Metric Dimension of Enhanced Hypercube Networks

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Abstract

Let $M = \{v_1, v_2 \ldots v_n\}$ be an ordered set of vertices in a graph G. Then $(d(u, v_1), d(u, v_2) \ldots d(u, v_n))$ is called the M-coordinates of a vertex u of G. The set M is called a **metric basis** if the vertices of G have distinct M-coordinates. A **minimum metric basis** is a set M with minimum cardinality. The cardinality of a minimum metric basis of G is called **minimum metric dimension**. This concept has wide applications in motion planning and in the field of robotics. In this paper we provide bounds for minimum metric dimension of certain class of enhanced hypercube networks

Keywords: metric basis, metric dimension, hypercube networks, robotics, image.

AMS Subject Classification: 05C12

1 Introduction

Major issues involved in the design of interconnection networks are quick communication among vertices, high robustness and rich structure in the sense of embeddable properties, fault tolerance and VLSI [17]. Hypercubes are widely studied as they meet several conflicting demands that arise in the design of interconnection networks. The hypercube has many excellent features, thus becomes the first choice for the topological structure of parallel processing and computing systems. The machines based on the hypercube such as the Cosmic Cube from Caltech, the iPSC/2 from Intel and Connection Machines have been implemented commercially [2]. Parallel algorithms based on the hypercube have been developed. The hypercube structure offers a rich interconnection with a large bandwidth and a short (logarithmic) diameter. Most of the hypercube variations focus on reducing diameter or message traffic.

Many variations of hypercube have been suggested to improve its performance. These variations support efficient embeddings, reduced diameter

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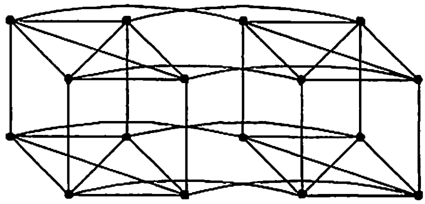

Figure 1: An Enhanced hypercube $Q_{4,\,2}$

and improve fault tolerance in comparison to the hypercube. The common features in the variations of hypercube are 2^n vertices; vertices are labeled with binary strings of length n; they are simple, connected and regular. The extra connections are made in a way that maximizes the improvement of the performance measure of interest under various traffic distributions [2].

2 An Overview of the Paper

Let $M = \{v_1, v_2 \ldots v_n\}$ be an ordered set of vertices in a graph G. Then $(d(u, v_1), d(u, v_2) \ldots d(u, v_n))$ is called the M-coordinates of a vertex u of G. The set M is called a **metric basis** if the vertices of G have distinct M-coordinates. A **minimum metric basis** is a set M with minimum cardinality [6]. If M is a metric basis then it is clear that for each pair of vertices u and v of $V \setminus W$, there is a vertex $w \in W$ such that $d(u, w) \neq d(v, w)$. The cardinality of a minimum metric basis of G is called **minimum metric dimension** and is denoted by $\beta(G)$; the members of a metric basis are called **landmarks** [8]. The **minimum metric dimension** (MMD) problem is to find a minimum metric basis.

This problem has application in the field of robotics. A robot is a mechanical device which is made to move in space with obstructions around. It has neither the concept of direction nor that of visibility. But it is assumed that it can sense the distances to a set of landmarks. Evidently, if the robot knows its distances to a sufficiently large set of landmarks its position in space is uniquely determined.

The concept of metric basis and minimum metric basis has appeared in the literature under a different name as early as 1975. Slater in [15] and later in [16] had called metric basis and minimum metric basis as **locating sets** and **reference sets** respectively. Slater called the cardinality of a reference set as the location number of G. He described the usefulness of these ideas when working with sonar and loran stations. Chartrand et al. [4] have called a metric basis and a minimum metric basis as a **resolving set** and **minimum resolving set**. We adopt the terminology of Harary and Melter.

