Rupture Degree of Mesh and Binary Trees

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Abstract

A well-designed interconnection network makes efficient use of scarce communication resources and is used in systems ranging from large supercomputers to small embedded systems on a chip. This paper deals with certain measures of vulnerability in interconnection networks. Let G be a non-complete connected graph and for $S \subseteq V(G)$ let $\omega(G-S)$ and $m(G-S)$ denote the number of components and the order of the largest component in $G-S$ respectively. The vertex-integrity of G is defined as $I(G) = \min\{|S|+m(G-S) : S \subseteq V(G)\}$. A set S is called an I-set of G if $I(G) = |S|+m(G-S)$.The rupture degree of G is defined by $r(G) = \max\{\omega(G-S) - |S| - m(G-S) : S \subseteq V(G), \omega(G-S) \geq 2\}$.A set is called an R-set of G if $r(G) = \omega(G-S) - |S| - m(G-S)$.In this paper, we compute the rupture degree of complete binary trees, and a class of meshes. We also study the relationship between an I-set and an R-set and find an upper bound for the rupture degree of Hamiltonian graphs.

Keywords: vertex - toughness, rupture degree, binary trees, minimum vertex cover, network vulnerability.

1 Introduction

Interconnection network plays a central role in determining the overall performance of a multicomputer system. They are used in systems ranging from large supercomputers to small embedded systems-on-a-chip (SoC) The knowledge and the ability to maintain a certain level of sustainable computational power is very important in the design of such networks. Thus the study of system reliability in general and network reliability in particular is critical to achieving performance goals. Among the relevant issue of importance we are particularly interested in one of vulnerabilities.

In an analysis of the vulnerability of a communication network to disruption, two qualities that come to mind are the number of elements that are not functioning and the size of the largest remaining sub-network within which mutual communication can still occur. In particular, in an adversarial relationship, it would be desirable for an opponent's network to be such that the two qualities can be made to be simultaneously small.

Thus, communication networks must be constructed to be as stable as possible, not only with respect to the initial disruption, but also with respect to the possible reconstruction of the network. Many graph theoretical parameters have been used in the past to describe the stability of communication networks. Most notably, the vertex-connectivity and the edge-connectivity have been frequently used. The difficulty with these parameters is that they do not take into account what remains after the graph is disconnected.

Consequently, a number of other parameters have been introduced in an attempt to overcome this difficulty, including toughness and edge-toughness [6], integrity and edge-integrity [2], tenacity and edge-tenacity [9]. Unlike the connectivity measures, each of these parameters shows not only the difficulty to break down the network but also the damage that has been caused.

Let G be a finite simple graph with vertex set $V(G)$ and edge set $E(G)$. For $S \subseteq V(G)$, let $\omega(G-S)$ and $m(G-S)$ respectively, denote the number of components and the order of a largest component in $G-S$. A set $S \subseteq V(G)$ is a cut set of G, if either $G-S$ is disconnected or $G-S$ has only one vertex. The Vertex-Connectivity $\kappa(G)$ of a non-complete graph G is defined as $\min\{|S| : S \subseteq V(G)$ is a cutset of $G\}$.The vertex toughness $\tau(G)$ of a non-complete graph G is defined as

$$\min\left\{\frac{|S|}{\omega(G-S)} : S \subseteq V(G) \text{ is a cutset of } G\right\}$$

Toughness of a complete graph is defined to be ∞.

For a subset S of $V(G)$, let $I(G,S) = |S| + m(G-S)$. Then the vertex-integrity $I(G)$ of G is defined as $I(G) = \min\{I(G,S)\}$ where the minimum is taken over all subsets S of $V(G)$ A set $S \subseteq V(G)$ is called an $I-$set if $I(G) = I(G,S)$. The Vertex-Tenacity $T(G)$ of G [9] is defined as $\min \frac{|S|+m(G-S)}{\omega(G-S)}$ where the minimum is taken over all cutsets S of G.

