GRAPH LABELLINGS, EMBEDDING AND NP-COMPLETENESS THEOREMS

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Abstract

In this paper, we establish the possibility of embedding a graph as an induced subgraph in an elegant graph, a harmonious graph, a felicitous graph, cordial graph, odd-graceful graph, polychrome graph and on strongly c-harmonious graph, each with a given property, leading to prove the NP-completeness of some parameters like chromatic number, clique number, domination number and independence number of these graphs.

Key Words: Harmonious graph, Elegant graph, Felicitous graph, Cordial graph, Odd-graceful graph, polychrome graph, strongly c-harmonious graph

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1 Introduction.

For standard terminology and notation in graph theory not given here, the reader may refer to Harary (1969). In this paper, by a *graph* we shall mean a finite undirected graph without loops or multiple edges. For aspects of computability, we refer the reader to Grey

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and Johnson (1979).

Graph labellings, where the vertices or edges are assigned elements of a given set or subsets of a given set (Acharya, 1983, 2001) subject to certain conditions, have often been motivated by practical considerations such as coding, X-ray crystallography, radar tracking, remote control, radio-astronomy, communication networks, network flows. Their theoretical applications too are numerous, not only within the theory of graphs but also in other areas of mathematics such as combinatorial number theory, linear algebra and group theory admitting a given type of labelling (*see* Gallian, 2005).

They are also of interest on their own right due to their abstract mathematical properties arising from various structural considerations of the underlying graphs. An enormous body of literature has grown around the theme. Over the past 30 years or so, more than 650 potential papers on various graph labelling methods have been published, identifying several classes of graphs admitting a given type of labelling (*see* Gallian, 2005). One of the intriguing problems on graphs admitting most of these labellings is to find 'good' characterizations of such graphs. While considering such problems, it is often felt that their complexity might be due to the possibility that the sets used to label the elements of a graph are mathematically more rich and deeper in their properties than the graphs that admit specific type of labellings.

A wide variety of commonly encountered combinatorial problems, such as the problem of the determination of the *clique number* of a graph, are now known to be NP-complete. Many graph theory problems have been shown to be NP-complete by Graham & Sloane (1980). The collection of such problems continues to grow almost daily. Indeed, the NP-completeness problems are now so pervasive that it is important for any one concerned with the computational aspects of these fields to be familiar with the meaning and implications of this concept. Theorems on embedding an arbitrary graph with a given property $\mathcal{P}$ in a graph admitting a specific type of labellings and having the property $\mathcal{P}$ can indeed be used to prove the NP-completeness of the determination of the parameters like the *chromatic number* χ, the *clique number* ω, the *domination number* γ and the *independence number* β_0 of graphs with the property $\mathcal{P}$. For instance, Acharya *et al.*, (2006) have proved that the problems of determining the chromatic number, the clique number, the domination number and the independence number of a *graceful graph* (*cf.*: Rosa, 1967; Golomb, 1972) are NP-complete.

In this paper, we establish the embedding theorems on the harmonious graphs, the elegant graphs, the felicitous graphs, the cordial graphs, the odd-graceful graphs, the polychrome graphs and the strongly c-harmonious graphs, each 'loaded' with a given property $\mathcal{P}$.

Nonstandard definitions of concern to us in this paper along with pertinent remarks and elementary observations are given below.

Definition 1. (Graham & Sloane, 1980): *A graph G with q edges, $q \geq 1$, is* harmonious *if there is an injection f from the vertices of G to the group of integers modulo q such that when each edge xy is assigned the label $(f(x)+f(y))(mod\ q)$, the resulting edge labels are all distinct; f is then a* harmonious labelling *of G.*

Liu and Zhang (1993) proved that *every graph is a subgraph of a harmonious graph.* Determining whether a graph has a harmonious labelling was shown to be NP-complete by Anuparajita *et al.* (2001).

Definition 2. (Chang *et al.*, 1981): *An* elegant labelling *f of a graph G with q edges is an injective function from the vertices of G to the set $\{0, 1, \ldots, q\}$ such that when each edge xy is assigned the label $(f(x)+f(y))(mod\ q+1)$ the resulting edge labels are all distinct positive integers; if G admits such a labelling, then G is an* elegant graph.

