INTEGER-MAGIC SPECTRA OF TREES OF DIAMETER FIVE

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ABSTRACT. For any $h \in \mathbb{Z}$, a graph $G = (V, E)$ is said to be h-magic if there exists a labeling $l : E(G) \to \mathbb{Z}_h - \{0\}$ such that the induced vertex set labeling $l^+ : V(G) \to \mathbb{Z}_h$ defined by

$$l^+(v) = \sum_{uv \in E(G)} l(uv)$$

is a constant map. For a given graph G, the set of all $h \in \mathbb{Z}_+$ for which G is h-magic is called the integer-magic spectrum of G and is denoted by $IM(G)$. In this paper, we will determine the integer-magic spectra of trees of diameter five.

Key Words: magic, non-magic, integer-magic spectrum.
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1. INTRODUCTION

In this paper all graphs are connected, finite, simple, and undirected. For an abelian group A, written additively, any mapping $l : E(G) \to A - \{0\}$ is called a *labeling*. Given a labeling on the edge set of G one can introduce a vertex set labeling $l^+ : V(G) \to A$ by

$$l^+(v) = \sum_{uv \in E(G)} l(uv).$$

A graph G is said to be *A-magic* if there is a labeling $l : E(G) \to A - \{0\}$ such that for each vertex v, the sum of the labels of the edges incident with v are all equal to the same constant; that is, $l^+(v) = c$ for some fixed $c \in A$. In general, a graph G may admit more than one labeling to become A-magic; for example, if $|A| > 2$ and $l : E(G) \to A - \{0\}$ is a magic labeling of G with sum c, then $\lambda : E(G) \to A - \{0\}$, the *inverse labeling* of l, defined by $\lambda(uv) = -l(uv)$ will provide another magic labeling of G with sum $-c$. A graph $G = (V, E)$ is called *fully magic* if it is A-magic for every abelian group A. For example, every regular graph is fully magic. A graph $G = (V, E)$ is called *non-magic* if for every abelian group A, the graph is not A-magic. The most obvious class of non-magic graphs is P_n $(n \geq 3)$, the path of order n. As a result, any graph with a pendant path of length $n \geq 3$ would be non-magic. Here is another example of a non-magic

graph: Consider the graph H depicted in Figure 1. Given any abelian group A, a potential magic labeling of H is illustrated in that figure. The combination of conditions $l^+(u) = l^+(v) = l^+(w) = x$ imply that $y = z = 0$, which is not an acceptable magic labeling. Thus H is not A-magic.

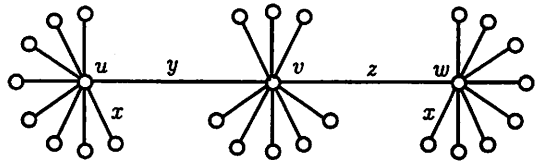

FIGURE 1. An example of a non-magic graph.

Certain classes of non-magic graphs are presented in [1]. The original concept of A-magic graph is due to J. Sedlacek [12, 13], who defined it to be a graph with a real-valued edge labeling such that

(1) distinct edges have distinct nonnegative labels; and
(2) the sum of the labels of the edges incident to a particular vertex is the same for all vertices.

Jenzy and Trenkler [3] proved that a graph G is magic if and only if every edge of G is contained in a (1-2)-factor. $\mathbb{Z}$-magic graphs were considered by Stanley [14, 15], who pointed out that the theory of magic labeling can be put into the more general context of linear homogeneous diophantine equations. Recently, there has been considerable research articles in graph labeling, interested readers are directed to [2, 16]. For convenience, the notation 1-magic will be used to indicate $\mathbb{Z}$-magic and $\mathbb{Z}_h$-magic graphs will be referred to as h-magic graphs. Clearly, if a graph is h-magic, it is not necessarily k-magic ($h \neq k$).

