A NEW CLASS OF GRACEFUL LOBSTERS

G. SETHURAMAN and J. JEBA JESINTHA
Department of Mathematics
Anna University, Chennai-600 025, INDIA.
E-mail : *jjesintha_75@yahoo.com*

Abstract

Pavel Hrnciar and Alfonz Havier [6] introduced a clever idea of transferring labeled pendant edges incident at a vertex of a graceful tree to some other suitable vertex of that tree, so that another graceful tree is obtained. This idea is further explored in this paper to generate graceful lobsters from a graceful caterpillar with n edges.

Key words: Trees; Graceful Labeling.

1 Introduction

A *graceful labeling* of a graph with n edges and vertex set V is an injection $f : V(G) \rightarrow \{0, 1, ..., n\}$ with the property that the resulting edge labels are also distinct where an edge incident with vertices u and v is assigned the label $|f(u) - f(v)|$. A graph which admits a graceful labeling is called a *graceful graph*. In 1963, Ringel [7] conjectured that K_{2n+1}, the complete graph on $2n + 1$ vertices, can be decomposed into $2n + 1$ isomorphic copies of a given tree with n edges. Kotzig [3], in 1965, conjectured that the complete graph K_{2n+1}, can be cyclically decomposed into $2n + 1$ copies of a given tree with n edges. In 1967, Rosa [8] proved the theorem that, if a tree T with n edges has a graceful labeling, then K_{2n+1} has a decomposition into $2n+1$ copies of T. From Rosa's Theorem it follows that both Ringel and Kotzig's conjectures are true if every tree is graceful. This led to the birth of the popular Ringel-Kotzig-Rosa conjecture or the graceful tree conjecture(GTC), 'All trees are graceful'. The graceful tree conjecture has been verified to be true for all trees with atmost 29 vertices [4] and for all trees of diameter less than or equal to 5 [6]. It is proved that caterpillars (trees whose removal of all the end-vertices produces a path) are graceful[[8]. Bermond [1] conjectured that lobsters (trees with the property that the removal of all the end-vertices produces a caterpillar) are graceful. Special classes of this conjecture are shown to be graceful in [5] and [9]. For an exhaustive survey on graceful trees refer the dynamic survey by Gallian [2]. In this paper we give a procedure that constructs a new family of graceful lobsters from a graceful caterpillar with n edges.

2 Main result

In this section we give a procedure for constructing graceful lobsters from a graceful caterpillar with n edges by using the idea of transferring pendant edges incident at a vertex of a graceful caterpillar to some other suitable vertex of that caterpillar. The transfers that are used to generate the class of graceful lobsters is based on the following Lemma proved in [6].

Let T be a tree with n edges and let f be a graceful labeling of the tree T. Then the set of labels assigned to all the edges of the tree T is $\{1, 2, \cdots, n\}$. Let T be a tree and let $uv \in E(T)$. We denote $T_{u,v}$ as the subtree of T induced by the set, $V(T_{u,v}) = \{w \in V(T)\colon w = u \text{ or } v \text{ is on a } u - w \text{ path}\}$.

Lemma [6]: Let T be a tree with a graceful labeling f and let u be a vertex adjacent with vertices u_1, u_2. Let T' be the subtree of T induced by the set
$V(T') = \{V(T) - [V(T_{u,u_1}) \cup V(T_{u,u_2})]\} \cup \{u\}$ and let $v \in V(T')$, $v \neq u$.

(a) If $u_1 \neq u_2$, $f(u_1) + f(u_2) = f(u) + f(v)$ and the tree T'' is obtained by gluing the trees T_{u,u_1}, T_{u,u_2} and T' in such a way that the vertex v of the tree T' is identified with vertex u of the trees T_{u,u_1}, T_{u,u_2} (see Figure 1.) then f is a graceful labeling of the tree T'' too.