Chang *et al.* (1981) proved that *every graph is a subgraph of an elegant graph.* Here onwards, the term *embedding* shall mean a mapping ς of the vertices of G into the set of vertices of a graph H such that the subgraph induced by the set $\{\varsigma(u) : u \in V(G)\}$ is isomorphic to G; for all practical purposes, we shall assume then that G is indeed a subgraph of H.

The following link between harmonious and elegant graphs has been discovered recently.

Theorem 3. (Acharya *et al.*, 2006): *Every elegant graph can be embedded into a harmonious graph H with* $|V(H)| \leq |V(G)| + 1$.

Definition 4. (Lee *et al.*, 1991): *An injective function f from the vertices of a graph G with q edges,* $q \geq 1$ *to the set* $\{0, 1, \ldots, q\}$ *is* felicitous *if the edge labels induced by* $(f(x) + f(y))(mod\ q)$ *for each edge xy are all distinct.*

Balakrishnan & Sampath Kumar (1994) proved that every graph is a subgraph of a felicitous graph. To the best of our knowledge the following observation is new.

Observation 5. *Every harmonious labelling of a harmonious graph is felicitous too; hence, every harmonious graph is felicitous.*

Definition 6. (Cahit, 1987): *Let f be a function from the vertices of G to the set* $\{0, 1\}$ *and for each edge xy assign the label* $|f(x) - f(y)|$. *Then f is a* cordial *labelling of G and G is a* cordial graph *if the number of vertices labelled* 0 *and the number of vertices labelled* 1 *differ by at most* 1, *as also the number of edges labelled* 0 *and the number of edges labelled* 1 *differ at most by* 1.

Cairnie & Edwards (2000) proved that deciding whether a graph admits a cordial labelling is NP-complete.

Definition 7. (Gnanajothi, 1991): *A graph G with q edges is* odd graceful *if there is an injection f from* $V(G)$ *into* $\{0, 1, \ldots, 2q - 1\}$ *such that, when each edge xy is assigned the label* $|f(x) - f(y)|$, *the resulting set of edge labels is* $\{1, 3, \ldots, 2q - 1\}$.

Definition 8. (Chang, 1981): *An injective labelling f of a graph G with q vertices is defined to be strongly* c-harmonious *or sequential if the vertex labels are from* $\{0, 1, \ldots, q\}$ *and the edge labels induced by* $f(x) + f(y)$ *for each edge xy are* $c, c+1, \ldots, c+q-1$. *Grace (1982) called such a labelling* sequential.

Definition 9. (Valentin, 1999): *For a graph* $G = (V, E)$ *and an additive Abelian group H a* polychrome labelling *of G by H is a bijection f from V to H such that the edge labels induced by* $f^+(uv) = f(u) + f(v)$, $uv \in E$, *are all distinct.*

2 Results on number labellings

We shall now embark on a study of embedding an arbitrary graph with a given property $\mathcal{P}$ in a graph admitting a specific type of labellings and having the property $\mathcal{P}$. In this context, first observe that in view of Theorem 3 and Observation 5 it is enough if we embed an arbitrary graph into an elegant graph. That this can be done has already been established as stated in Gallian (2005). Yet, our proof of this result has the potential to be valid when loaded with certain main-stream graph properties and hence is produced below.

Theorem 10. *Every (p,q)-graph G can be embedded in a connected elegant graph H with 2^{p+1} edges and $2^{p+1}-q+1$ vertices.*

Proof. Let G be a graph with $V(G) = \{u_1, u_2, \ldots, u_p\}$. We will embed the graph G in a graph H with $|V(H)| = 2^{p+1} - q + 1$ and $|E(H)| = 2^{p+1}$. Consider the set $Z_{2^{p+1}}$ of integers addition modulo 2^{p+1}. Label the vertices u_i by 2^{i-1} where $1 \leq i \leq p$. Then, the edge labels of G are $2^{i-1} + 2^{j-1}$ where $1 \leq i \leq j \leq p$, whenever u_iu_j is an edge. All the edge labels so obtained are clearly distinct. Now, introduce $p+2$ vertices $v_1, v_2, \ldots, v_{p+2}$ and label them by $0, 2^p, 2^p+1+2^0, 2^p+1+2^1, \ldots, 2^p+1+2^{p-1}$ respectively. Join v_1 to v_k where $2 \leq k \leq p+2$.

If G is connected, join v_1 to u_1 and v_2 to all v_k, $4 \leq k \leq p+2$.