If G has p vertices then it is clear that $1 \leq \beta(G) \leq p-1$. Harary et al. [6] have shown that for the complete graph K_p, the cycle C_p and the complete bipartite graph $K_{m,\,n}$, the minimum metric dimensions are $\beta(K_p) = p - 1$, $\beta(C_p) = 2$ and $\beta(K_{m,\,n}) = m + n - 2$. This problem has been studied for grids [9], trees, multi-dimensional grids [8], Petersen graphs [1], De Bruijn graphs [13], Kautz networks [14], Torus networks [10], Benes and Butterfly networks [11] and Honeycomb networks [12].

Garey and Johnson [5] proved that the minimum metric dimension problem is NP-complete for general graphs by a reduction from 3-dimensional matching. Recently Manuel et al. [11] have proved that the minimum metric dimension problem is NP-complete for bipartite graphs by a reduction from 3-SAT, thus narrowing down the gap between the polynomial classes and NP-complete classes of the minimum metric dimension problem.

In this paper, we discuss the minimum metric dimension problem for n-dimensional enhanced hypercubes $Q_{n,\,2}$.

3 Topological Properties of Enhanced Hypercube Networks

Several operations exist to combine two copies G_1, G_2 of a graph G. One such operation yields the permutation graph $P(G, \alpha)$, where α is a permutation of $V(G)$. It is obtained from G_1 and G_2 by joining vertex v of G_1 with the vertex $\alpha(v)$ of G_2. Many important classes of permutation graphs can be constructed by judiciously choosing G and α. A prime example is the n-dimensional hypercube $Q_n = P(Q_{n-1}, \alpha)$, where α is the identity permutation of $V(Q_{n-1})$.

The main consideration here for defining hypercube is that the degree of a node in G rises as a slow-growing function of $|V(G)|$. Inversely, we want that $|V(G)|$ be a fast growing function of the degree of a node in G. Constructing different hypercube-like topologies, as well as other inspirations, holds out the promise of finding a simply-implementable network with low diameter and high connectivity.

The chief property of the hypercube network is the efficient interconnection of nodes. The n-cube is particularly compact. The worst case distance between any two nodes is only the dimension of the structure. This logical structure is extremely useful because of the wide range of algorithms that fit it particularly well.

Let Q_n denote the graph of the n-dimensional hypercube, $n \geq 1$. The vertex set $V(Q_n) = \{(x_0 x_1 \ldots x_{n-1}) : x_i = 0 \text{ or } 1\}$. Two vertices $(x_0 x_1 \ldots x_{n-1})$ and $(y_0 y_1 \ldots y_{n-1})$ are adjacent if and only if they differ exactly in one position.

The definition of Q_n recursively in terms of the cartesian product is as follows:

$Q_1 = K_2$, $Q_n = Q_{n-1} \times Q_1 = K_2 \times K_2 \times \ldots \times K_2$ of n identical complete graph K_2.

A hypercube of order n is n-regular, bipartite, with 2^n vertices, $n2^{n-1}$ edges and diameter n. The graph is hamiltonian if $n \geq 2$ and eulerian if n is even.

The **enhanced hypercube** $Q_{n,k}$, $0 \leq k \leq n-1$, is a graph with vertex set $V(Q_{n,k}) = V(Q_n)$ and edge set $E(Q_{n,k}) = E(Q_n) \cup \{(x_0x_1x_2 \ldots x_{k-2}x_{k-1}x_k \ldots x_{n-1}, x_0x_1x_2 \ldots x_{k-2}\overline{x}_{k-1}\overline{x}_k \ldots \overline{x}_{n-1})\}$. The edges of Q_n in $Q_{n,k}$ are **hypercube edges** and the remaining edges of $Q_{n,k}$ are called **complementary edges**. See Figure 1. The set is empty when $k = 0$. Hence $Q_{n,0}$ reduces to the n-dimensional hypercube.

The enhanced hypercubes $Q_{n,k}$, $0 \leq k \leq n-1$, proposed by Tzeng and Wei [17] are $(n+1)$-regular. They have 2^n vertices and $(n+1)2^{n-1}$ edges.