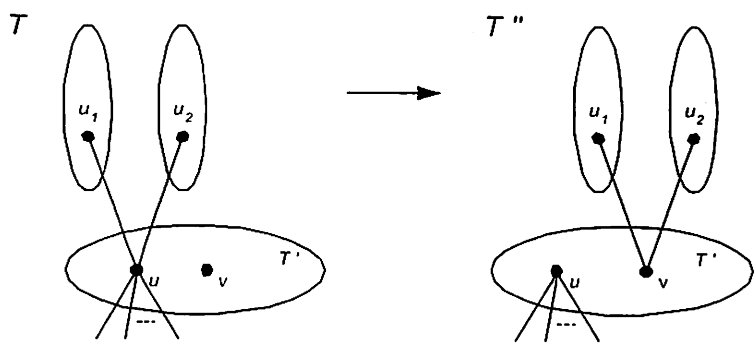

Fig. 1. Transferring subtree at u to v

(b) If $u_1 = u_2$, $2f(u_1) = f(u) + f(v)$ and the tree T'' is obtained by gluing the trees T' and T_{u,u_1} in such a way that the vertex v of the tree T' is identified with vertex u of the tree T_{u,u_1}, then f is a graceful labeling of the tree T'' too.

We call those set of edges which satisfy the summation defined in Lemma as *eligible edges* at u. Note that only those eligible edges at a vertex u which satisfy the summations defined in Lemma can be transferred from the vertex u to the vertex v. In other words, as long as the pendant edges incident at u are eligible, it can be transferred to any vertex v which gives the eligible summation. Then the resultant tree is always graceful. This is an important idea that we frequently use for constructing graceful lobsters from graceful caterpillars.

Here in this paper we construct graceful lobsters from graceful caterpillars by applying a number of transfers on the eligible edges. Thus by the Lemma, it is guaranteed that we always obtain graceful trees(lobsters). Hereafter, we will not distinguish between a vertex and its label for any given graceful labeling which we consider. We now recall the following transfers defined in [6].

First type transfer. A transfer from a vertex i to a vertex j denoted as $i \rightarrow j$ transfer, is called a transfer of the first type if the end-vertices of the transferred end-edges form a continuous sequence $k, k+1, k+2, \cdots, k+m$.

Second type transfer. A transfer from a vertex i to a vertex j denoted as $i \rightarrow j$ transfer, is called a transfer of the second type if the end-vertices of the transferred end-edges form two sections namely, $k, k+1, k+2, \cdots, k+m$ and $l, l+1, l+2, \cdots, l+m$.

We denote successive transfers $a \rightarrow b$, $b \rightarrow c$, $c \rightarrow d$, $\cdots$ as $a \rightarrow b \rightarrow c \rightarrow d \rightarrow \cdots$. The Figure 3 illustrates the effect of first and second type transfers on the graceful tree given in Figure 2.

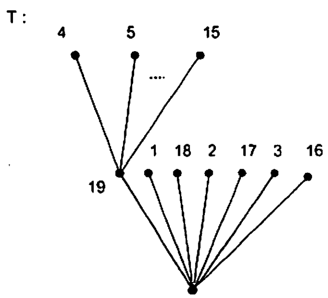

Figure 2. A graceful tree T.

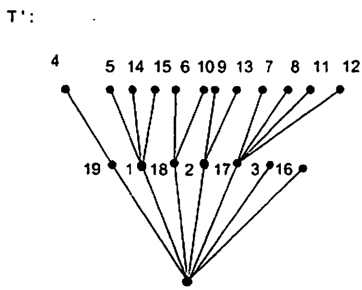

Figure 3. A graceful tree T'.

Consider a graceful tree T with $n = 19$ edges as shown in Figure 2. On the tree T first we carry out two transfers $19 \rightarrow 1$, $1 \rightarrow 18$ of the first type and then two transfers $18 \rightarrow 2$, $2 \rightarrow 17$ of the second type. At the beginning 12 vertices namely, 4, 5, $\cdots$, 15 are adjacent to vertex 19. After the first transfer $19 \rightarrow 1$, the vertices 5, 6, $\cdots$, 15 are transferred from 19 to 1 and hence they are now adjacent with 1. Therefore, the vertex 4 is only retained at vertex 19. Now we apply the transfer $1 \rightarrow 18$. After this transfer, the vertices 6, 7, $\cdots$ 13 are transferred to vertex 18 and the vertices 5, 14 and 15 are retained at vertex 1. The effect of the transfers $19 \rightarrow 1$, $1 \rightarrow 18$, $18 \rightarrow 2$, $2 \rightarrow 17$ (i.e. the sequence of transfers $19 \rightarrow 1 \rightarrow 18 \rightarrow 2 \rightarrow 17$) is described in Table 1.