If G is disconnected and has t components, say $C_1, C_2, \ldots, C_t$, join v_1 to exactly one vertex u_{j_i} of C_i, for each i, $1 \leq i \leq t$, where the indices j_i run through the labels of the vertices in the component C_i, v_1 and v_2 to each v_k, $3 \leq k \leq p+2$, $k \neq j_i + 2$, $1 \leq i \leq t$. Then, the new edge labels are $2^0, 2^1, \ldots, 2^{p-1}, 2^p, 2^p+2^0, 2^p+2^1, \ldots, 2^p+2^{p-1}$ are all distinct and are distinct from edge labels of G. The elements of $Z_{2^{p+1}}$ which are not edge labels so far are all numbers not of the form $2^i + 2^j$ where $0 \leq i < j \leq p$, and all numbers of the form $2^k + 2^l$ where u_{k+1} and u_{l+1}, $0 \leq k < l < p$ are not adjacent in G. Clearly, these numbers are not vertex labels so far and they are $2^{p+1} - q(G) - 1 - 2p$ in number. Now, introduce $2^{p+1} - q(G) - 1 - 2p$ new vertices and label them by the elements of $Z_{2^{p+1}}$ which are not edge labels so far and 2^{p+1} except 0 in an injective manner and join them to the vertex v_1 whose label is 0. Here the new graph H with 2^{p+1} edges and $2^{p+1} - q + 1$ vertices is elegant and G is an induced subgraph of H. □

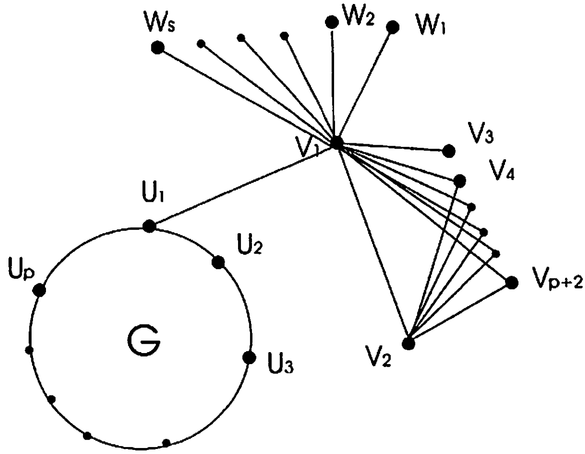


Figure 1:

From the proof of Theorem 10, the following results are evident.

Corollary 11. *If G is a planar elegant graph, so is H.*

Corollary 12. *If the chromatic number, clique number and independence number of G is ≥ 3, then the chromatic number $\chi(H) = \chi(G)$, the clique number $\omega(H) = \omega(G)$ and the independence number $\beta_0(H) = \beta_0(G) + 2^{p+1} - q - 1$. Therefore, the problems of the determination of the chromatic number, the clique number and the independence number of a connected elegant graph, when each of these numbers is at least three, are NP-complete.*

Corollary 13. *In the theorem, if we avoid the edge between v_1 and u_1 and join v_2 to v_3, we get a disconnected elegant graph H^* for which G is an induced subgraph. Hence, if $\gamma(G) \geq 3$, then the domination number $\gamma(H^*) = \gamma(G) + 1 \geq 4$. Therefore, the problem of the determination of the domination number of a elegant graph, whose domination number is at least three, is NP-complete.*

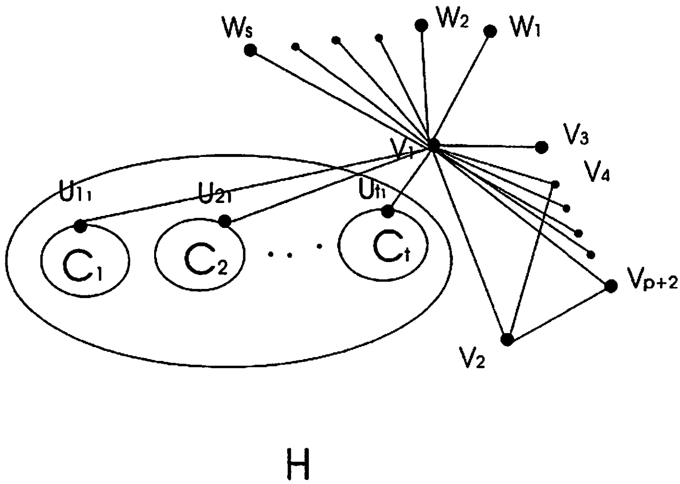


Figure 2:

Corollary 14. *Every (p, q)-graph G can be embedded in a connected harmonious graph H with 2^{p+1} edges and $2^{p+1} - q$ vertices.*

Corollary 15. *If G is a planar harmonious graph, so is H.*

Corollary 16. *If the chromatic number, clique number and independence number of G is at least three, then the chromatic number $\chi(H) = \chi(G)$, the clique number $\omega(H) = \omega(G)$ and the independence number $\beta_0(H) = \beta_0(G) + 2^{p+1} - q - 3$. Therefore, the problems of the determination of the chromatic number, the clique number and the independence number of a connected harmonious graph, whenever each of them is required to be at least three, are NP-complete.*

Corollary 17. *In the corollary 13, if we do not introduce the edge between v_1 and u_1 and instead join v_1 to v_3, we get a disconnected harmonious graph H^* for which G is an induced subgraph. Hence, if $\gamma(G) \geq 3$, then the domination number $\gamma(H^*) = \gamma(G) + 1 \geq 4$. Therefore, the problem of the determination of the domination number of a harmonious graph, whose domination number is at least three, is NP-complete.*

Observation 18. *Every harmonious graph is a felicitous graph.*

Corollary 19. *Every (p, q) graph G can be embedded in a connected felicitous graph L with 2^{p+1} edges and $2^{p+1} - q$ vertices.*

Corollary 20. *If G is a planar felicitous graph, so is L.*

Corollary 21. *If the chromatic number, clique number and independence number of G is at least three, then the chromatic number $\chi(L) = \chi(G)$, the clique number $\omega(L) = \omega(G)$, and the independence number $\beta_0(L) = \beta_0(G) + 2^{p+1} - q - 3$. Therefore, the problems of the determination of the chromatic number, the clique number and the independence number of a connected felicitous graph, whenever each of them is required to be at least three, are NP-complete.*

Corollary 22. *In the corollary 13, if we do not introduce the edge between v_1 and u_1 and instead join v_1 to v_3, we get a disconnected felicitous graph L^* for which G is an induced subgraph. Here, if $\gamma(G) \geq 3$, then domination number $\gamma(L^*) = \gamma(G) + 1$. Therefore, the problem of the determination of the domination number of a felicitous graph is NP-complete.*

We now study two different embeddings of a graph into a cordial graph and study NP-completeness of the problems of the determination of the clique number, the independence number, the domination number and the chromatic number of a cordial graph.

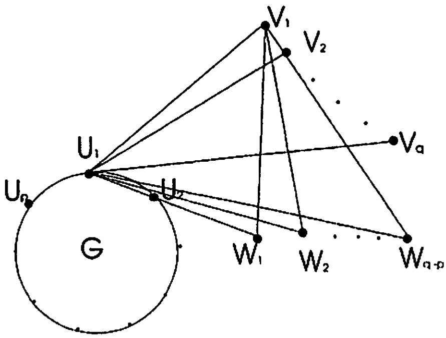
Theorem 23. *Every connected graph can be embedded into a connected cordial graph.*

Proof. Let G be a (p, q)-graph with $V(G) = \{u_1, u_2, \ldots, u_p\}$.

Case 1: $q > p$.

Label u_i by 0, for some i, $1 \leq i \leq p$. Add $2q - p$ new vertices $v_1, v_2, \ldots, v_q$, $w_1, w_2, \ldots, w_{q-p}$. Label v_j, $1 \leq j \leq q$ by 1 and w_l, $1 \leq l \leq q - p$ by 0. Fix one vertex of G say u_1. Join u_1 to v_j, for $1 \leq j \leq p$ and to w_l, $1 \leq l \leq q - p$. Join w_l to the vertex v_1. The new graph H is a cordial graph with q vertices of label 0, q vertices of label 1, $2q-p$ edges of label 0 and $2q-p$ edges of label 1.