4 Minimum Metric Dimension of Enhanced Hypercube Networks $Q_{n,2}$

In this section, we provide an upper bound for the minimum metric dimension of the enhanced hypercubes $Q_{n,2}$. We begin with a few observations and definitions. There are four copies of $Q_{n-2,2}$ in $Q_{n,2}$. We denote these $(n-2)$-dimensional enhanced hypercubes as A, A_1^1, A_1^2 and A_2. Clearly $A \cup A_1^1$ and $A \cup A_1^2$ are isomorphic to $Q_{n-1,2}$. Figure 2 shows the four copies of $Q_{3,2}$ in $Q_{5,2}$.

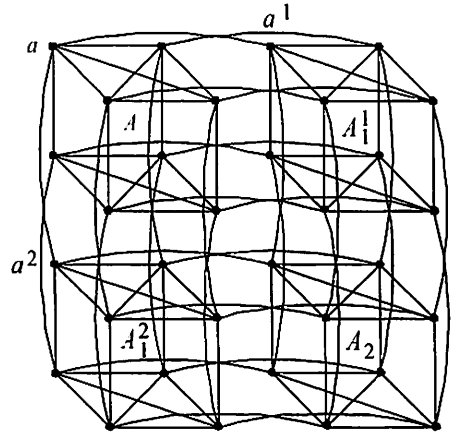

Figure 2: Four copies of $Q_{3,2}$ in $Q_{5,2}$

Let $x \in A$. A vertex y belonging to A_1^1 or A_1^2 is called an **image** of x if

$d(x,y) = 1$. For example, the images of the vertex a in A are a^1 in A_1^1 and a^2 in A_1^2. See Figure 2. Observe that vertices in A, at distance 1 from a, are not considered as images of a.

If $P = x_0x_1 \ldots x_n$ is a path in A then the path $P^1 = x_0^1x_1^1 \ldots x_n^1$ where x_i^1 is the image of x_i in A_1^1 is called the **image** of P in A_1^1; and P is called the **preimage** of P^1. See Figure 3.

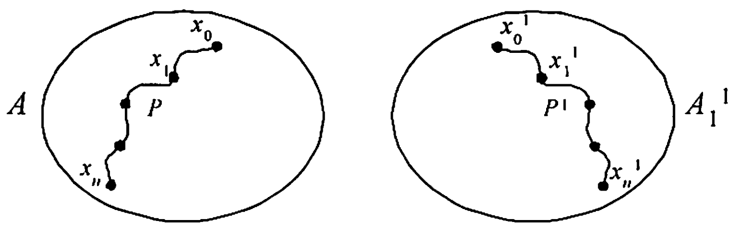

Figure 3: Images of paths

Lemma 1 Let $x \in A_1^1$ and $x^1 \in A$ be the preimage of x. Let w be any vertex of A. Then $d(x,w) = 1 + d(x^1,w)$.

Proof. Let $d(x^1,w) = s$. Let P be a shortest path from x to w, not passing through x^1. Then $P = P^1 \circ (y, y^1) \circ P^2$ where P^1 is the shortest path from x to some $y \in A_1^1$(or A_2) and P^2 is a shortest path from y^1 to w, where $y^1 \in A$(or A_1^2). The possible cases are depicted in Figures 4 and 5. Let Q^1 be the preimage of P^1 in $A \cup A_1^2$. Then the length of the path Q^1 is equal to the length of the path P^1.

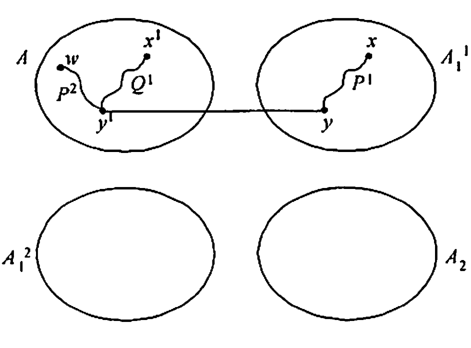

Figure 4: Case $y \in A_1^1$and $y^1 \in A$

We claim that $(V(P^2) \cap V(Q^1)) \setminus \{y^1\} = \phi$. Suppose not; let $z \in V(P^2) \cap V(Q^1) \setminus \{y^1\}$. Since $z \in V(Q^1)$ and $z \neq y^1$, $d(x^1,z) < d(x^1,y^1)$. Similarly, since $z \in V(P^2)$ and $z \neq y^1$, $d(w,z) < d(w,y^1)$. Hence the image of the (x^1,z)-section of Q^1 lying in A_1^1, followed by the edge (z^1,z) and the (z,w)-section of P^2 is a shorter (x,w)-path, a contradiction.