For any cutset S of G, let $r_s(G) = \{\omega(G-S) - |S| - m(G-S) : S \subseteq V(G), \omega(G-S) \geq 2\}$. Then the rupture degree $r(G)$ of a non-complete connected graph G is defined by $r(G) = \max\{r_s(G)\}$where the maximum is taken over all the cutsets S of G. In particular, the rupture degree of a complete graph K_n is defined to be $1-n$.

2 An Overview of the Paper

In the integrity model, the basic assumption is that some intelligent enemy is trying to disrupt the network by destroying its elements. The cost on his part is measured by the number of elements he would destroy, and his success in incapacitating the network is measured by the order (i.e. number of nodes) of the largest connected component in the remaining network. The enemy of course wants both to be small. Therefore, the minimum attainable sum of these two quantities is considered as a measure of vulnerability of the network. In [2] Barefoot et al proved that the integrity of - the complete graph K_p is p; the star $K_{1,n}$ is 2; the cycle C_p is $2\sqrt{p}-1$; the path Pp is $2\sqrt{p+1}-2$ and the complete bipartite graph $K_{m,n}$ is $[1+\min\{m,n\}]$. Clark et al [10] proved that the determination of the integrity of a graph is NP-complete.

The concept of rupture degree was first introduced in [12], and the rupture degrees of several classes of graphs were determined including certain join graphs. The rupture degree can be regarded as the additive dual of vertex-tenacity. It is proved that the rupture degree is a better parameter of vulnerability than the vertex – tenacity [12]. Computing the rupture degree of a graph is NP-complete in general [11]. Hence, it becomes an interesting question to calculate the rupture degrees for some special classes of interesting or practically useful graphs.

In this paper, we compute the rupture degree of binary trees and a class of meshes. We also obtain an upper bound for the rupture degree of Hamiltonian graphs. ¤

3 Rupture degree of some graphs

In the following theorem, we obtain an upper bound for the rupture degree of Hamiltonian graphs.

Theorem 1 If G is Hamiltonian then $r(G) \leq -1$
Proof. If G is a Hamiltonian graph then, $\omega(G-S) \leq |S|$ for any cutset S of G[3]. Also $m(G-S) \geq 1$ for any S. Hence, $\omega(G-S)-|S|-m(G-S) \leq -1$ for any cutset S of G. ■

The following theorem explores the conditions when an integrity set of G becomes a rupture set.

Theorem 2 Let $S \subseteq V(G)$ be the minimum vertex cover of a simple connected graph G of n vertices such that $|S| = r$. If S is an I - set , then it is also an R-set and $r(G) = n-2r-1$.
Proof. Since S is a minimum vertex cover of G, $m(G-S) = 1$, $\omega(G-S) = n-r$. Hence, $\omega(G-S)-|S|-m(G-S) = n-2r-1$. Now, let $\hat{S}$ be any other

cutset of G. Then, $\omega(G-\hat{S}) \leq n-r$. Also, $|\hat{S}|+m(G-\hat{S}) \geq |S|+m(G-S)$. Hence, $\omega(G-\hat{S})-|\hat{S}|-m(G-\hat{S}) \leq \omega(G-S)-|S|-m(G-S)$. ■

The following two theorems gives the conditions which facilitates the computation of the rupture degree of a supergraph when the rupture degree of a subgraph is known. Using these two theorems, we compute the rupture degree of a class of mesh and binary trees in the following two sub-sections.

Theorem 3 Let $G = (V, E)$ be a graph and S be an R-set of G. Let $S^* = \{v_1, v_2, v_3, ...v_n\}$ be a set of independent vertices and $\hat{G} = [V \cup S^*]$. If $N(S^*) \subseteq S$, then S is an R-set of $\hat{G}$ and $r(\hat{G}) = r_s(G) + n$.

Proof. Let $\hat{S}$ be any other cutset of $\hat{G}$. We have the following three cases.