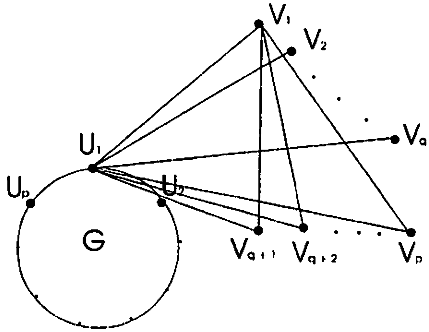
Case 2: $q < p$.



H

Figure 3:

Let G be a (p,q)-graph with $V(G) = \{u_1, u_2, \ldots, u_p\}$. label u_i by 0, for some i, $1 \leq i \leq p$. Add p new vertices $v_1, v_2, \ldots, v_q, v_{q+1}, \ldots, v_p$ and label them by 1. Fix one vertex of G say u_1. Join u_1 to v_j, for $1 \leq j \leq p$, and join v_1 to v_k, $q+1 \leq k \leq p$. The new graph H is a cordial graph with number of vertices with label 0, number of vertices with label 1, number of edges with label 0 and number of edges with label 1 are all equal to p. Clearly G is an induced subgraph of H and hence the theorem follows. □



H

Figure 4:

Corollary 24. *If G is planar cordial graph, so is H.*

Corollary 25. *If the chromatic number or the clique number of G is at least three, then the chromatic number $\chi(H) = \chi(G)$ and the clique number $\omega(H) = \omega(G)$ as the case may be. Therefore, the problems of the determination of the chromatic number and the clique number of a connected cordial graph, whenever they are at least three, are NP-complete.*

To prove that the determination of some more parameters of cordial graphs is NP-complete, we will give another embedding as follows.

Theorem 26. *Every graph can be embedded into a cordial graph.*

Proof. Let G be a (p,q)-graph with $V(G) = \{u_1, u_2, \ldots, u_p\}$.
Case 1: $q > p$.
Label u_i by 0, for some i, $1 \leq i \leq p$.
Add $2q-p$ new vertices $v_1, v_2, \ldots, v_q, w_1, w_2, \ldots, w_{q-p}$. Label v_j, $1 \leq j \leq q$ by 1 and w_l, $1 \leq l \leq q-p$ by 0. Join the vertex w_1 to v_j, for $1 \leq j \leq q$. Then the new graph H is a cordial graph with q vertices of label 0, q vertices of label 1, q edges of label 0 and q edges of label 1. Note that the number of isolated vertices is this construction is $q-p-1$.

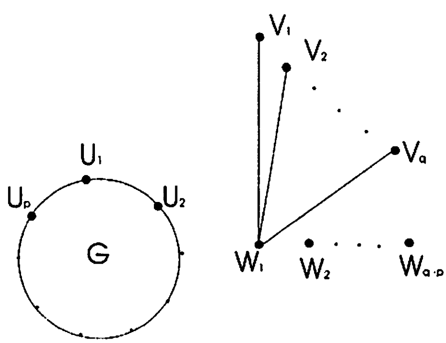


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Figure 5:

Case 2: $q < p$.
Let G be a (p,q)-graph with $V(G) = \{u_1, u_2, \ldots, u_p\}$. Label u_i by 0,

for some i, $1 \leq i \leq p$. Add $p+2$ new vertices $v_1, v_2, \ldots, v_{p+2}$. Label v_1 by 0 and v_i, $2 \leq i \leq p+2$ by 1. Join v_1 to v_j, $2 \leq j \leq q+1$. The new graph H is a cordial graph with the number of vertices with label 0 and the number of vertices with label 1 both equal to $p+1$. Number of edges with label 0 and number of edges with label 1 are both equal to q. Note that the number of isolated vertices in this case is $p-q+1$. Clearly, G is an induced subgraph of H. Hence the theorem. □

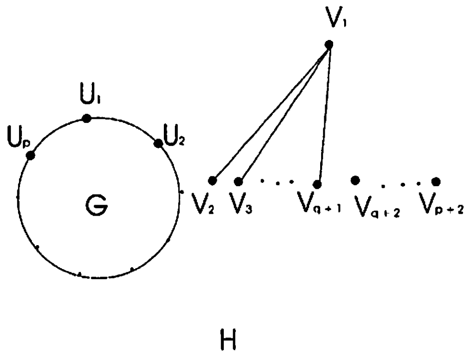


Figure 6:

Corollary 27. *If G is a triangle free cordial graph, then so is H.*

Corollary 28. *If the domination number or independence number of G is at least three, then the domination number $\gamma(H) = \gamma(G)+p-q$ if $q > p$ and $\gamma(H) = \gamma(G)+p-q+2$ if $q < p$, the independence number $\beta_0(H) = \beta_0(G) + 2q - p - 1$ if $q > p$ and $\beta_0(H) = \beta_0(G) + p + 1$ if $q < p$. Therefore, the problems of determining the domination number and the independence number of a cordial graph under the said constraints are NP-complete.*

In the embeddings of graphs with labellings dealt above, there are vertices of small degrees in H. The following theorem proves the existence of a connected cordial graph with minimum degree greater than a pre-assigned number.

