Detour Graphs

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Abstract

Let $G = (V, E)$ be a connected simple graph. Let $u, v \in V(G)$. The detour distance, $D(u, v)$, between u and v is the distance of a longest path from u to v. E.Sampathkumar defined the detour graph of G , denoted by $D(G)$ as follows: $D(G)$ is an edge labelled complete graph on n vertices where $n = |V(G)|$, the edge label for uv , $u, v \in V(K_n)$ being $D(u, v)$. Any edge labelled complete graph need not be the detour graph of a graph. In this paper, we characterize detour graphs of a tree. We also characterise graphs for which the detour distance sequences are given.

Definition:- 1 *Let $G = (V, E)$ be a simple graph. The detour graph of G denoted by $D(G)$, is an edge labelled complete graph*

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$K_{|V(G)|}$ *on the same vertex set* $V(G)$ *such that for any two vertices* a, b *in* G *with* $D(a,b) = k$, *the edge label of* ab *is* k *in* $K_{|V(G)|}$.

Result:- 1 *If a detour graph of* G *has all the labels 1, then* $G = K_2$.

Proof :-

Assume $|V(G)| \geq 3$. If $d(G) = 1$, then all the labels are $(n-1)$ and $(n-1) \geq 2$. If $d(G) = k \geq 2$, there exists a label $r \geq 2$, a contradiction. Therefore $G = K_2$.

Result:- 2 *If a detour graph has all the edge labels 2, then the graph is* K_3.

Proof :-

Assume $|V(G)| \geq 4$. Since there is no label 1, $\delta(G) \geq 2$. Therefore, G has a cycle. If $d(G) \geq 3$, then there is an edge with label greater than or equal to 3 in the detour graph, a contradiction.

If $d(G) = 1$, then we get a complete graph and all the labels are greater than or equal to 3, a contradiction. Therefore, $d(G) = 2$.

If G has a cycle C_4 or C_5 as a subgraph, there exists an edge which has label greater than or equal to 3, a contradiction. Therefore, the cycles in G are all triangles. Let G have two triangles T_1 and T_2. If T_1 and T_2 have common edge, we have a C_4, a contradiction. If T_1 and T_2 have a common vertex, then there exists an edge with label greater than or equal to 4, a contradiction. Therefore, G cannot have two triangles. Therefore, G has a unique triangle. Since $|V(G)| \geq 4$, there exists an edge label greater than or equal to 3, a contradiction. Therefore, $G = K_3$.

Result:- 3 *If a graph is any one of the following, then the detour distance between any two points in the graph is* $(n-1)$.

(i.) $G_1 + G_2$ *where* G_1 *and* G_2 *are Hamiltonian graphs and*
$|V(G_1)| + |V(G_2)| = n$.

(ii.) $G_1 + K_r$ *where the number of components of* G_1 *is less than or equal to* r. *(that is,* $\omega(G) \leq r$*) and* $|V(G_1)| = n - r$.

(iii.) K_n.

Result:- 4

(i.) If $G = C_{2n}$, then $D(C_{2n}) = nK_2$.

(ii.) If $G = C_{2n+1}$, then $D(C_{2n+1}) = C_{2n+1}$.

Result:- 5 *If* $G = P_n$, *then* $D(P_n) = P_n$ *for all* n.

Result:- 6 *If* $G = K_{1,r}$, *then* $D(K_{1,r}) = K_{r+1}$.

Result:- 7 *The maximum number of distinct integers that one can get in a detour distance sequence is* $n - 1$.

Proof :-

Let $G = P_n$. Then the detour distance sequence is $1^{(n-1)}, 2^{(n-2)}, \cdots, (n-1)^{(1)}$ and this has maximum number of distinct integers. (Note that the number of distinct integers in the detour distance sequence of any graph on n vertices is atmost $n - 1$).

Theorem:- 1 *A graph* G *on* n *vertices is a tree if and only if the detour graph* $D(G)$ *contains* $(n-1)$ *1's.*

Proof :-

Let the detour graph of G have $(n-1)$ 1's. Let $V(G) = \{u_1, u_2, u_3, \cdots, u_n\}$. If G contains a cycle C with vertices $u_{r_1}, u_{r_2}, u_{r_3}, \cdots, u_{r_k}$, then all the edges of this cycle will have labels greater than 1. Then the $(n-k)$ points, other than those in the cycle C will contribute a maximum of $(n-k-1)$ 1's to the edges of the complete graph K_n on the n-vertices $\{u_1, u_2, u_3, \cdots, u_n\}$. Therefore, the number of 1's in $D(G)$ is atmost $(n-k-1)$ 1's, a contradiction. Therefore, G has no cycles. Since there are $(n-1)$ 1's in $D(G)$, there are exactly $(n-1)$edges in G. Therefore G is a connected graph. Therefore G is a connected acyclic graph. That is, G is a tree. The converse is obvious. ■