Case 1. $S^* * \hat{S}$ and $N(S^*) \subset \hat{S}$

$\hat{S}$ is also a cutset of G and since $N(S^*) \subset \hat{S}$, $\omega(\hat{G}-\hat{S}) = \omega(G-\hat{S})+n$ and $m(\hat{G}-\hat{S}) \geq m(G-\hat{S})$.Hence, $r_{\hat{S}}(\hat{G}) = \omega(\hat{G}-\hat{S})-|\hat{S}|-m(\hat{G}-\hat{S}) \leq \omega(G-\hat{S})+n-|\hat{S}|-m(G-\hat{S}) = r_{\hat{S}}(G)+n \leq r_s(G)+n = r_s(\hat{G})$.

Case 2. $S^* * \hat{S}$ and $N(S^*) * \hat{S}$.

$\hat{S}$ is also a cutset of $\hat{G}$ and since $N(S^*) * \hat{S}$, $\omega(\hat{G}-\hat{S}) < \omega(G-\hat{S})+n$, and $m(\hat{G}-\hat{S}) \geq m(G-\hat{S})$.Hence, $r_{\hat{S}}(\hat{G}) = \omega(\hat{G}-\hat{S})-|\hat{S}|-m(\hat{G}-\hat{S}) \leq \omega(G-\hat{S})+n-|\hat{S}|-m(G-\hat{S}) = r_{\hat{S}}(G)+n \leq r_s(G)+n = r_s(\hat{G})$.

Case 3. $S^* \subset \hat{S}$

Then $S^{**} = \hat{S}-S^*$ is a cutset for $G.r_{\hat{S}}(\hat{G}) = \omega(\hat{G}-\hat{S})-|\hat{S}|-m(\hat{G}-\hat{S}) = \omega(G-S^{**})-(|S^{**}|+n)-m(G-S^{**}) = r_{s**}(G)-n < r_s(G)$. ■

Theorem 4 Let $G = (V, E)$ be a graph with S as the R-set of G/v where $v \in V$. Suppose $N(v) \subseteq G-S$, then $S^* = S \cup \{v\}$ is an R-set of G and $r(G) = r(G-v)-1$.

Proof. Let $\hat{G} = G-v$ and $\hat{S}$ be any other cutset of G. We claim that $r_{s*}(G) \geq r_{\hat{S}}(G)$. We consider the following two cases.

Case 1. $v \in \hat{S}$

Let $\acute{S} = \hat{S}-\{v\}$.Then $\acute{S}$ is a cutset for G..

$r_{\hat{S}}(G) = \omega(G-\hat{S})-|\hat{S}|-m(G-\hat{S}) \leq \omega(\hat{G}-\acute{S})+n-(|\acute{S}|+1)-m(\hat{G}-\acute{S}) = r_{\acute{S}}(\hat{G})-1 \leq r_s(\hat{G})-1 = r_{s*}(G)$.

Case 2. $v \notin \hat{S}$

$\hat{S}$ is a cutset for $\hat{G}$. Since S is the only rupture set of $\hat{G}$, $r_S(G) < r_s(\hat{G}) = r_{s*}(G)+1$.Hence, $r_{s*}(G) \geq r_{\hat{S}}(G)$. ■

3.1 Mesh

Let P_n be a path on n vertices. Then an $m \times n$ mesh is defined to be $P_m \times P_n$ denoted by $M_{m \times n}$.

Theorem 5 The rupture degree of $M_{m \times n}$ is either 0 or -1

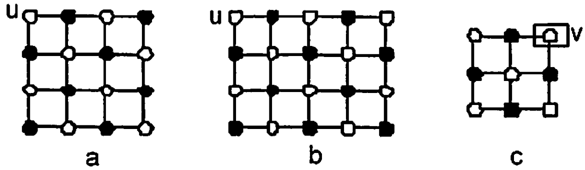

Figure 1: a. $M_{4\times 4}$ b. $M_{4\times 5}$ c. $M_{3\times 3}$

Proof. We consider two cases.

Case 1. At least one of m, n is even. Without loss of generality, let m be even. Consider a Hamiltonian cycle in $M_{m\times n}$beginning and ending with vertex u in the first row, first column position of $M_{m\times n}$. Let S be the set of alternate vertices on this Hamiltonian cycle beginning from a vertex adjacent to u (see Figure 1). This is possible since the Hamiltonian cycle is even. This set S has $n/2$ vertices and $r(M_{m\times n}, S) = -1$. By theorem 1. $r(M_{m\times n}) = -1$, when m is even.