Theorem 29. *For any graph G and for any θ, an arbitrary integer, there exists a connected cordial graph C with minimum degree $\delta(C) > \theta$.*

Proof. Let G be a graph. Theorem 26 can embed G into a connected cordial graph H. We can obtain connected cordial graphs H_k, $k \geq 1$ as follows.

Put $H_1 = H + K_1$, where $V(K_1) = \{v_1\}$, which was labelled by 0; $H_2 = H_1 + K_1$, where $V(K_1) = \{v_2\}$, with v_2 labelled 1; $H_{3'} = H_2 + K_1$, where $V(K_1) = \{v_3\}$, with v_3 labelled 0. Now, obtain H_3 by removing the edge between v_2 and v_3. Let $H_4 = H_3 + K_1$, where $V(K_1) = \{v_4\}$, with v_4 labelled 1 ; $H_{5'} = H_4 + K_1$, where $V(K_1) = \{v_5\}$, with v_5 labelled 0. Now, to obtain H_5 remove the edge between v_4 and v_5. Proceeding like this, we get connected cordial graphs H_k, $k \geq 1$ having minimum degree 1 more than the graph H_{k-1}. We can thus choose an integer l so that $H_l = C$ with minimum degree $> \theta$. This completes the proof of the theorem. □

It is easy to see that every odd-graceful graph is bipartite. Further, Gallian (2005) quotes Barrientos's conjecture that *every bipartite graph is odd-graceful.* In the absence of a proof of this conjecture, a study of embedding an arbitrary bipartite graph into an odd graceful graph might be of some value as it does lead to decide about the NP-completeness of the problems of determining the independence number and the chromatic number of an odd-graceful graph.

Theorem 30. *Every bipartite graph can be embedded into an odd-graceful graph.*

Proof. Let $G = (V_1, V_2)$ be a bipartite graph with bipartition V_1 and V_2 where $|V_1| = m$ and $|V_2| = n$. Without loss of generality, assume that $m \geq n$. Let $V_1 = \{u_1, u_2, \ldots, u_m\}$ and $V_2 = \{w_1, w_2, \ldots, w_n\}$. Label the vertex u_i, $1 \leq i \leq m$ by $4^i + 4^{i-1} + \cdots + 4^0$ and w_l, $1 \leq l \leq n$ by 4^l. Then, all edge labels of G are distinct. Define the set S as the set of all even integers less than $4^m + 4^{m-1} + \cdots + 4^0$ and not equal to 4^l, $0 \leq l \leq n$. Then $|S| = \frac{4^m + 4^{m-1} + \cdots + 4^1}{2} - n = \frac{4(4^m - 1)}{6} - n$. Now, add $|S| + 2$ new vertices $v_1, v_2, \ldots, v_{|S|+2}$. Label v_1 by 1 and v_2 by 0 and all other v_k, $3 \leq k \leq |S| + 2$ by elements from S. Join v_1 to all v_k, $2 \leq k \leq |S| + 2$.Under the condition that if x is an edge label in G then we do not join the vertex $v_j$labelled by $x + 1$ to v_1. To get edges with label $4^l - 1$, join the vertex corresponding to the label $4^l + 4^{l-1} + \cdots + 4^1$ to the vertex u_{l-1} which was labelled by $4^{l-1} + 4^{l-2} + \cdots + 4^0$.The new graph H is an odd-graceful graph. □

Corollary 31. *If G is planar odd-graceful graph, so is H.*

Corollary 32. *Every tree can be embedded in an odd-graceful tree.*

Corollary 33. *If the chromatic number or independence number of G is at least three, then accordingly the chromatic number $\chi(H) = \chi(G)$ or the independence number $\beta_o(H) = \beta_o(G) + |S|$. Therefore, the problems of the determination of the chromatic number and the independence number of an odd-graceful graph, whenever the corresponding invariant is at least three, are NP-complete.*

We now study the embedding of a graph into a polychrome graph with the Abelian group H as the set of all integers modulo 2^{p+1} and study NP-completeness of the problems of the determination of the clique number, the independence number, the domination number and the chromatic number of a polychrome graph.

