ON THE FEKETE-SZEGÖ PROBLEM FOR SOME SUBCLASSES OF ANALYTIC FUNCTIONS DEFINED BY CONVOLUTION

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ABSTRACT. In this present investigation, the authors obtain Fekete-Szegö's inequality for certain normalized analytic functions $f(z)$ defined on the open unit disk. As a special case of this result, Fekete-Szegö's inequality for a class of functions defined through fractional derivatives is obtained. The Motivation of this paper is to give a generalization of the Fekete-Szegö inequalities obtained by Srivastava and Mishra and Ma and Minda.

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1. INTRODUCTION

Let $\mathcal{A}$ denote the class of all *analytic* functions $f(z)$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (z \in \Delta := \{z \in \mathbb{C} : |z| < 1\}) \tag{1.1}$$

and $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of univalent functions. Let $\phi(z)$ be an analytic function with positive real part on Δ with $\phi(0) = 1$, $\phi'(0) > 0$ which maps the unit disk Δ onto a region starlike with respect to 1 which is symmetric with respect to the real axis. Let $S^*(\phi)$ be the class of functions in $f \in \mathcal{S}$ for which $\dfrac{zf'(z)}{f(z)} \prec \phi(z)$, $(z \in \Delta)$ and $C(\phi)$ be the class of functions in $f \in \mathcal{S}$ for which $1 + \dfrac{zf''(z)}{f'(z)} \prec \phi(z)$, $(z \in \Delta)$, where $\prec$ denotes the subordination between analytic functions. These classes were introduced and studied by Ma and Minda [7]. They have obtained the Fekete-Szegö inequality for the functions in the class $C(\phi)$. Since $f \in C(\phi)$ if and only if $zf'(z) \in S^*(\phi)$, we get the Fekete-Szegö inequality for functions in the class $S^*(\phi)$. Fekete-Szegö problem for different subclasses has been obtained earlier by Ravichandran *et al.* [9] and also by Shanmugam and Sivasubramanian [12]. For a brief history of Fekete-Szegö problem for the class of starlike, convex and close-to-convex functions, see the recent paper by Srivastava *et al.* [13] (see also the references cited by them).

For $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=2}^{\infty} g_n z^n \in \mathcal{A}$, the Hadamard product(or convolution product) is given by $(f * g)(z) := z + \sum_{n=2}^{\infty} a_n g_n z^n$. For various choices of $g(z)$ we get different operators and are listed below.

(1) For $g(z) = z + \sum_{n=2}^{\infty} \frac{(\alpha_1)_{n-1}(\alpha_2)_{n-1}, \dots, (\alpha_q)_{n-1}}{(\beta_1)_{n-1}(\beta_2)_{n-1}, \dots, (\beta_s)_{n-1}(1)_{n-1}} z^n$, we get the Dziok–Srivastava operator $H_{q,s}(\alpha)f(z)$ introduced by Dziok and Srivastava [4].

(2) For $g(z) = \phi(a, c, z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} z^n$, we get the Carlson-Shaffer operator $L(a, c)f(z)$ introduced by Carlson-Shaffer [1].

(3) For $g(z) = \frac{z}{(1-z)^{\lambda+1}}$, we get the Ruscheweyh operator $D^\lambda f(z)$ introduced by Ruschweyh [10].

(4) For $g(z) = z + \sum_{n=2}^{\infty} n^m z^n \quad (m \geq 0)$, we get the Sălăgean operator $D^m f(z)$ introduced by Sălăgean [11].

(5) For $g(z) = z + \sum_{n=2}^{\infty} \left(\frac{n+\lambda}{1+\lambda}\right)^k z^n \quad (\lambda \geq 0;\ k \in \mathbb{Z})$, we get the multiplier transformation $I(\lambda, k)$ introduced by Cho and Srivastava [3].

(6) For $g(z) = z + \sum_{n=2}^{\infty} n \left(\frac{n+\lambda}{1+\lambda}\right)^k z^n \quad (\lambda \geq 0;\ k \in \mathbb{Z})$, the multiplier transformation $I(\lambda, k)$ introduced by Cho and Kim [2].

Motivated essentially by the above works, we obtain the Fekete-Szegö inequality for the estimate $\frac{(f*g)(z)}{(f*h)(z)}$, where g and h are fixed functions such that $g_n > 0$, $h_n > 0$, with $g_n - h_n > 0$ where $h(z) = z + \sum_{n=2}^{\infty} h_n z^n \in \mathcal{A}$. For special choices of g and h we get all the estimates which we have mentioned earlier. Also, for various choices of $g(z)$ and $h(z)$ we get various subclasses of $\mathcal{A}$.

