Parametric Programming to the Analysis of Bulk Arrival Queuing Model

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Abstract

The purpose of this paper is to construct the membership functions of performance measures in bulk arrival queuing system with arrival rate and service rate are fuzzy numbers. Thus this paper develop the parametric programming approach to derive the membership functions of the steady state performance measures in bulk arrival queuing system with varying batch size. On the basis of a cut representation and extension principle, a parametric programming is formulated to describe the family of crisp bulk arrival queues. The performance measures are expressed by membership functions rather by crisp values, they completely conserve the fuzziness of input information when some data of bulk arrival queuing systems are ambiguous.

Keywords: Fuzzy Queues, Bulk arrival, α-cut, Parametric Programming.

1 Introduction

Bulk arrival queuing models have been widely applied to many practical situations such as production manufacturing system, communication systems and computer networks. For example, when the operation in a production, manufacturing system will not begin until a specified number of new materials are accumulated during an ideal period. We often analyse this system by a bulk arrival queuing model [4] that provides a powerful tool for evaluating the system performance.

In actual practice, the arrival rate, service rate are frequently described as "fast", "slow" or "moderately slow" and so on, which are linguistic fuzzy terms and can be best described by using fuzzy sets.

The problem of fuzzy queues has been analysed by Prade [9] and Li and Lee [6, 7] through the use of extension principle. Buckley [1] considers a elementary queuing system with multiple parallel server with finite or infinite system capacity and calling source, whose arrivals and departure are restricted by arbitrary possibility distribution. Negi and Lee [8] proposed two types of approaches namely α-cut and analytical representation of fuzzy numbers.

In this paper, we adopt the α-cut approach to decompose a fuzzy queue to a family of crisp queue. As the α value varies, the parametric programming technique [3] is applied to describe the family of crisp queues. The solution of the parametric programs is to derive the membership function of the usual crisp bulk arrival queue can be extended to fuzzy bulk arrival queue, bulk arrival queuing model would have wider application.

2 Fuzzy Numbers

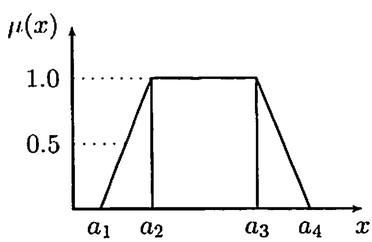
Trapezoidal fuzzy number is used to represent the more fuzziness trapezoidal numbers [6] are an ideal compromise between complexity and over simplification.

2.1 Trapezoidal Fuzzy Numbers

A trapezoidal fuzzy number $A = (a_1, a_2, a_3, a_4)$ is defined by the membership function.

$$\mu_{\tilde{A}(x)} = \begin{cases} (x - a_1)/(a_2 - a_1), & \text{if } a_1 \leq x \leq a_2 \\ 1, & \text{if } a_2 \leq x \leq a_3 \\ (x - a_4)/(a_3 - a_4), & \text{if } a_3 \leq x \leq a_4 \\ 0, & \text{Otherwise} \end{cases}$$

This is represented diagrammatically as:



2.2 Notations Used

L_s: Expected numbers of customers in the system.
L_q: Expected number of customers in the queue.
W_s: Expected waiting time in system.
W_q: Expected waiting time in queue.
$\tilde{\lambda}$: Fuzzy Arrival rate.
$\tilde{\mu}$: Fuzzy service rate.
$E(K)$: Expected batch size of arrival.
$\mu_{\tilde{L}_q(z)}$: Membership function of $\tilde{L}_q(z)$.
$\mu_{\tilde{L}(z)}$: Membership function of $\tilde{L}(z)$.
$\mu_{\tilde{W}_S(z)}$: Membership function of $\tilde{W}_S(z)$.
$\mu_{\tilde{W}(z)}$: Membership function of $\tilde{W}(z)$.

3 Fuzzy Bulk Arrival Queues with Varying Batch Size

Consider a queuing system in which customers arrive at a single server facility in batches as a Poisson process with group arrival rate $\tilde{\lambda}$, where $\tilde{\lambda}$ is a fuzzy number and all service times are independent and identically distributed according to exponential distribution with fuzzy service rate $\tilde{\mu}$. The actual number of customers in any arriving module is stochastically equivalent to a generic random variable k, which may take on any positive integer with probability $f(k)$. Customers are served according to a first-come-first served discipline and both the size of calling population and the system capacity is infinite. This model with hereafter denoted by $\mathrm{FM}^{[K]}/\mathrm{FM}/1$.