Table 1: The sequence of transfers applied on the tree T given in Figure 2

Graceful labels of the end-vertices of the *transferred* pendant edges incident at the vertex appearing just below in the sequence of transfers, *before* the transfer.	4-15	5-15	6-13	7-9,11-13	7-8,11-12
Sequence of transfers	19 →	1 →	18 →	2 →	17
Graceful labels of the end-vertices of the pendant edges *retained* at the vertex appearing just above in the sequence of transfers, *after* the transfer.	4	5,14,15	6,10	9,13	7,8,11,12

Note that the first type transfer will leave an *odd* number of pendant edges at each vertex after the transfer. Whereas the second type transfer will leave an *even* number of pendant edges at each vertex after the transfer. Also note that *all or some*, (depending on the requirement), of the eligible edges can be transferred from vertex i to vertex j thereby leaving an *odd* number of pendant edges at the vertex i after the transfer in the case of the first type transfers and leaving an *even* number of pendant edges at the vertex i after the transfer in the case of the second type transfers.

Also note that after a first type transfer one can go on with another first type transfer or with a second type transfer. After a second type transfer one can go on with another second type transfer only, since after the second type transfer the labels of the end-vertices adjacent with that vertex form two sections and thus the continuous sequence of labels could not be formed. It is also noted that in order to carry out a second type transfer from a vertex i to a vertex j, a minimum of 4 edges should be incident at vertex i.

In general, we start with a graceful caterpillar, apply the sequence of transfers and obtain a graceful lobster. Whenever we apply the transfer, the vertices of the graceful tree are not distinguished from their labeling. Or in other words, vertices are referred with its graceful label.

We now give a procedure for generating graceful lobsters from a graceful caterpillar with n edges by using only the first type transfer.

2.1. Procedure :

Let n be an integer with $n \geq 6$ and let d be an integer such that $3 \leq d \leq n$. Let l be an integer such that $2 \leq l \leq (n-4)$. Then we construct a graceful lobster T with n edges from a graceful caterpillar (with n edges) of length l with the property that each internal vertex and one of the penultimate vertices having a fixed degree d and the other penultimate vertex having the degree $(d+t)$, where $t = n - [(l-1)(d-1)+1]$ and $t \geq 3$, by applying the first type transfers.

Let T denote a graceful caterpillar with n edges and with graceful labels as shown in Figures 4 and 5. Note that t edges are incident with the vertex $k(d-1)$ or $[n-(k-1)(d-1)]$ of the caterpillar depending on whether $k = (l-1)/2$ or $k = l/2$ respectively, that is, depending on whether l is odd or even.

To generate different graceful lobsters we start applying a sequence of first type transfers on the caterpillar T starting from the vertex $k(d-1)$ or $[n-(k-1)(d-1)]$ depending on whether l is odd or even respectively as indicated in the Table 2.

In the Table 2, we give the different first type transfer sequences that are to be carried out on T depending on the nature of l. The Table 2 indicates the range for the number of possible transfers m and the transfer sequences.

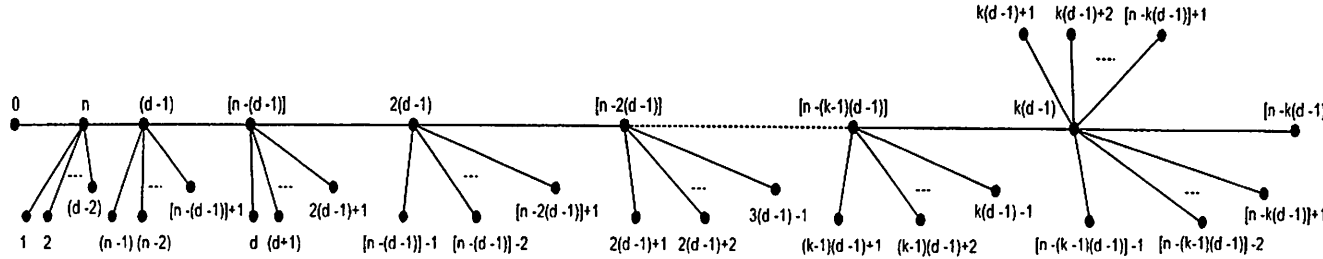

Figure 4. Caterpillar of odd length l with graceful labels.

