Graceful Labelings of Directed Graphs

Alison M. Marr
Department of Mathematics and Computer Science
Southwestern University, Georgetown, TX 78626
marra@southwestern.edu

Abstract

A graceful labeling of a directed graph D with e edges is a one-to-one map $\theta : V(D) \to \{0, 1, \ldots, e\}$ such that $\theta(y) - \theta(x) \bmod (e+1)$ is distinct for each $(x, y) \in E(D)$. This paper summarizes previously known results on graceful directed graphs and presents some new results on directed paths, stars, wheels, and umbrellas.

1 Introduction

One of the most popular problems in graph labeling is the Graceful Tree Conjecture. Since its introduction in the late 60's, hundreds of papers have been written exploring graceful labelings. Gallian keeps an up-to-date survey of the work that has been done with graceful and other labelings [4]. Almost all of the labelings mentioned in Gallian's survey focus on undirected graphs. However, in a series of papers in the early 1980's Bloom and Hsu defined graceful labelings for directed graphs and opened the door for a whole new route of study for labeling problems [1],[2],[3]. Since their original papers on graceful labelings, no other work on labelings of directed graphs has been published.

This paper will summarize the previously known results on graceful labelings of directed graphs and present some new results. Section 2 will review the previously known results. Section 3 will discuss a new result for directed paths and stars. Section 4 will explore new ideas for directed wheels and umbrellas. Section 5 will give directions for future research.

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2 Properties

Definition 2.1. *A graceful labeling on a directed graph D with e edges is a one-to-one map $\theta : V(D) \to \{0, 1, \ldots, e\}$ such that $\theta(y) - \theta(x) \bmod (e+1)$ is distinct for every $(x, y) \in E(D)$.*

Figure 1 shows an example of a graceful labeling on the unidirectional P_4.

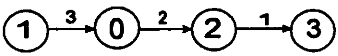

Figure 1: Graceful labeling of P_4

It is useful to note that if θ is a graceful labeling of a digraph D then so is ϕ where $\phi(x) = \theta(x) + k \bmod (e+1)$ where $k \in \{1, 2, \ldots, e\}$. This will be useful when proving graceful labelings exist for certain types of graphs.

Definition 2.2. *Two graceful labelings θ and ϕ on a digraph D with e edges are equivalent if there exists a $k \in \{1, 2, \ldots, e\}$ such that $\theta(x) = \phi(x) + k \bmod (e+1)$ for all $x \in V(D)$.*

Lemma 2.1. *Given any graceful labeling θ of a digraph D and any specified vertex v of D there exists an equivalent graceful labeling ϕ such that $\phi(v) = k$ for any value $k \in \{0, 1, \ldots, e+1\}$.*

Table 1 summarizes the previously known results on graceful directed graphs [1]. The letter G is used to signify a class of digraphs is graceful.

3 Trees

As shown in Table 1, the only trees that were previously studied were the unidirectional paths, $\overrightarrow{P}_n$. In [2], a labeling for $\overrightarrow{P}_n$ for n even is given by $\theta(a_i) = (-1)^{i+1}\lfloor \frac{n}{2} \rfloor \pmod n$ for consecutive vertices of the path $a_1, a_2, \ldots, a_n$. This labeling can actually be used to produce a labeling for the alternating path P_n for n odd. Suppose θ is the previously mentioned labeling for some unidirectional P_n with n even. Then, the alternating path P_{n+1} can be labeled with ϕ by letting $\phi(a_1) = n$ and $\phi(a_i) = \theta(a_{i-1})$ for all $i = 2, \ldots, n+1$. An example of this construction is given in Figure 2.

Another class of digraphs that was not previously studied is the various orientations of stars, $K_{1,n}$. Although we know undirected stars $K_{1,n}$ are graceful for every n, the following classifies all orientations of any given star.