(1) For $g(z) = \frac{z}{(1-z)^2}$, and $h(z) = \frac{z}{(1-z)}$, $\frac{(f*g)(z)}{(f*h)(z)} \equiv \frac{zf'(z)}{f(z)}$.

(2) For $g(z) = \frac{z+z^2}{(1-z)^3}$, and $h(z) = \frac{z}{(1-z)^2}$, $\frac{(f*g)(z)}{(f*h)(z)} \equiv 1 + \frac{zf''(z)}{f'(z)}$.

In the present paper, we obtain the Fekete-Szegö inequality for functions in a more general class $M_{g,h}(\phi)$ of functions which we define below. The Motivation of this paper is to give a generalization of the Fekete-Szegö inequalities of Srivastava and Mishra [13].

Definition 1. *Let $\phi(z)$ be a univalent starlike function with respect to 1 which maps the open unit disk Δ onto a region in the right half plane and is symmetric with respect to the real axis, $\phi(0) = 1$ and $\phi'(0) > 0$. A function $f \in \mathcal{A}$ is in the* *class* $M_{g,h}(\phi)$ *if*

$$\frac{(f*g)(z)}{(f*h)(z)} \prec \phi(z) \quad (g_n > 0,\, h_n > 0,\, g_n - h_n > 0).$$

We remark here that the assumptions $g_n > 0$ and $h_n > 0$ are taken to make sure that the absolute value in our main results is non-negative.

To prove our main result, we need the following:

Lemma 1. [7] *If* $p_1(z) = 1 + c_1 z + c_2 z^2 + \cdots$ *is an analytic function with positive real part in* Δ, *then*

$$|c_2 - vc_1^2| \le \begin{cases} -4v+2 & \text{if } v \leqq 0 \\ 2 & \text{if } 0 \leqq v \leqq 1 \\ 4v-2 & \text{if } v \ge 1 \end{cases}$$

When $v < 0$ *or* $v > 1$, *the equality holds if and only if* $p_1(z)$ *is* $(1+z)/(1-z)$ *or one of its rotations. If* $0 < v < 1$, *then the equality holds if and only if* $p_1(z)$ *is* $(1+z^2)/(1-z^2)$ *or one of its rotations. If* $v = 0$, *the equality holds if and only if*

$$p_1(z) = \left(\frac{1}{2} + \frac{1}{2}\lambda\right)\frac{1+z}{1-z} + \left(\frac{1}{2} - \frac{1}{2}\lambda\right)\frac{1-z}{1+z} \quad (0 \leqq \lambda \leqq 1)$$

or one of its rotations. If $v = 1$, *the equality holds if and only if* p_1 *is the reciprocal of one of the functions such that the equality holds in the case of* $v = 0$.

Also the above upper bound is sharp, and it can be improved as follows when $0 < v < 1$:

$$|c_2 - vc_1^2| + v|c_1|^2 \leqq 2 \quad (0 < v \leqq 1/2)$$

and

$$|c_2 - vc_1^2| + (1-v)|c_1|^2 \leqq 2 \quad (1/2 < v \leqq 1).$$

2. Fekete-Szegö Problem

Our main result is the following:

Theorem 1. *Let* $\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$. *If* $f(z)$ *given by (1.1) belongs to* $M_{g,h}(\phi)$, *then*

$$|a_3 - \mu a_2^2| \leqq \begin{cases} \dfrac{B_2}{g_3 - h_3} - \dfrac{\mu}{(g_2-h_2)^2}B_1^2 + \dfrac{g_2h_2 - h_2^2}{(g_2-h_2)^2(g_3-h_3)}B_1^2 & \mu \leqq \sigma_1 \\ \dfrac{B_1}{2(g_3-h_3)} & \sigma_1 \leqq \mu \leqq \sigma_2 \\ -\dfrac{B_2}{g_3 - h_3} + \dfrac{\mu}{(g_2-h_2)^2}B_1^2 - \dfrac{g_2h_2 - h_2^2}{(g_2-h_2)^2(g_3-h_3)}B_1^2 & \mu \ge \sigma_2 \end{cases}$$

where $\sigma_1 := \dfrac{(g_2-h_2)^2(B_2 - B_1) + h_2(g_2-h_2)B_1^2}{(g_3-h_3)B_1^2}$

and $\sigma_2 := \dfrac{(g_2-h_2)^2(B_2 + B_1) + h_2(g_2-h_2)B_1^2}{(g_3-h_3)B_1^2}$. *The result is sharp.*