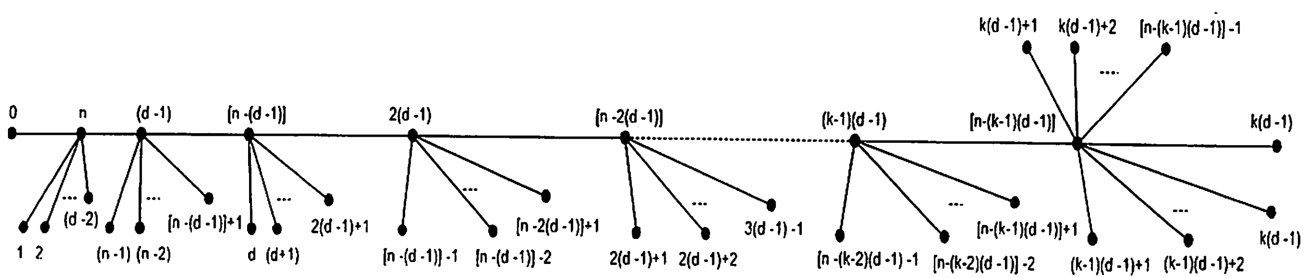

Figure 5. Caterpillar of even length l with graceful labels.

Table 2. Sequence of transfers to generate graceful lobsters.

Nature of l	Number of transfers 'm'	Sequence of transfers to be carried out
l is odd	choose any odd 'm' in the range $1 \leq m \leq (l-1)(d-1)-1$.	$k(d-1) \rightarrow [n-k(d-1)]+1 \rightarrow k(d-1)-1 \rightarrow [n-k(d-1)]+2 \rightarrow k(d-1)-2 \rightarrow \cdots \rightarrow k(d-1)-i \rightarrow [n-k(d-1)]+(i+1)$. where $i=(m-1)/2$ and $k=(l-1)/2$.
	choose any even 'm' in the range $2 \leq m \leq (l-1)(d-1)-2$.	$k(d-1) \rightarrow [n-k(d-1)]+1 \rightarrow k(d-1)-1 \rightarrow [n-k(d-1)]+2 \rightarrow k(d-1)-2 \rightarrow \cdots \rightarrow [n-k(d-1)]+i \rightarrow k(d-1)-i$, where $i=m/2$ and $k=(l-1)/2$.
l is even	choose any odd 'm' in the range $1 \leq m \leq (l-1)(d-1)-(d-2)$.	$[n-(k-1)(d-1)] \rightarrow k(d-1)-1 \rightarrow [n-(k-1)(d-1)]+1 \rightarrow k(d-1)-2 \rightarrow [n-(k-1)(d-1)]+2 \rightarrow \cdots \rightarrow [n-(k-1)(d-1)]+i \rightarrow k(d-1)-(i+1)$. where $i=(m-1)/2$ and $k=l/2$.
	choose any even 'm' in the range $2 \leq m \leq (l-2)(d-1)$.	$[n-(k-1)(d-1)] \rightarrow k(d-1)-1 \rightarrow [n-(k-1)(d-1)]+1 \rightarrow k(d-1)-2 \rightarrow [n-(k-1)(d-1)]+2 \rightarrow \cdots \rightarrow k(d-1)-i \rightarrow [n-(k-1)(d-1)]+i$, where $i=m/2$ and $k=l/2$.

Illustrative examples of the labeling given in the procedure are provided in Figures 6 to 9.

Illustration for the case when l is odd and m is odd with the parameters $n = 67$, $l = 5$, $d = 7$.