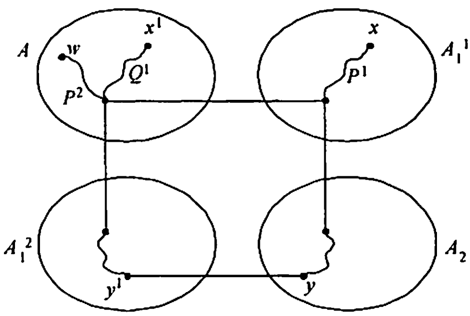

Figure 5: Case $y \in A_2$ and $y^1 \in A_1^2$

Now, if the length of the path $Q^1 \circ P^2$ is less than s, then $d(x^1,w) < s$ which is not possible. Hence the length of $Q^1 \circ P^2$ is at least s and consequently the distance between x and w is at least $s+1$. Since there is an (x,w)-path of length $s+1$, $d(x,w) = s+1$. ¤

Lemma 2 Let x be a vertex of A. Let x^1, x^2 be the images of x in A_1^1, A_1^2 respectively. Then x^1, x^2 are equidistant from every vertex of A.

Proof. By definition, $d(x,x^1) = 1 = d(x,x^2)$. Then

$$\begin{aligned} d(y,x^1) &= d(y,x) + d(x,x^1) \\ &= d(y,x) + d(x,x^2) \\ &= d(y,x^2). \end{aligned}$$ ¤

We now provide an upper bound for the minimum metric dimension of enhanced hypercubes $Q_{n,2}$. In what follows P_n will denote a path on n vertices. ¤

Theorem 3 There exist $2n-5$ points in $Q_{n-1,2}$ inducing a path P_{2n-5} whose alternate vertices beginning with the first form a metric basis for $Q_{n,2}$, $n \geq 5$.

Proof. We prove by induction on n.

Base Case: Consider the vertices w_1, w_2, w_3, w_4, w_5 in $Q_{5,2}$. See Figure 6. These vertices induce a path P_5. By calculating the distance of all the vertices of $Q_{5,2}$ from w_1, w_3 and w_5, it is verified that the set $M = \{w_1, w_3, w_5\}$ forms a metric basis for $Q_{5,2}$.

Assume that the result is true for all enhanced hypercubes $Q_{k,2}$, $k \leq n$. Then there are $2k-5$ vertices in $Q_{k-1,2}$ inducing a path P_{2k-5} whose alternate vertices form a metric basis for $Q_{k,2}$.

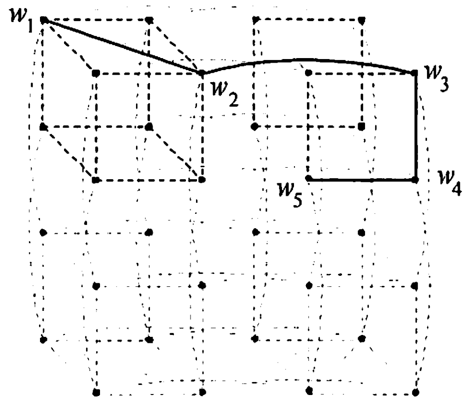

Figure 6: $Q_{5,\,2}$ with $M = \{w_1, w_3, w_5\}$

Consider $Q_{k+1,\,2}$. As mentioned earlier there are four copies of $Q_{k-1,\,2}$ denoted as A, A_1^1, A_1^2, A_2 with $A \cup A_1^1$ and $A \cup A_1^2$ isomorphic to $Q_{k,\,2}$. By induction hypothesis, there exists a path $P_{2k-5} = w_1 w_2 \ldots w_{2k-6} w_{2k-5}$ in A whose alternate vertices forms a metric basis for $A \cup A_1^1$ and $A \cup A_1^2$. Let $w_{2k-4} \in A_1^2$ be the image of $w_{2k-5} \in A$.