Theorem 34. *Every graph can be embedded in a connected polychrome graph.*

Proof. Let G be a (p, q) graph. Consider the group H as the set of all integers modulo 2^{p+1}. Then $H = Z_{2^{p+1}}$. Let $V(G) = \{u_1, u_2, \ldots, u_p\}$. Label u_i by 2^{i-1}, $1 \leq i \leq p$. Let $S_1 = \{$elements of$H$ of the form2^i, $0 \leq i \leq p\}$ $S_2 = \{$elements of$H$ of the form $2^p + 2^j$, $0 \leq j \leq p-1\}$, $S_3 = \{$elements of$H$of the form $2^l + 2^k, 2^p + 2^l + 2^k$ where $u_{l+1}u_{k+1}$ is an edge of $G\}$. Then $|S_1| = p + 1$, $|S_2| = p$, $|S_3| = 2q$. Define $S = Z_{2^{p+1}} - \{0\} - \{S_1 \cup S_2 \cup S_3\}$. Then $|S| = 2^{p+1} - 2p - 2q - 2$. Now, introduce $2^{p+1} - p$ new vertices $v_1, v_2, \ldots, v_{2^{p+1}-p}$. Label v_1 by 0, v_2 by 2^p, v_t, $3 \leq t \leq p+2$ by elements of S_2, v_r, $p+3 \leq r \leq p+2+2q$ by elements of S_3 and v_m, $p+2q+3 \leq m \leq 2^{p+1} - p$ by elements of S. Join v_1 to u_i, $0 \leq i \leq p-1$. Join v_1 to v_i, $2 \leq i \leq p+2$. Join v_2 to v_i, $p+3 \leq i \leq p+2+2q$. Join v_1 to v_i, $p+2q+3 \leq i \leq 2^{p+1} - p$. The new graph M has 2^{p+1} vertices and $2^{p+1} + q - p - 1$ edges. Here $V(M) = Z_{2^{p+1}}$ and edge labels of M are all distinct. □

Corollary 35. *If the chromatic number, the clique number,the domination number or the independence number of G is at least three,*

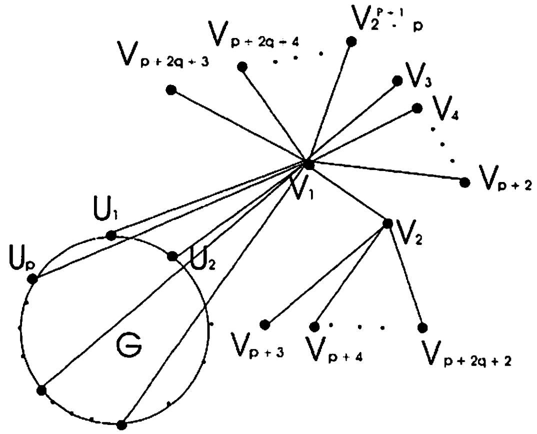


Figure 7:

then accordingly the chromatic number $\chi(H) = \chi(G) + 1$, *the clique number* $\omega(H) = \omega(G) + 1$, *the domination number*$\gamma(H) = \gamma(G) + 2$ *and the independence number* $\beta_0(H) = \beta_0(G) + 2^{p+1} - p - 2$. *Therefore, the problems of determining the chromatic number, the clique number, the domination number and the independence number of a connected polychrome graph, whenever the corresponding parameter is required to be at least three, are NP-complete.*

We now study the embedding of a graph into a *strongly c-harmonious graph* and study NP-completeness of the problems of the determination of the clique number, the independence number, the domination number and the chromatic number of a strongly c-harmonious graph.