Definition 1.1. *For a given graph G the set of all positive integers h for which G is h-magic is called the* integer-magic spectrum *of G and is denoted by $IM(G)$.*

Since any regular graph is fully magic, then it is h-magic for all positive integers $h \geq 2$; therefore, $IM(G) = \mathbb{N}$. On the other hand, the graph H, Figure 1, is non-magic, hence $IM(H) = \emptyset$. In determining the integer-magic spectra of graphs the following observations will be useful:

Observation 1.2. *If a graph G has an $\mathbb{N}$-magic labeling $l : E(G) \to \mathbb{N}$, then G is k-magic as long as k does not divide $l(e)$ for every $e \in E(G)$.*

Observation 1.3. *If a graph G has a $\mathbb{Z}$-magic labeling $l : E(G) \to \mathbb{Z}$, then G is k-magic as long as k does not divide $l(e)$ for every $e \in E(G)$.*

Proof. In order to construct a k-magic labeling, we start with the $\mathbb{Z}$-magic labeling of G, and replace every edge label $l(e)$ with $l(e) \pmod k$. Since k does not divide any $l(e)$, none of these new labels are 0. □

Observation 1.4. *If G is $\mathbb{Z}$-magic, then G is k-magic for sufficiently large k.*

Proof. If G has a $\mathbb{Z}$-magic labeling ℓ, then G is k-magic as long as $k > \ell(e)$ for every edge e. So G is k-magic for every k larger than $max\{\ell(e)\}$ □

The integer-magic spectra of certain classes of graphs have been studied in [5, 6, 7, 8, 9, 10, 11]. In particular, in [6] the integer-magic spectra of the trees of diameter at most four have been characterized.

2. Trees of diameter at most four

Trees of diameter two are the complete bipartite graphs $K(1,n)$ also called *stars* and is denoted by $ST(n)$. Note that $K(1,1) = P_2$ has diameter 1, and $K(1,2) = P_3$ is non-magic.

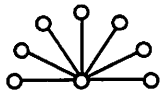

FIGURE 2. A typical tree of diameter 2.

Theorem 2.1. [6] *Let $n \geq 3$, and $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be the prime factorization of $n-1$. Then*

$$IM(\,ST(n)\,) = \bigcup_{i=1}^{k} p_i \mathbb{N}.$$

Trees of diameter 3 are double-stars. These graphs have two central vertices u and v plus leaves. We will use $DS(m,n)$ to denote the double-star whose two central vertices have degrees m and n, respectively. Note that if $m = 2$ or $n = 2$, then $DS(m,n)$ is non-magic. Therefore we will assume that $m \geq n > 2$.

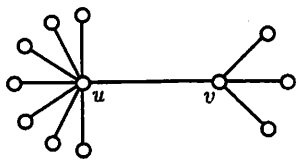

FIGURE 3. A typical tree of diameter 3.

Theorem 2.2. [6] *Let $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ and $p_1^{\beta_1} p_2^{\beta_2} \cdots p_k^{\beta_k}$ be the prime factorizations of $m-n$ and $n-2$, respectively. Then $IM(\,DS(m,n)\,) = \cup_{i=1}^{k} A_i$, where*

$$A_i = \begin{cases} p_i^{1+\beta_i} \mathbb{N} & \text{if } \alpha_i > \beta_i \geq 0; \\ \emptyset & \text{if } \beta_i \geq \alpha_i \geq 0. \end{cases}$$

Trees of diameter four, denoted by $T_4(a_1, a_2, \cdots, a_n)$, consist of n stars $ST(a_1)$, $ST(a_2), \cdots, ST(a_n)$ exactly one of their end vertices identified. The common vertex is the center of the tree and will be denoted by c. Equivalently, $T_4(a_1, a_2, \cdots, a_n)$ is a tree with center-vertex c, in which n edges cu_k $(1 \leq k \leq n)$ are emanated from c, and $\deg(u_i) = a_i$ for each $i = 1, 2, \cdots, n$, see Figure 4. In order to have a tree of diameter four, one needs $n \geq 2$ and $a_i \geq 2$ for at least two values of i.