Graph	Known Results
$\overleftrightarrow{K_n}$	G $\Leftrightarrow$ $\exists$ cyclic (v,k,λ)-difference set with $v = n^2 - n + 1,\ k = n,\ \lambda = 1$
$\overrightarrow{P_n}$	G $\Leftrightarrow$ n is even $\Leftrightarrow$ Z_n is sequenceable
$\cup_{i=1}^{t}\overrightarrow{C_i}$ Union of t identical unicycles on n vertices	G if $t = 1$ and n even, $t = 2$ or $n = 2$ or 6 not G if tn is odd
$(\cup_{i=1}^{r}\overrightarrow{C_{k_i}}) \cup (\cup_{j=1}^{s}\overrightarrow{P_{h_j}})$	G if $s = 1$ and e is odd
$\overrightarrow{D + K_m^C}$ Digraph formed by directing an edge from each of m isolated vertices to each of the vertices of D	G $\forall$ graceful D with n vertices and $n - 1$ edges
$(t)\overrightarrow{C_3}$ Unicyclic windmill with t vanes	G $\Leftrightarrow$ t even
$\overrightarrow{W_n}$ Unicyclic outspoken wheel on n vertices	conjectured G for all n shown true for some special cases of n
$\overrightarrow{U_n}$ Unicyclic outspoken umbrella on n vertices	G if n even

Table 1: Summary of previously known results

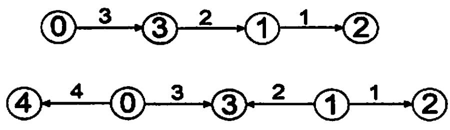

Figure 2: Constructing a labeling of alternating P_5 from unidirectional P_4

Theorem 3.1. *A directed $K_{1,n}$ with center vertex v has a graceful labeling iff*

1. *n is odd or*

2. *n is even and $d_{in}(v), d_{out}(v) \in 2\mathbb{Z}$*

Proof. By Lemma 2.1, if a labeling of $K_{1,n}$ exists, 0 can be the center label. This leaves the numbers $1, 2, \ldots, n$ to label the remaining pendant vertices $v_1, \ldots v_n$. Since 0 is the center label, any inward edge $(v_i, 0)$ will have label $-v_i \pmod{n+1}$ and any outward edge $(0, v_j)$ will have label v_j. Thus, a labeling will exist iff each element a and it's additive inverse mod $n+1$ are placed on edges facing the same direction with respect to the center vertex. If n is odd, there exists an element whose inverse is itself. Thus, since either the number of edges directed in or the number of edges directed out must be odd, we can place this label on the extra odd edge and match up the other pairs so that their edges are in the same direction. But, if n is even, every element has an inverse that is not itself. Hence, in order for a labeling to exist in this case, $d_{in}(v)$ and $d_{out}(v)$ must both be even as each inverse pair needs to be placed on edges with the same orientation. □

4 Wheels and Umbrellas

Definition 4.1. *A unicyclic outspoken wheel $\overrightarrow{W}_n$ is a wheel whose outer cycle is directed in one direction and whose spokes are directed outward.*

Definition 4.2. *A unicyclic outspoken umbrella $\overrightarrow{U}_n$ is a unicyclic outspoken wheel $\overrightarrow{W}_n$ with an extra edge directed from the center vertex to a new vertex.*

Figure 3 shows examples of these two directed graphs.

The following conjecture was presented in [1].

Conjecture 4.1. *All unicyclic wheels are graceful.*

In [1] it was shown that all unicyclic wheels W_n with $n \leq 11$ and $n \neq 6, 10$ are graceful.

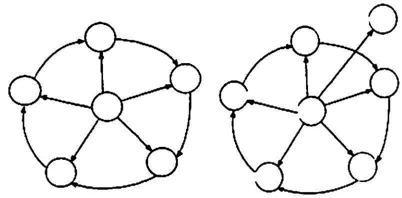

Figure 3: Examples of unicylcic outspoken W_5 and U_5

Theorem 4.1. *The unicyclic outspoken wheel $\overrightarrow{W}_n$ has a graceful labeling iff there exists a sequence $(a_1, a_2, \dots a_n)$ such that $a_i \in [2n]$ for each i, and $\{a_i, a_{i+1} - a_i \text{ mod } (2n+1),\ 1 \leq i \leq n, \text{ where } a_{n+1} = a_1\} = [2n]$.*

Proof. First, suppose such a sequence exists. Place 0 as the label on the center vertex by Lemma 2.1. Then the a_i elements will be the labels on the cycle in the given order. Then, since 0 is the center label, the spokes will have the labels a_i and the edges of the cycle will have the labels $a_{i+1} - a_i$ mod $(2n+1)$. And, by the properties of the sequence we have a graceful labeling of $\overrightarrow{W}_n$. Necessity holds because if a labeling of $\overrightarrow{W}_n$ exists, there is one with 0 as the center vertex label. Then, the sequence given would be required in order for a graceful labeling to exist.