Theorem 36. *Every connected* (p,q)*-graph can be embedded in a connected strongly c-harmonious graph, for some positive integer c.*

Proof. Let G be a (p,q)-graph with $V(G) = \{u_1, u_2, \ldots, u_p\}$. Let $c = 2^i + 1$. Label u_1 by 2^0, u_k by 2^{i+j}, $2 \le k \le p$, $0 \le j \le p-2$. Now, we will embed G into a graph H with $2^{i+p-1} - 1$ edges. Define $S_1 = \{2^k,\ 0 \le k \le i+p-2\} - \{2^k,\ i \le k \le i+p-2\}$, $S_2 = \{2^{k_1} + 2^{k_2},\ 0 \le k_2 < k_1 \le i+p-2\} - \{2^{k_1} + 2^{k_2}$, the number is an edge label in $G\}$ $S_l = \{2^{k_1}+2^{k_2}+\cdots+2^{k_l}, 0 \le k_l < k_{l-1} < \cdots < k_1 \le i+p-2\}$. Define $S = \cup_{n=1}^{l} S_n$. Here, $|S_1| = i-1$, $|S_2| = (i+p-1)C_2 - q$, $|S_3| = (i+p-1)C_3, \ldots, |S_{i+p-1}| = (i+p-1)C_{i+p-1}$. Then $|S| = 2^{i+p-1} - q - p - 1$. Let $\theta = \{$ all elements of S that are $\ge c\}$. Now, introduce $|S|+1$ new vertices, $v_1, w_1, w_2, \ldots, w_{|S-\theta|}, x_1, x_2, \ldots, x_{|\theta|}$. Label v_1 by 0, label x_i by elements from θ in an increasing and injective manner and label w_j by elements from $S - \theta$ in an increasing and injective manner. Join v_1 to u_j, $3 \le j \le p$ and v_1 to x_i, $1 \le i \le |\theta|$;Join u_1, u_2 to $x_{|\theta|}$ whenever $c = 2^i + 1$. Join v_1 to u_j, $2 \le j \le p$;v_1 to x_i, $1 \le i \le |\theta|$ and u_1 to $x_{|\theta|}$whenever $c < 2^i + 1$. Also join w_j, $1 \le j \le |S - \theta|$ to $x_{|\theta|}$. The new graph H has $2^{i+p-1} - q$ vertices and $2^{i+p-1} - 1$ edges, with edge values $c, c+1, \ldots, c + 2^{i+p-1} - 2$. Clearly, G is an induced subgraph of H. □

The following figure illustrates the above construction whenever $c = 2^i + 1$ and whenever $c < 2^i + 1$

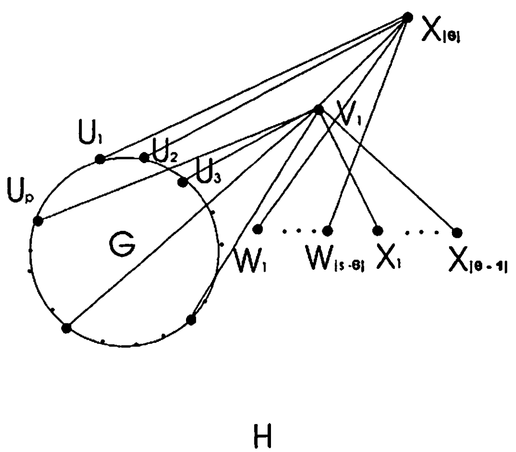
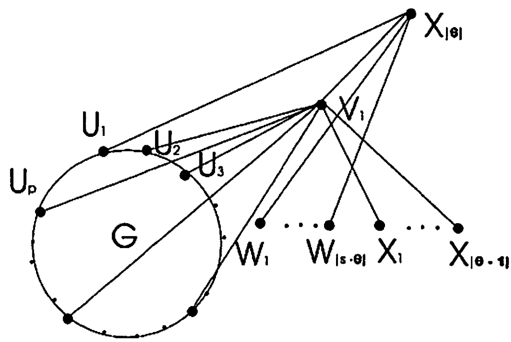


Figure 8:

Corollary 37. *If the chromatic number, clique number or the independence number of G is at least three, then accordingly, the chromatic number* $\chi(H) = \chi(G)$*, the clique number* $\omega(H) = \omega(G) + 2$*, and the independence number* $\beta_0 H) = \beta_0(G) + 2^{i+p+1} - q - p - 2$*. Therefore, the problems of the determination of the chromatic number, the clique number, and the independence number of a connected*



H

Figure 9:

strongly c-harmonious graph, whenever the corresponding parameter is at least three, are NP-complete.

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