The following theorem [6], has been an attempt to identify the integer-magic spectra of trees of diameter four:

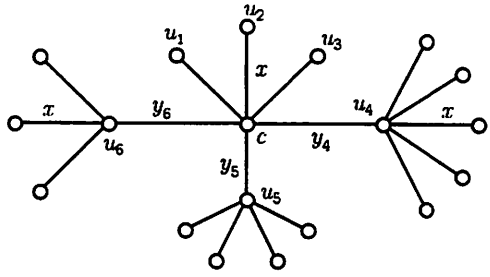

FIGURE 4. A typical tree of diameter 4.

Theorem 2.3. *Consider the tree of diameter four* $G = T_4(a_1, a_2, \cdots, a_n)$ $(n \geq 3)$, *and let* $\pm p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ *be the prime factorization of* $\sigma = 1 - 2n + \sum_{i=1}^{n} a_i$. *Then the integer-magic spectrum of* G *is*

$$IM(G) = \begin{cases} \emptyset & \text{if } \sigma | (a_i - 2) \text{ for some } i > k \\ \mathbb{N} - D & \text{if } \sigma = 0 \\ \bigcup_{i=1}^{k} p_i \mathbb{N} - D & \text{otherwise,} \end{cases}$$

where $D = \{d : d \text{ is a divisor of } a_i - 2 \text{ for some } i \ (1 \leq i \leq n)\}$.

When we apply the above theorem to $G = T_4(5, 7, 8)$, it will produce $IM(G) = 3\mathbb{N} \cup 5\mathbb{N}$. However, it is easy to see that this graph is not 6 or 10-magic. In fact, $IM(G) = 15\mathbb{N}$. The correct version of the theorem is as follows:

Theorem 2.4. *Given a tree of diameter four* $G = T_4(a_1, a_2, \ldots, a_n)$, *let* $\sigma = 1 - 2n + \sum_{i=1}^{n} a_i$ *and let* C *be the set of all divisors of* $a_i - 2$ $\forall i = 1, \cdots, n$. *Then*

$$IM(G) = \begin{cases} \emptyset & \text{if } \sigma \in C; \\ \mathbb{N} - C & \text{if } \sigma = 0; \\ \bigcup_{d \in D} d\mathbb{N} & \text{otherwise,} \end{cases}$$

where D *is the set of all positive divisors* d *of* σ *with the property that* $d \notin C$.

Caterpillar is a tree having the property that the removal of its end-vertices results in a path (the spine). We use $CR(a_1, a_2, \cdots, a_n)$ to denote the caterpillar with a P_n-spine, where the i^{th} vertex of P_n has degree a_i.

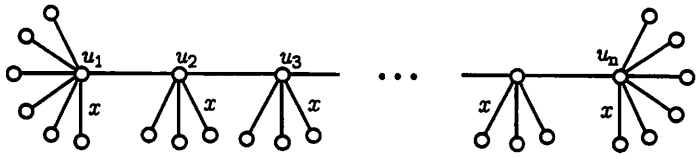

FIGURE 5. A Caterpillar of diameter $n + 1$ (P_n-spine).

The following theorem [10] provides the integer-magic spectra of caterpillars:

Theorem 2.5. *Given a caterpillar* $G = CR(a_1, a_2, \ldots, a_n)$, *let* $\sigma = \dfrac{(-1)^n - 1}{2} - \sum_{i=1}^{n}(-1)^i a_i$ *and* $c_i = \dfrac{3-(-1)^i}{2} - a_i + a_{i-1} - \cdots + (-1)^i a_1$ $(1 \le i \le n-1)$. *Also, let* C *be the set of all divisors of* c_i $\forall i = 1, \cdots, n-1$. *Then*

$$IM(G) = \begin{cases} \emptyset & \text{if } \sigma \in C; \\ \mathbb{N} - C & \text{if } \sigma = 0; \\ \bigcup_{d \in D} d\mathbb{N} & \text{otherwise,} \end{cases}$$

where D *is the set of all positive divisors* d *of* σ *with the property that* $d \notin C$.