Theorem:- 2 *A graph G is a path if and only if the detour distance sequence is* $1^{(n-1)}, 2^{(n-2)}, \cdots, (n-1)^{(1)}$

Proof :-

Suppose the detour distance sequence of a graph G is $1^{(n-1)}, 2^{(n-2)}, \cdots, (n-1)^{(1)}$. Then G is a tree. Since the label $(n-1)$ appears exactly once, there exists exactly one longest path(say) between u and v of length $(n-1)$. Let $\{u = u_1, u_2, u_3, \cdots, u_n = v\}$, be the longest path. If u or v has degree greater than 1, then there will be a longest path of length greater than $(n-1)$. But there exists no label in the detour distance sequence greater than $(n-1)$. Therefore, u and v are pendent vertices. Further, all the n-points are in this longest path. Therefore, G is a path. The converse is obvious. ∎

Remark:- 1 *In the above proof, we use only two facts. In the detour distance sequence, (i) there are exactly $(n-1)$ 1's and (ii) one $(n-1)$. These two facts are enough to determine that the graph is a path. That is, if the detour distance sequence of G contains $1^{(n-1)}$ and $(n-1)^{(1)}$, then 2 exactly appear $(n-2)$ times, 3 appears exactly $(n-3)$ times,$\cdots$ $(n-2)$ appears exactly 2 times.*

Theorem:- 3 *A graph G is a star if and only if the detour distance sequence of G is* $1^{n-1}, 2^{(n_{c_2}-(n-1))}$

Proof :-

Since the detour distance sequence of G contains $1^{(n-1)}$, G is a tree.

Claim :- There exists a point of degree $(n-1)$.

Let u be a point of G of degree greater than 1. Let $v, w \in N(u)$. If $|V(G)| = 3$, then G is a star with center u. Suppose $x \in V(G) - \{u, v, w\}$. Suppose x is not adjacent to u. Then the detour distance between x and v is greater than or equal to 3, a contradiction (since the detour distance sequence contains no 3). Therefore, u is adjacent to every vertex of G. Since G is a tree, all the vertices other than u are pendent vertices. Therefore, G is a star. The converse is obvious. ∎

Theorem:- 4 *A graph G is a cycle of length 2n if and only if the detour distance sequence of G is* $n^{(n)}, (n+1)^{(2n)}, \cdots, (2n-1)^{(2n)}$.

Proof :-

Suppose G is acyclic. The absence of ∞ in the detour distance sequence ensures that G is a tree. 1 must occur $(n-1)$ times in the detour distance sequence. But 1 is not present in the detour distance sequence. Therefore, G contains a cycle.

G has no pendent vertex since 1 is absent in the detour distance sequence of G. Therefore, $\delta(G) \geq 2$.

Since $(2n-1)^{(2n)}$ occurs in the detour distance sequence of G, there are exactly $2n$ longest paths of length $(2n-1)$. Let u and v be two points, the length of the longest path between them being $(2n-1)$. From the sequence it is clear that G has order $2n$. Let $V(G) = \{u_1, u_2, u_3, \cdots, u_{2n}\}$. Let, without loss of generality, the longest path between u_1 and u_{2n} be $u_1, u_2, u_3, \cdots, u_{2n}$. Since there are $2n$ longest paths of length $(2n-1)$, there exist u_i and u_j $(1 \leq i < j \leq 2n)$ such that the longest path between u_i and u_j is $(2n-1)$. This shows that u_i and u_j are adjacent and u_1 and u_{2n} are adjacent. Hence G is a cycle of length $2n$. The converse is obvious. ■

Remark:- 2 *A graph G is a cycle of length* $(2n+1)$ *if and only if the detour distance sequence of G is* $(n+1)^{(2n+1)}, (n+2)^{(2n+1)}, (n+3)^{(2n+1)}, \cdots, (2n)^{(2n+1)}$.

References:-

1. G. Chartrand, H.Escuadro and P.Zhang, Detour Distance in Graphs, JCMCC, 53,2005,75-94.

2. G. Chartrand and P.Zhang, Distance in Graphs - Taking the Long View, AKCE J. Graphs and Combin.,1, No.1,2004, 1-13.

3. F.Harary, Graph Theory, 1972, Addison Wesley, New york.