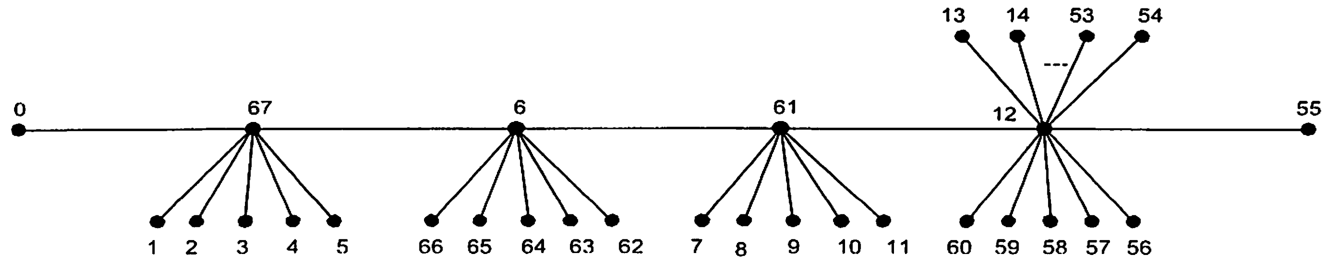

Figure 6. Caterpillar T of odd length l with graceful labels.

Choosing an odd $m = 9$ in the range $1 \leq m \leq 23$ such that $i = 4$ and $k = 2$, perform the sequence of first type transfers $12 \to 56 \to 11 \to 57 \to 10 \to 58 \to 9 \to 59 \to 8 \to 60$

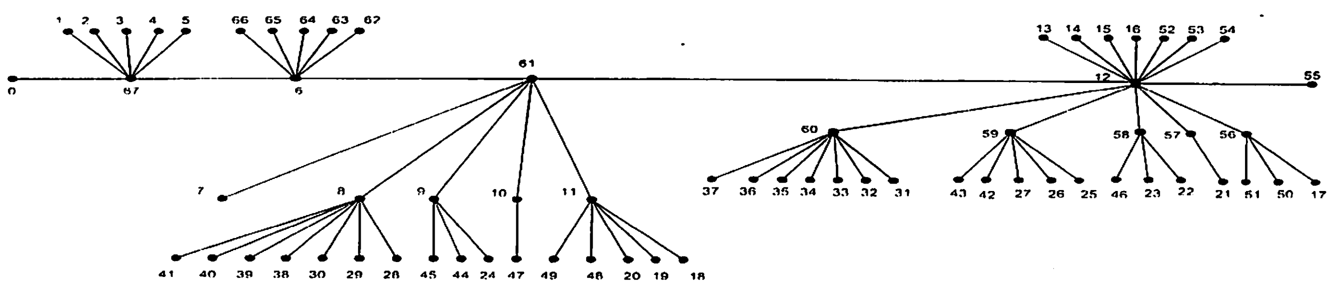

Figure 7. Graceful Lobster T with n edges.

Illustration for the case when l is even and m is even with the parameters $n = 66$, $l = 6$, $d = 4$.

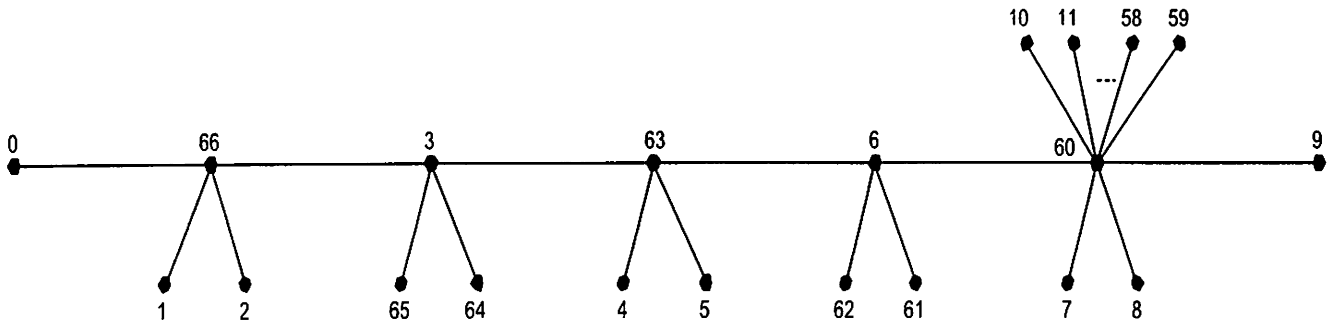

Figure 8. Caterpillar T of even length l with graceful labels.