3. Trees of Diameter Five

In every tree T of diameter five, there are exactly two adjacent vertices with minimum eccentricity 3. The subgraph induced by these two vertices, also known as center of the T, is isomorphic with P_2. One can utilize this fact to give another characterization for T : A tree of diameter five can be viewed as P_2 with a number of stars one of their leaves identified with exactly one of the vertices of P_2. We will use $T_5(a_1, a_2, \cdots, a_{m-1}; b_1, b_2, \cdots, b_{n-1})$ to denote a tree of diameter five whose central vertices u, v have degrees m and n respectively. Furthermore, there are $m-1$ stars $ST(a_1), ST(a_2), \cdots, ST(a_{m-1})$ with centers $u_1, u_2, \cdots, u_{m-1}$ one of their leaves identified with u, and there are $n-1$ stars $ST(b_1), ST(b_2), \cdots, ST(b_{n-1})$ with centers $v_1, v_2, \cdots, v_{n-1}$ one of their leaves identified with v.

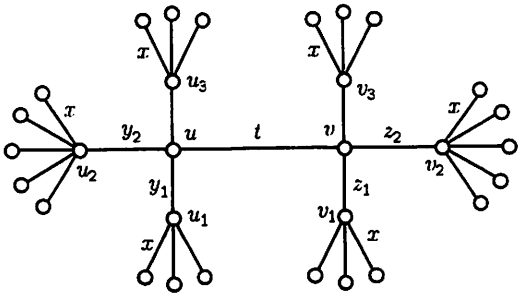

FIGURE 6. A typical tree of diameter 5.

Note that if $m = 1$ or $n = 1$, then the resulting tree will have diameter four. Similarly, if $b_j = 1$ for all $j = 1, \cdots, n-1$, then the resulting tree will have diameter four. Therefore, in what follows, we will assume that $m, n > 1$ and $a_i, b_j > 1$ for at least one value of i and j.

Theorem 3.1. *Given a tree* $G = T_5(a_1, \ldots, a_{m-1}; b_1, \cdots, b_{n-1})$ *of diameter five, let* $\sigma = 2m - 2n + \sum_{i=1}^{n-1} b_i - \sum_{i=1}^{m-1} a_i$ *and let* C *be the set of all divisors of the numbers*

$a_i - 2$ $(i = 1, \cdots, m-1)$, $b_j - 2$ $(j = 1, \cdots, n-1)$, *and* $3 - 2m + \sum_{i=1}^{m-1} a_i$. *Then*

$$IM(G) = \begin{cases} \emptyset & \text{if } \sigma \in C; \\ \mathbb{N} - C & \text{if } \sigma = 0; \\ \bigcup\limits_{d \in D} d\mathbb{N} & \text{otherwise,} \end{cases}$$

where D is the set of all positive divisors d of σ with the property that $d \notin C$.

Proof. Let $l : E(G) \to \mathbb{Z}_h$ be a magic labeling of G and let $l(uu_i) = y_i$, $l(uv) = t$, and $l(vv_j) = z_j$, as illustrated in Figure 6. Note that in any magic labeling of G all the terminal edges have the same label x, which is then equal to the vertex sum. The graph G is h-magic if and only if we can find nonzero elements $t, x, y_i, z_j \in \mathbb{Z}_h$ such that $l^+(u_i) = l^+(v_j) = l^+(u) = l^+(v) = x$. This will provide a homogeneous system of $m+n$ equations with $m+n$ unknowns $(a_i - 2)x + y_i \equiv 0 \pmod{h}$, $(b_i - 2)x + z_i \equiv 0 \pmod{h}$, $(3 - 2m + \sum_{i=1}^{m-1} a_i)x - t \equiv 0 \pmod{h}$, $(3 - 2n + \sum_{i=1}^{n-1} b_i)x - t \equiv 0 \pmod{h}$, which will result in

$$\sigma x \equiv 0 \pmod{h}; \tag{3.1}$$

$$(a_i - 2)x + y_i \equiv 0 \pmod{h}; \tag{3.2}$$

$$(b_i - 2)x + z_i \equiv 0 \pmod{h}; \tag{3.3}$$

$$\left(3 - 2m + \sum_{i=1}^{m-1} a_i\right)x - t \equiv 0 \pmod{h}, \tag{3.4}$$

where $\sigma = 2m - 2n + \sum_{i=1}^{n-1} b_i - \sum_{i=1}^{m-1} a_i$.