3.1 Problem Formulation

Consider a bulk arrival queuing system with one server. The inter arrival time and service time are approximately known and represented by fuzzy set,

$$\tilde{\lambda} = \{x, \mu_{\tilde{\lambda}}(x)/x \in X\} \quad \tilde{\mu} = \{y, \mu_{\tilde{\mu}}(y)/y \in Y\}$$

where X and Y are crisp sets of the inter arrival times and service times and $\mu_{\tilde{\lambda}}(x)$ and $\mu_{\tilde{\lambda}}(y)$ are the respective membership functions.

The α-cut of $\tilde{\lambda}$ and $\tilde{\mu}$ are

$$\lambda(\alpha) = \{x \in X;\, \mu_{\tilde{\lambda}}(x) \geq \alpha\} \quad \mu(\alpha) = \{y \in Y;\, \mu_{\tilde{\mu}}(x) \geq \alpha\}$$

Both $\lambda(\alpha)$ and $\mu(\alpha)$ are crisp sets. Hence a fuzzy queue can be reduced to a family of crisp with different α levels cut

$$\{\lambda(\alpha) : 0 \le \alpha \le 1\} \quad \text{and} \quad \{\mu(\alpha) : 0 \le \alpha \le 1\}$$

These two sets represent sets of movable boundaries and they form nested for expressing the relationship between the crisp sets and fuzzy sets. Let the confidence intervals of the fuzzy sets $\tilde{\lambda}$ and $\tilde{\mu}$ are $(l_{\lambda(\alpha)}, u_{\lambda(\alpha)})$ and $(l_{\mu(\alpha)}, u_{\mu(\alpha)})$ respectively. When both inter arrival time and service time are fuzzy numbers based on Zadeh's extension principle [11] the membership function of the performance measure $p(x, y)$ is defined as,

$$\mu_{p(\tilde{\lambda},\tilde{\mu})}(z) = \sup_{\substack{x \in X \\ y \in Y}} \left\{ \min\left(\mu_{\tilde{\lambda}}(x), \mu_{\tilde{\mu}}(y)\right) / z = p(x, y) \right\}$$

The corresponding parametric programming techniques for finding the lower bound and upper bound of the α-cut of $\mu_{p(\tilde{\lambda},\tilde{\mu})}(z)$ are

$$\begin{aligned} l_{p(\alpha)} &= \min p(x, y) \quad \text{such that } l_{\lambda(\alpha)} \le x \le u_{\lambda(\alpha)},\ l_{\mu(\alpha)} \le y \le u_{\mu(\alpha)} \text{ and} \\ u_{p(\alpha)} &= \max p(x, y) \quad \text{such that } l_{\lambda(\alpha)} \le x \le u_{\mu(\alpha)} \text{ and } l_{\mu(\alpha)} \le y \le u_{\mu(\alpha)}. \end{aligned}$$

If both $l_{p(\alpha)}$ and $u_{p(\alpha)}$ are invertible with respect to α, then the left shape function $L(z) = l^{-1}_{p(\alpha)}$ and right shape function $R(z) = u^{-1}_{p(\alpha)}$ can be obtained, from which the membership function $\mu_{p(\tilde{\lambda},\tilde{\mu})}(z)$ is constructed.

$$\mu_{p(\tilde{\lambda},\tilde{\mu})}(z) = \begin{cases} L(z), & z_1 \le z \le z_2 \\ 1, & z_2 \le z \le z_3 \\ R(z), & z_3 \le z \le z_4 \end{cases}$$

where $z_1 \le z_2 \le z_3 \le z_4$ and $L(z_1) = R(z_4) = 0$.

3.2 $(FM^{[K]}/FM/1)$:$(\infty/FCFS)$ Queues

From the knowledge of traditional queuing theory [2, 5, 10] under the steady state condition $p = \dfrac{xE(K)}{y} < 1$ where $E(K)$ denote the expectation of K, the expected number of customers in the queue of a crisp queuing system with bulk arrival is