Case 2. Now, consider $M_{m\times n}$ where both n, m are odd (see Figure1). By removing one corner-most vertex v (blocked by a square) the graph becomes Hamiltonian. As in Case 1, the set S of vertices shaded in black is an R-set of $(M_{m\times n} - v)$ and $r(M_{m\times n} - v) = -1$. We observe that $N(v) \subset S$ and hence by using theorem 3 with the case n = 1,$r(M_{m\times n}) = r(M_{m\times n} - v) + 1 = 0$. ■

So far, the rupture degree has been defined only for connected graphs. Now we extend the concept of a rupture set to disconnected graphs. If G is a disconnected graph with $H_1, H_2, H_3 ... H_n$ as its components then, we define the rupture set of G as the union of the rupture sets of $H_1, H_2, H_3 ... H_n$. We now define the rupture degree of a disconnected graph.

Definition 1 Let G be a disconnected graph with $H_1, H_2, H_3 ... H_n$ as its components. If $S_1, S_2, S_3 ... S_n$ are the rupture sets of $H_1, H_2, H_3 ... H_n$ respectively, then the rupture degree of G is defined as $r(G) = \{\omega(H_1 - S_1) + \omega(H_2 - S_2) + \omega(H_3 - S_3) + ...\omega(H_n - S_n) - (|S_1| + |S_2| + |S_3| + ..|S_n|) - \max\{m(H_1 - S_1), m(H_2 - S_2), m(H_3 - S_3), ...m(H_n - S_n)\}\}$

3.2 Complete Binary Tree

A complete binary tree of height h has $2^{h+1} - 1$ vertices and $2^{h+1} - 2$ edges.Let the root be at level 0 and S_t be the set of vertices of a complete binary tree at any one level say, t. Then S_t is an independent set and $|S_t| = 2t$.

Theorem 6 Let G be a complete binary tree of height h. Then

$$r(G) = \begin{cases} \frac{2^{h+1}-4}{3} & if \quad \text{h is odd} \\ \frac{2^{h+1}-2}{3} & if \quad \text{h is even} \end{cases}$$

Proof. One can easily verify that the rupture degree of the complete binary tree of height 1 is 0 and the rupture set is the root vertex at level zero. Similarly the rupture degree of the complete binary tree of height 2 is 2 and the rupture set is the set of vertices at level one. Now consider the binary tree of height 3 say B_3. The removal of its root vertex say, $\{v\}$disconnects the graph into two binary trees say, B_2 and $B_{2'}$ whose rupture sets are the level one vertices. The root vertex of B_3 is adjacent to the roots of B_2 and $B_{2'}$ and hence by using theorem 4, the rupture set of B_3 is the set of level 2 vertices $\cup\{v\}$ and $r(B_3) = 4$.

By a similar argument and using theorem 3 with the case $n = 1$, we find that the rupture degree of B_4 to be 10. Thus by using theorem 3 and 4 alternatively, we find that the rupture set of a binary tree of height h to be $S_{h-1} \cup S_{h-3} \cup S_{h-5} \cup .... S_3 \cup S_1$ if h is even and $S_{h-1} \cup S_{h-3} \cup S_{h-5} \cup .... S_2 \cup S_0$ if h is odd. ■

4 Conclusion

The rupture degree of a graph , to some extent, represents a trade-off between the amount of work done to damage the network and how badly the network is damaged. Hence, the rupture degree can be used to measure the vulnerability of networks. So clearly, it is of prime importance to determine this parameter for a graph. In this paper, we have obtained the upper bound for the rupture degree of Hamiltonian graphs to be -1 and the rupture degree of a class of mesh and binary trees. To make further progress in this direction, one could try to characterize the hamiltonian graphs with rupture degree -1. Also, determining the rupture degree of graphs like the butterfly, pyramid, prism, hybrid networks is under consideration. ¤

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