Consider the path $P_{2k-5} \circ (w_{2k-5}, w_{2k-4}) \circ (w_{2k-4}, w_{2k-3})$ where w_{2k-3} is any vertex in A_1^2 adjacent to w_{2k-4}. The graph $Q_{6,\,2}$ shown in Figure 7 exhibits P_{2k-5} with $k = 6$. Let us denote this path as P_{2k-3}. We claim that the set M of all the alternate vertices of P_{2k-3} beginning with the first vertex forms a metric basis for $Q_{k+1,\,2}$. Equivalently, given any two vertices x, y in $Q_{k+1,\,2}$ we need to find a vertex $w \in M$ such that $d(x,w) \neq d(y,w)$. Since $2k-3$ is odd, we observe that both ends of P_{2k-3} are in M.

Since alternate vertices in P_{2k-5}, beginning with the first vertex in P_{2k-5}, already forms a metric basis for $A \cup A_1^1$ as well as $A \cup A_1^2$, it is enough to consider the two cases namely $(x,y) \in A_1^1 \times A_1^2$ and $(x,y) \in A_2 \times A_2$.

Case 1: Suppose $x \in A_1^1$, $y \in A_1^2$. Let $x^1 \in A$ and $y^1 \in A$ be the preimages of x and y respectively. Then $d(x^1,x) = 1 = d(y^1,y)$.

Subcase $\{x^1 \neq y^1\}$: Since x^1, $y^1 \in A$, there exists $w \in P_{2k-5}$ such that $d(x^1,w) \neq d(y^1,w)$. Then by Lemma 1, $1 + d(x^1,w) \neq 1 + d(y^1,w)$ or $d(x,w) \neq d(y,w)$.

Subcase $\{x^1 = y^1\}$: By Lemma 2, x, y are equidistant from every vertex of A. Since the path P_{2k-5} lies in A, x and y are equidistant from every vertex of P_{2k-5}. Now consider the path $P_{2k-3} = P_{2k-5} \circ (w_{2k-5}, w_{2k-4}) \circ (w_{2k-4}, w_{2k-3})$. We show that x and y are at unequal distances from w_{2k-3}. Let $d(x, w_{2k-3}) = s$ and let T denote the corre-

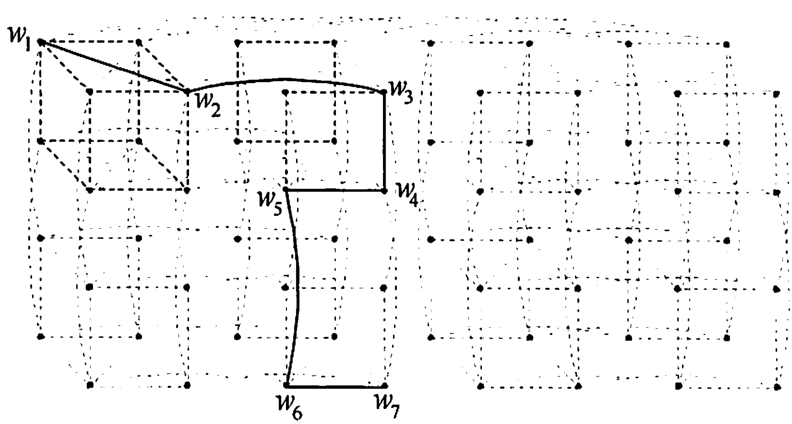

Figure 7: $Q_{6,\,2}$ with $M = \{w_1,\, w_3,\, w_5,\, w_7\}$ (only a few hypercube edges in $Q_{6,\,2}$ are shown)

sponding $(x, w_{2k\,-\,3})$-path. Assume that $T = P \circ (a, b) \circ Q \circ (c, c^1) \circ R$ where P is a shortest (x, a)-path in A_1^1, $a \in A_1^1$, $b \in A_2$ is the image of a, Q is a shortest (b, c)-path in A_2, $c \in A_2$, $c^1 \in A_1^2$ is the image of c and R is a shortest $(c^1, w_{2k\,-\,3})$-path.