$$L_q = \frac{x[yE(K^2) + 2x(E(K))^2 - yE(K)]}{2y[y - xE(K)]}$$

and the expected number of customers in the system is

$$L = \frac{x[E(K) + E(K^2)]}{2[y - xE(K)]}.$$

The expected waiting time in the queue is

$$W_q = \frac{yE(K^2) + 2x(E(K))^2 - yE(K)}{2y[y - xE(K)]}.$$

The waiting time of the system is

$$W_s = \frac{x[E(K) + E(K^2)]}{2[y - xE(K)]}.$$

4 Numerical Example

Consider a local postal route in a central mail handling system. Postal workers collect mail from mailboxes with fixed pick-up times and delivers the mail to the local post office. The workers at the local post office collect mail up to a certain point and then send batches to the central mailing handling office. The number of parcels send each time follows a geometric distribution with parameter $p = 0.5$. That is the probability that A parcels are sent is $P(A = K) = 0.5K$, $K = 1, 2, \ldots$. The mails are arriving to the postal office with Poisson process and the service time follows an exponential distribution. Both the group arrival rate and service rate are trapezoidal numbers represented by $\tilde{\lambda} = [3, 4, 5, 6]$ and $\tilde{\mu} = [19, 20, 21, 22]$.

The postal system officer wants to evaluate the performance measures such as the expected number of mails in the queue. We have $E(K) = 2$, $\text{Var}(K) = 2$ and $\text{Var}(K) = E(K^2) - [E(K)]^2$; $E(K^2) = \text{Var}K + [E(K)]^2 = 6$. α-cut of and are $[3 + \alpha,\ 6 - \alpha]$ and $[19 + \alpha,\ 22 - \alpha]$ respectively.

$$\rho = \frac{xE(K)}{y}, \qquad L_q = \frac{x[yE(K^2) + 2x[E(K)]^2 - yE(K)]}{2y[y - xE(K)]}$$

when x reaches its lower bound and y reaches its upper bound, $L_q(\alpha)$ attains its minimum.

$$l_{L_q} = \frac{2\alpha^2 + 62\alpha + 168}{3\alpha^2 - 82\alpha + 352}$$

The inverse function of l_{L_q} exists.

$$z = \frac{2\alpha^2 + 62\alpha + 168}{3\alpha^2 - 82\alpha + 352}$$

$$\alpha = \frac{(41z+31) \pm 25\sqrt{z^2+6z+1}}{3z-2}$$

$$u_{L_q} = \max\left[\frac{4x^2+2xy}{y^2-2xy}\right]$$

On the contrary, to maximize $L_q(\alpha)$ it is desired that x increases to its upper bound and y decreases to its lower bound

$$u_{L_q} = \frac{2\alpha^2 - 74\alpha + 372}{3\alpha^2 + 64\alpha + 133}$$

$u_{L_{q(\alpha)}}$ is also invertible.

$$z = \frac{2\alpha^2 - 74\alpha + 372}{3\alpha^2 + 64\alpha + 133}$$

$$\alpha = \frac{-(32z+37) \pm \sqrt{625z^2+3750z+625}}{3z-2} \leq 1$$

The membership function of $\tilde{L}_q(z)$ is

$$\mu_{\tilde{L}_q(z)} = \begin{cases} \frac{(41z+31)-25(z^2+6z+1)^{1/2}}{3z-2}, & \frac{21}{44} \leq z \leq \frac{232}{273} \\ 1, & \frac{232}{273} \leq z \leq \frac{3}{2} \\ \frac{-(32z+37)+25(z^2+6z+1)^{1/2}}{3z-2}, & \frac{3}{2} \leq z \leq \frac{372}{133} \end{cases}$$

$$L = \frac{x[E(K)+E(K^2)]}{2[y-2x]}$$

Similarly,

$$\mu_{\tilde{L}(z)} = \begin{cases} \frac{16z-12}{3z+4}, & \frac{3}{4} \leq z \leq \frac{16}{13} \\ 1, & \frac{16}{13} \leq z \leq 2 \\ \frac{24-4z}{7+3z}, & 2 \leq z \leq \frac{24}{7} \end{cases}$$

$$W_S = \frac{E(K)+E(K^2)}{2(y-2x)}$$

$$l_{W_S}(\alpha) = \min\left[\frac{4}{y-2x}\right] = \frac{4}{16-3\alpha}$$

$$u_{W_S(\alpha)} = \max\left[\frac{4}{y-2x}\right]$$

$$\mu_{\tilde{W}_S(z)} = \begin{cases} \frac{16z-4}{3z}, & \frac{1}{4} \leq z \leq \frac{4}{13} \\ 1, & \frac{4}{13} \leq z \leq \frac{2}{5} \\ \frac{4-7z}{3z}, & \frac{2}{5} \leq z \leq \frac{4}{7} \end{cases}$$