Proof. For $f(z) \in M_{g,h}(\phi)$, let

$$p(z) := \frac{(f*g)(z)}{(f*h)(z)} = 1 + b_1 z + b_2 z^2 + \cdots. \tag{2.1}$$

From (2.1), we obtain

$$a_2 g_2 - a_2 h_2 = b_1 \quad \text{and} \quad (g_3 - h_3)a_3 = b_2 + a_2^2\left(g_2h_2 - h_2^2\right).$$

Since $\phi(z)$ is univalent and $p \prec \phi$, the function

$$p_1(z) = \frac{1+\phi^{-1}(p(z))}{1-\phi^{-1}(p(z))} = 1 + c_1 z + c_2 z^2 + \cdots$$

is analytic and has positive real part in Δ. Also we have

$$p(z) = \phi\left(\frac{p_1(z)-1}{p_1(z)+1}\right) \tag{2.2}$$

and from this equation (2.2), we obtain

$$b_1 = \frac{1}{2}B_1 c_1, \text{ and } b_2 = \frac{1}{2}B_1(c_2 - \frac{1}{2}c_1^2) + \frac{1}{4}B_2 c_1^2.$$

Therefore we have

$$a_3 - \mu a_2^2 = \frac{B_1}{2(g_3 - h_3)}\left\{c_2 - v c_1^2\right\} \tag{2.3}$$

where $v := \frac{1}{2}\left[1 - \frac{B_2}{B_1} + \frac{1}{(g_2-h_2)^2}\left[h_2^2 - g_2 h_2 + \mu(g_3 - h_3)\right] B_1\right]$. Our result now follows by an application of Lemma 1. To show that the bounds are sharp, we define the functions K^{ϕ_n} $(n = 2, 3, \ldots)$ by

$$\frac{(K^{\phi_n} * g)(z)}{(K^{\phi_n} * h)(z)} = \phi(z^{n-1}), \quad K^{\phi_n}(0) = 0 = [K^{\phi_n}]'(0) - 1$$

and the function F^λ and G^λ $(0 \leqq \lambda \leqq 1)$ by

$$\frac{(F^\lambda * g)(z)}{(F^\lambda * h)(z)} = \phi\left(\frac{z(z+\lambda)}{1+\lambda z}\right), \quad F^\lambda(0) = 0 = (F^\lambda)'(0) - 1$$

and

$$\frac{(G^\lambda * g)(z)}{(G^\lambda * h)(z)} = \phi\left(-\frac{z(z+\lambda)}{1+\lambda z}\right), \quad G^\lambda(0) = 0 = (G^\lambda)'(0) - 1.$$

Clearly the functions $K^{\phi_n}, F^\lambda, G^\lambda \in M_{g,h}(\phi)$. Also we write $K^\phi := K^{\phi_2}$.

If $\mu < \sigma_1$ or $\mu > \sigma_2$, then the equality holds if and only if f is K^ϕ or one of its rotations. When $\sigma_1 < \mu < \sigma_2$, the equality holds if and only if f is K^{ϕ_3} or one of its rotations. If $\mu = \sigma_1$ then the equality holds if and only if f is F^λ or one of its rotations. If $\mu = \sigma_2$ then the equality holds if and only if f is G^λ or one of its rotations. This can be verified by the following. Let

$$K_{\phi n}(z) = \frac{z}{(1-z^{n-1})^{\frac{2}{n-1}}} = z + \frac{2}{(n-1)}z^n + \frac{(n+1)}{(n-1)^2}z^{2n-1} + \cdots$$

and let $g(z) = z + b_2 z^2 + b_3 z^3 + \cdots + b_n z^n + \cdots$. Hence,

$$K_{\phi n}(z) * g(z) = z + \frac{2b_n}{(n-1)}z^n + \frac{(n+1)b_{2n-1}}{(n-1)^2}z^{2n-1} + \cdots.$$

Also for, $h(z) = z + c_2 z^2 + c_3 z^3 + \cdots + c_n z^n + \cdots$, we have

$$K_{\phi n}(z) * h(z) = z + \frac{2c_n}{(n-1)}z^n + \frac{(n+1)c_{2n-1}}{(n-1)^2}z^{2n-1} + \cdots.$$

A simple computation yields

$$\frac{K_{\phi n}(z) * g(z)}{K_{\phi n}(z) * h(z)} = 1 + \frac{2(b_n + c_n)}{(n-1)}z^{n-1} + \cdots \simeq \phi(z^{n-1}).$$

Instead of taking $K_{\phi n}(z)$, if we take $F_\lambda = \dfrac{z}{\left(1 - \dfrac{z(z+\lambda)}{1+\lambda z}\right)^{2/(2-\lambda)}}$, we can obtain a similar result, as for the choice of $\lambda = 1$, $F_1 = K_{\phi 2}(z) = K_\phi$ and $\lambda = 0$, $F_0 = K_{\phi 3}(z)$.