Let P^1 be the image of P in A, Q^1 be the image of Q in A_1^2. Then $(x, x^1) \circ P^1 \circ (a^1, b^1) \circ Q^1 \circ R$ is also a shortest $(x, w_{2k\,-\,3})$-path and hence is of length s. Now $P^1 \circ (a^1, b^1) \circ Q^1 \circ R$ is a shortest $(x^1, w_{2k\,-\,3})$-path and is of length $s - 1$. Let P^2 be the image of P^1 lying in A_1^2. Then $P^2 \circ Q^1 \circ R$ is a shortest $(y, w_{2k\,-\,3})$-path whose length $s - 2$. Hence $d(y, w_{2k\,-\,3}) = s - 2$. Consequently $d(x, w_{2k\,-\,3}) \neq d(y, w_{2k\,-\,3})$. See Figure 8.

Case 2: The case $x,\ y \in A_2$ is easy, because the preimages of x and y namely x^1 and y^1 respectively will satisfy one of the following conditions:

(i) $x^1, y^1 \in A_1^1$.

(ii) $x^1, y^1 \in A_1^2$.

(iii) $x^1 \in A_1^1$, $y^1 \in A_1^2$ or vice versa.

The proof then follows by **Case 1** and the discussion preceeding it. ¤

The number of alternate vertices of $P_{2n\,-\,5}$ beginning with the first is $n - 2$. Hence

Theorem 4 $\beta(Q_{n,\,2}) \leq n - 2$, for $n \geq 5$. ¤

Remark 1 $\beta(Q_{n,\,2}) = 3$, when $n = 3, 4$.

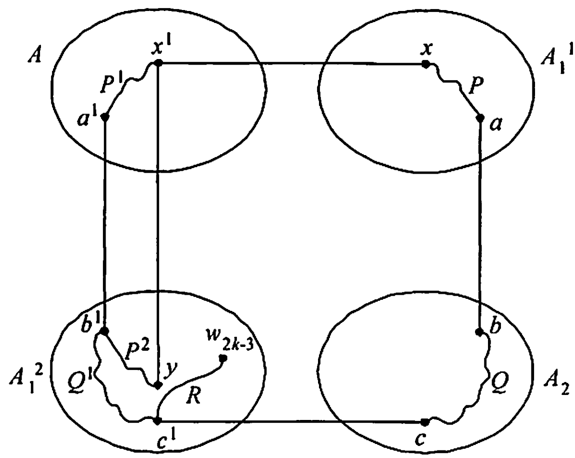

Figure 8: Case $x^1 = y^1$

4.1 Folded Hypercube Networks

The Folded hypercube $Q_{n,\,1}$ proposed by El-Amawy and Latifi [3] is $(n+1)$-regular, has 2^n vertices, $(n+1)2^{n-1}$ edges; $Q_{n,\,1}$ has diameter $\lceil n/2 \rceil$ and connectivity $n+1$. The graph shown in Figure 9 is a 3-dimensional folded hypercube $Q_{3,\,1}$, where the complementary edges are (000, 111), (001, 110), (010, 101) and (011, 110). Folded hypercubes are nothing but enhanced hypercubes $Q_{n,\,1}$.

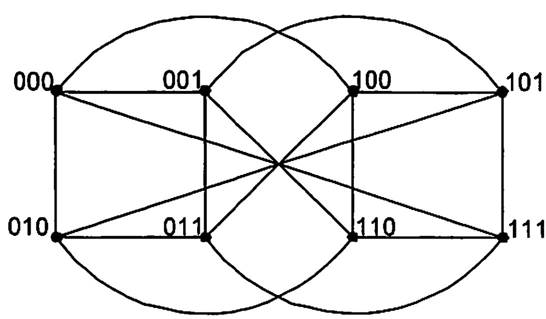

Figure 9: A 3-dimensional folded hypercube $Q_{3,\,1}$

Conjecture 5 $\beta(Q_{n,\,1}) \leq n+3$ for $n \geq 2$. ¤

5 Conclusion

We have obtained an upper bound for minimum metric dimension of enhanced hypercubes $Q_{n,\,2}$. The problem for folded hypercube $Q_{n,\,2}$ and enhanced hypercubes $Q_{n,\,k}$, $3 \leq k \leq n-1$ is under investigation.

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