2, \cdots, C_l$ are the components of $H - T$. If $f(H) = l$, then $H = H'$, and so (8) holds. Assume that $f(H) > l$. Then at least one of the components, say C, has at least s vertices. Using the same notation as in the construction above, we let $n' = |V(C)|$. Consider the graph H/C. Then H/C can be constructed in the procedure above via the case $n' = 1$ instead of the $n' \geq s$ cases. Hence by induction with $\lambda_2(H) = s$, and by $E(H) - E(C) = E(G/C)$,

$$|E(H)| - |E(C)| = \frac{\lambda_2(H)|V(H/C)|}{2} = \frac{\lambda_2(H)(|V(H)| - |V(C)| + 1)}{2}. \quad (9)$$

It is routine to check that in the construction procedure above, in each of the cases when $n' \geq s$, we always have

$$|E(C)| = \frac{\lambda_2(H)(|V(C)| - 1)}{2}. \quad (10)$$

Thus combining (9) and (10), we obtain (8). This complete the proof of Theorem 4.1. □

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