Choosing an even $m = 10$ in the range $2 \leq m \leq 12$ such that $i = 5$ and $k = 3$, perform the sequence of first type transfers $60 \to 8 \to 61 \to 7 \to 62 \to 6 \to 63 \to 5 \to 64 \to 4 \to 65$

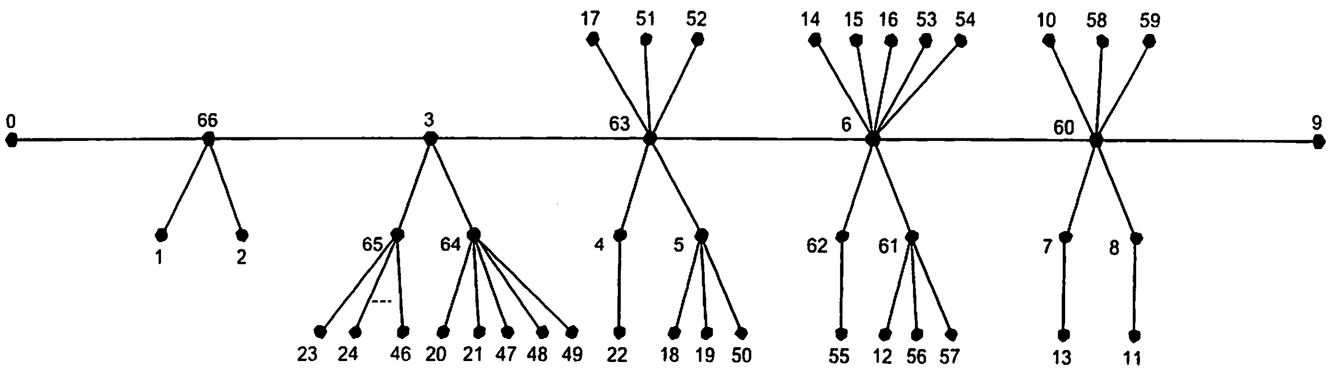

Figure 9. Graceful Lobster T with n edges.

3. Conclusion

The lobsters obtained from the original starting caterpillar given in Figures 4 and 5 have the property that end-vertices of the caterpillar at which the edges were transferred to, will always have the even degree after the transfer and the penultimate vertex of the caterpillar from which the edges were transferred will be of either odd or even degree after the transfer.

In this paper we have generated graceful lobsters from a particular type of a graceful caterpillar, with n edges, of length l with the property that each internal vertex and one of the penultimate vertices having a *fixed* degree d and the other penultimate vertex having the degree $(d + t)$, where $t = n - [(l - 1)(d - 1) + 1]$ with $t \geq 3$, using the first type transfer sequences. Similarly other family of lobsters can be constructed by using second type transfer.

References

[1] Bermond J.C., "Graceful graphs, radio antennae and French windmills", Graph Theory and Combinatorics, Pitman, San Fransisco, 1979, pp. 18–37.

[2] Gallian J.A., "A dynamic survey of graph labeling", The Electronic Journal of Combinatorics, Vol.5, DS6, http://www.combinatorics.org 2007.

[3] Kotzig A., "Decompostions of a complete graph into 4k-gons (in Russia)", Matematicky Casopis, Vol.15, pages 229–233, 1965.

[4] Michael Horton, "Graceful Trees: Statistics and Algorithms", Master's Thesis http://eprints.comp.utas.edu.au81/archive/00000019/01.,2001.

[5] Ng H.K., "Gracefulness of a class of lobsters", Notices AMS, Vol. 7, page 247, 1986.

[6] Pavel Hrnciar, Alfonz Havier, "All trees of diameter five are graceful", Discrete Math, Vol. 233, pages 133–150, 2001.

[7] Ringel G. , Problem 25 in Theory of Graphs and its Applications, Proc. Symp. Smolenice, Prague, 1963, page 162.

[8] Rosa A., "On certain valuations of the vertices of a graph", Theory of Graphs, (Internat. Symposium, Rome, July 1966), Gordon and Breach, New York, 1967, pp. 349-355. 5submitted to Discrete Mathematics.

[9] Wang J.G. , Jin D.J. , Lu X.G, Zhang D. , "Gracefulness of a class of lobster trees", Math. Comput. Modelling, Vol. 20, pages 105–110, 1994.