We observe that if $\sigma \in C$; that is, σ be a divisor of $(a_i - 2)$, $(b_j - 2)$, or $3 - 2m + \sum_{i=1}^{m-1} a_i$, then $y_i = 0$, $z_j = 0$, or $t = 0$ and the graph would be nonmagic. In particular, if $a_i = 2$ for some i, or $b_j = 2$ for some j, or $\sum_{i=1}^{m-1} a_i = 2m - 3$, then the graph is nonmagic. Assume that $a_i, b_j \neq 2$ and $\sum_{i=1}^{m-1} a_i \neq 2m - 3$.

If $\sigma = 0$, then equation (3.1) is automatically satisfied. Choose $x = 1$ and note that equations (3.2), (3.3), and (3.4) have nonzero solutions if and only if $h \notin C$. Therefore, to avoid the zero solutions we must exclude all the elements of C. In this case, the integer-magic spectrum of G would be $\mathbb{N} - C$.

Finally, suppose $\sigma \neq 0$ and $\sigma \notin C$. We claim that $IM(G) = \bigcup\limits_{d \in D} d\mathbb{N}$, where D is the set of all positive divisors d of σ with the property that $d \notin C$.

Suppose $h \in IM(G)$. Then equation (3.1) has a nonzero solution for x if and only if $\gcd(\sigma, h) = d > 1$, and h/d divides x. Also, $d \notin C$. Because, if $d \in C$, for example $d|(a_i - 2)$ some $i = 1, \cdots, m-1$, then $d(h/d)|(a_i - 2)x$ or $h|y_i$ and $y_i \equiv 0 \pmod{h}$. Therefore, $h = dq$, where $d \in D$.

On the other hand, let $h = dq$ with $d \in D$ and $q \in \mathbb{N}$. Note that $d \in D$ implies that $d > 1$. We choose $x = h/d \not\equiv 0 \pmod{h}$. Since $d \notin C$, then $d \nmid (a_i - 2)$, Therefore,

$d(h/d) \nmid (a_i - 2)x$ or $h \nmid y_i$ and $y_i \not\equiv 0 \pmod{h}$. Similarly, equations (3.3), and (3.4) have nonzero solutions and $h \in IM(G)$. □

Corollary 3.2. *Using the notations of theorem 3.1, G is nonmagic if and only if $\sigma \in C$.*

Examples 3.3.

(a) The graph in Figure 1 is the caterpillar $CR(9,10,9)$. Here, $\sigma = 7$ and $a_1 - 2 = 7$. Therefore, $IM(CR(9,10,9)) = \emptyset$.

(b) For $T_4(5,7,8)$ we have $\sigma = 15$ and $C = \{1,3,5,6\}$. The positive divisors of σ are $1,3,5,15$ and only $15 \notin C$. Therefore, $IM(T_4(5,7,8)) = 15\mathbb{N}$.

(c) $IM(T_4(1,1,3,6)) = \emptyset$. Here, $\sigma = 4$, $a_4 - 2 = 4$ and $\sigma | (a_4 - 2)$.

(d) The graph in Figure 6 is $G = T_5(4,4,6;4,4,6)$. Here, $\sigma = 0$ and $C = \{1,2,4,11\}$. Therefore, $IM(G) = \mathbb{N} - \{1,2,4,11\}$.

(e) For $G = T_5(4,5,6;5,6,7)$ we have $\sigma = 3$ and $C = \{1,2,3,4,5,6,12\}$. Since $\sigma \in C$, then $IM(G) = \emptyset$.

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