Similarly, we can prove for G_λ by taking $G_\lambda = \dfrac{z}{\left(1 - \dfrac{z(z-\lambda)}{1-\lambda z}\right)^{2/(2-\lambda)}}$. □

For $g(z) = \dfrac{z}{(1-z)^2}$, and $h(z) = \dfrac{z}{(1-z)}$, Theorem 1 reduces to the following result for the class $S^*(\phi)$.

Corollary 1. *If f given by (1.1) belongs to $S^*(\phi)$, then*

$$|a_3 - \mu a_2^2| \leqq \begin{cases} \dfrac{B_2}{2} - \mu B_1^2 + \dfrac{1}{2}B_1^2 & \text{if} \quad \mu \leqq \sigma_1 \\ \dfrac{B_1}{2} & \text{if} \quad \sigma_1 \leqq \mu \leqq \sigma_2 \\ -\dfrac{B_2}{2} + \mu B_1^2 - \dfrac{1}{2}B_1^2 & \text{if} \quad \mu \geq \sigma_2 \end{cases}$$

where,

$$\sigma_1 \quad := \frac{(B_2 - B_1) + B_1^2}{2B_1^2}, \ \sigma_2 \quad := \frac{(B_2 + B_1) + B_1^2}{2B_1^2}.$$

The result is sharp.

Corollary 2. *If $g(z) = \dfrac{z+z^2}{(1-z)^3}$, and $h(z) = \dfrac{z}{(1-z)^2}$, the Theorem 1, coincides with the following result obtained for the class $C(\phi)$ by Ma and Minda [7].*

3. Applications to Functions Defined by Fractional Derivatives

Definition 2 (see [8]). *Let $f(z)$ be analytic in a simply connected region of the z-plane containing the origin. The* fractional derivative *of f of order λ is defined by*

$$D_z^\lambda f(z) := \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^\lambda} d\zeta \quad (0 \leqq \lambda < 1)$$

where the multiplicity of $(z-\zeta)^\lambda$ is removed by requiring that $\log(z-\zeta)$ *is real for $z - \zeta > 0$.*

Using the above Definition 2 and its known extensions involving fractional derivatives and fractional integrals, Owa and Srivastava [8] introduced the operator $\Omega^\lambda : \mathcal{A} \to \mathcal{A}$ defined by $(\Omega^\lambda f)(z) = \Gamma(2-\lambda) z^\lambda D_z^\lambda f(z)$, $(\lambda \neq 2, 3, 4, \ldots)$. If

$$g(z) = z + \sum_{n=2}^{\infty} n \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} z^n, \ h(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} z^n,$$

Theorem 1 reduces to the following theorem in terms of the fractional derivative.

Theorem 2. *Let the function $\phi(z)$ be given by $\phi(z) = 1+B_1z+B_2z^2+B_3z^3+\cdots$ and let $\lambda < 2$. If $f(z)$ given by (1.1) belongs to $M_{g,h}(\phi)$, then*

$$|a_3 - \mu a_2^2| \leqq \begin{cases} \dfrac{(2-\lambda)(3-\lambda)}{6}\gamma & \text{if} \quad \mu \leqq \sigma_1 \\ \dfrac{(2-\lambda)(3-\lambda)}{6}\dfrac{B_1}{2} & \text{if} \quad \sigma_1 \leqq \mu \leqq \sigma_2 \\ -\dfrac{(2-\lambda)(3-\lambda)}{6}\gamma & \text{if} \quad \mu \geq \sigma_2 \end{cases}$$

where

$$\gamma := \frac{B_2}{2} - \frac{3(2-\lambda)}{2(3-\lambda)}\mu B_1^2 + \frac{1}{2}B_1^2$$

$$\sigma_1 := \frac{2(3-\lambda)}{3(2-\lambda)}.\frac{(B_2-B_1)+B_1^2}{2B_1^2}, \quad \sigma_2 := \frac{2(3-\lambda)}{3(2-\lambda)}.\frac{(B_2+B_1)+B_1^2}{2B_1^2}.$$

The result is sharp.

Remark 1. *When $B_1 = 8/\pi^2$ and $B_2 = 16/(3\pi^2)$, the above Theorem 2 reduces to a recent result of Srivastava and Mishra*[13, Theorem 8, p. 64] *for a class of functions for which $\Omega^\lambda f(z)$ is a parabolic starlike function*[5].

Remark 2. *For the choices $\lambda = 1$, $B_1 = \dfrac{8}{\pi^2}$ and $B_2 = \dfrac{16}{3\pi^2}$, Theorem 2 with the result obtained by Ma and Minda* [6].

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