$$W_q = \frac{yE(K^2) + 2x[E(K)]^2 - yE(K)}{2y[y - xE(K)]}$$

$$l_{W_q} = \min\left[\frac{2y + 4x}{y^2 - 2xy}\right]$$

$$\mu_{\tilde{W}_q}(z) = \begin{cases} \frac{(41z+1)-(625z^2+250z+1)^{1/2}}{3z}, & \frac{7}{44} \le z \le \frac{58}{273} \\ 1, & \frac{58}{273} \le z \le \frac{3}{10} \\ \frac{-(32z+1)+(625z^2+250z+1)^{1/2}}{3z}, & \frac{3}{10} \le z \le \frac{62}{183} \end{cases}$$

5 Conclusion

This paper applies the concept of α-cuts and Zadeh extension principle to a batch arrival queuing model with single server to construct the membership function of the expected waiting time in the queue, expected number of customers in the system and expected length of time in the queue and systems using parametric programming. From Table 1, maximum length of the system is 3.4286. From Table 2, the maximum length of the queue is 2.7970. From Table 3, waiting time of the system ranges from 0.2500 to 0.5714. From Table 4, the waiting time of the queue ranges from 0.1591 to 0.4662. Following the proposed α-cut of the membership function are found and their interval limits inverted to attain explicit closed form expression for the system characteristic. Since the performance measures are expressed by the membership function rather than by a crisp value, it maintains the fuzziness of input information and the results can be used to represent the fuzzy system more accurately.

6 References

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A Appendix

Table 1:

α	$l_{x(\alpha)}$	$u_{x(\alpha)}$	$l_{y(\alpha)}$	$u_{y(\alpha)}$	$l_{L(\alpha)}$	$u_{L(\alpha)}$
0.0	3.0000	6.0000	19.0000	22.0000	0.6875	3.4286
0.1	3.1000	5.9000	19.1000	21.9000	0.7261	3.2329
0.2	3.2000	5.8000	19.2000	21.8000	0.7662	3.0526
0.3	3.3000	5.7000	19.3000	21.7000	0.8079	2.8861
0.4	3.4000	5.6000	19.4000	21.6000	0.8514	2.7317
0.5	3.5000	5.5000	19.5000	21.5000	0.8966	2.5882
0.6	3.6000	5.4000	19.6000	21.4000	0.9437	2.4545
0.7	3.7000	5.3000	19.7000	21.3000	0.9928	2.3297
0.8	3.8000	5.2000	19.8000	21.2000	1.0441	2.2128
0.9	3.9000	5.1000	19.9000	21.1000	1.0977	2.1031
1.0	4.0000	5.0000	20.0000	21.0000	1.1538	2.0000

Table 2:

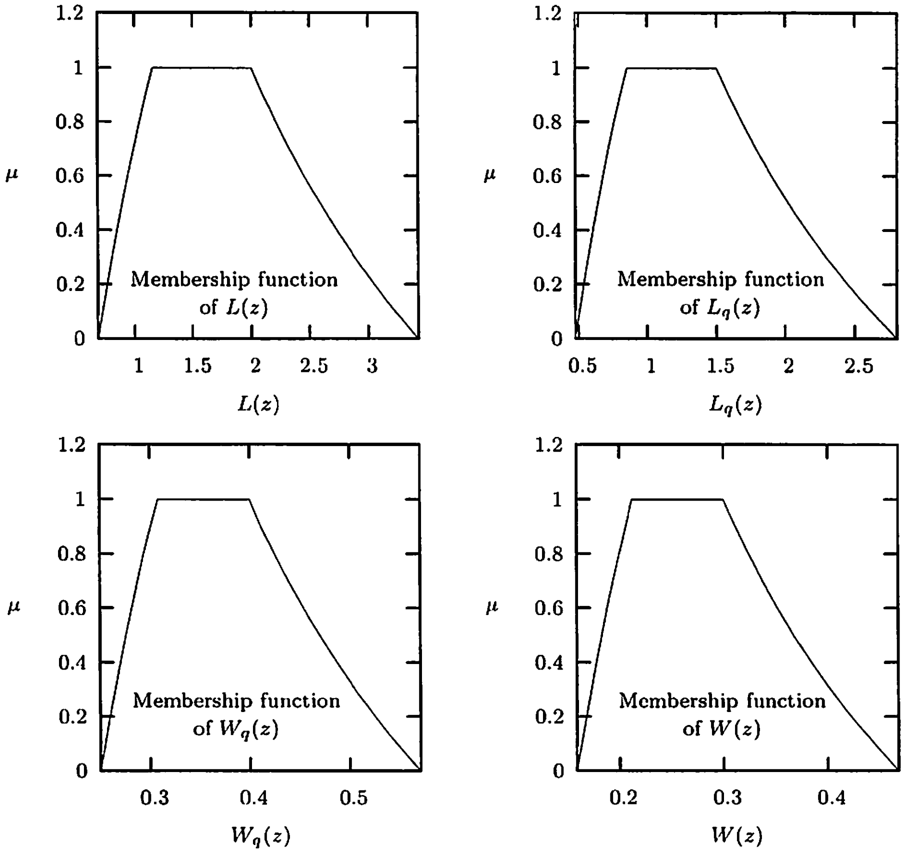
α	$l_{x(\alpha)}$	$u_{x(\alpha)}$	$l_{y(\alpha)}$	$u_{y(\alpha)}$	$l_{L_q(\alpha)}$	$u_{L_q(\alpha)}$
0.0	3.0000	6.0000	19.0000	22.0000	0.4773	2.7970
0.1	3.1000	5.9000	19.1000	21.9000	0.5067	2.6151
0.2	3.2000	5.8000	19.2000	21.8000	0.5376	2.4485
0.3	3.3000	5.7000	19.3000	21.7000	0.5700	2.2954
0.4	3.4000	5.6000	19.4000	21.6000	0.6041	2.1544
0.5	3.5000	5.5000	19.5000	21.5000	0.6399	2.0241
0.6	3.6000	5.4000	19.6000	21.4000	0.6776	1.9035
0.7	3.7000	5.3000	19.7000	21.3000	0.7173	1.7916
0.8	3.8000	5.2000	19.8000	21.2000	0.7592	1.6875
0.9	3.9000	5.1000	19.9000	21.1000	0.8033	1.5905
1.0	4.0000	5.0000	20.0000	21.0000	0.8498	1.5000

Table 3:

α	$l_{x(\alpha)}$	$u_{x(\alpha)}$	$l_{y(\alpha)}$	$u_{y(\alpha)}$	$l_{W_s(\alpha)}$	$u_{W_s(\alpha)}$
0.0	3.0000	6.0000	19.0000	22.0000	0.2500	0.5714
0.1	3.1000	5.9000	19.1000	21.9000	0.2548	0.5479
0.2	3.2000	5.8000	19.2000	21.8000	0.2597	0.5263
0.3	3.3000	5.7000	19.3000	21.7000	0.2649	0.5063
0.4	3.4000	5.6000	19.4000	21.6000	0.2703	0.4878
0.5	3.5000	5.5000	19.5000	21.5000	0.2759	0.4706
0.6	3.6000	5.4000	19.6000	21.4000	0.2817	0.4545
0.7	3.7000	5.3000	19.7000	21.3000	0.2878	0.4396
0.8	3.8000	5.2000	19.8000	21.2000	0.2941	0.4255
0.9	3.9000	5.1000	19.9000	21.1000	0.3008	0.4124
1.0	4.0000	5.0000	20.0000	21.0000	0.3077	0.4000

Table 4:

α	$l_{x(\alpha)}$	$u_{x(\alpha)}$	$l_{y(\alpha)}$	$u_{y(\alpha)}$	$l_{W(\alpha)}$	$u_{W(\alpha)}$
0.0	3.0000	6.0000	19.0000	22.0000	0.1591	0.4662
0.1	3.1000	5.9000	19.1000	21.9000	0.1635	0.4432
0.2	3.2000	5.8000	19.2000	21.8000	0.1680	0.4221
0.3	3.3000	5.7000	19.3000	21.7000	0.1727	0.4027
0.4	3.4000	5.6000	19.4000	21.6000	0.1777	0.3847
0.5	3.5000	5.5000	19.5000	21.5000	0.1828	0.3680
0.6	3.6000	5.4000	19.6000	21.4000	0.1882	0.3525
0.7	3.7000	5.3000	19.7000	21.3000	0.1939	0.3380
0.8	3.8000	5.2000	19.8000	21.2000	0.1998	0.3245
0.9	3.9000	5.1000	19.9000	21.1000	0.2060	0.3119
1.0	4.0000	5.0000	20.0000	21.0000	0.2125	0.3000



Membership function of characteristics of bulk arrival queue system