□

Also, it has been shown that all unicyclic umbrellas $\overrightarrow{U}_n$ are graceful when n is even. However, the question was posed as to whether these are the only graceful unicyclic umbrellas. Once again, there is a nice representation for a graceful labeling of a unicyclic umbrella.

Theorem 4.2. *The unicyclic outspoken umbrella $\overrightarrow{U}_n$ has a graceful labeling iff there exists a sequence $(a_1, a_2, \dots a_n)$ such that $a_i \in [2n+1]$ for all i and $\{a_i, a_{i+1} - a_i \text{ mod } (2n+2),\ 1 \leq i \leq n \text{ where } a_{n+1} = a_1\} = [2n+1] \backslash \{h\}$ for some $h \in [2n+1]$.*

For each wheel (or umbrella) every possible labeling (sequence) was generated, and a computer program was used to check if each sequence satisfied the conditions stated in the theorems. Table 2 shows the number of non-equivalent graceful labelings for unicyclic outspoken wheels and umbrellas for $n \leq 10$.

Hence, we now know graceful labelings exist for $\overrightarrow{W}_6$ and $\overrightarrow{W}_{10}$. In addition, we were able to computationally show that labelings of $\overrightarrow{U}_n$ exist for values of n that are both even and odd. But, we are still not able to prove the case for n odd. However, this does lead to the new conjecture.

Conjecture 4.2. *All unicyclic umbrellas are graceful.*

n	Wheels	Umbrellas
2	2	5
3	2	8
4	6	34
5	24	76
6	64	542
7	88	1552
8	940	9,192
9	4,728	60,728
10	34,644	427,468

Table 2: Number of non-equivalent graceful unicyclic W_n and U_n

While looking at the list of labelings for $\overrightarrow{W}_n$ for even n a pattern was discovered. It seems that for even $n \geq 10$ there is a labeling of the form $(1, 3, a_3, \ldots, a_{n/2-1}, n+1, 2n, 2n-2, a_{n/2+3}, \ldots, a_{n-1}, n)$. These sequences have many interesting properties. First, they are symmetric. The first $n/2$ numbers are just the additive inverses of the second $n/2$ elements mod $(2n + 1)$ in the same order. Hence, the differences between consecutive elements are also symmetric. Second, the sum of the first $n/2$ differences is always equal to $2n - 1$. Third, if we represent the first $n/2$ elements and their corresponding differences as elements from $\{-n, \ldots, -1, 1, 2, \ldots, n\}$ and then take their absolute values, we get the set $[n]$.

After running a program to check if such sequences exist for all even n, we find they do exist for even n with $10 \leq n \leq 20$. Table 3 shows the number of labelings of this type for each even $n \leq 18$.

n	Symmetric Wheels
2	0
4	0
6	1
8	0
10	1
12	4
14	14
16	95
18	898

Table 3: Number of non-equivalent symmetric wheels

5 Future work

Although the work presented here is moving this area of research forward, there is still much more to do. Hopefully, there will be a proof that all $\overrightarrow{W}_{2n}$ are graceful using the information already gathered. In addition, it would be nice to prove that all unicyclic wheels and umbrellas are indeed graceful.

Also, there are many classes of trees that can be examined to find orientations that are graceful. These are just a couple of ideas for future work, but there are many more directions to explore in the area of directed graceful graphs.

References

[1] G.S. Bloom, D.F. Hsu. On graceful directed graphs. *SIAM J. Algebraic Discrete Methods* **6** (1985), no. 3, 519–536.

[2] G.S. Bloom, D.F. Hsu. On graceful digraphs and a problem in network addressing. *Congr. Numer.* **35** (1982), 91–103.

[3] G.S. Bloom, D.F. Hsu. On graceful directed graphs that are computational models of some algebraic systems. *Graph theory with applications to algorithms and computer science (Kalamazoo, Mich., 1984)*, 89–102, Wiley-Intersci. Publ., *Wiley, New York*, 1985.

[4] J.A. Gallian. A dynamic survey of graph labeling. *Electron. J. Combin.* **14** (2007), Dynamic Survey 6, 180 pp. (electronic).