On Efficiently Roman Dominatable Graphs

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Abstract

A (2,2) packing on a graph G is a function $f : V(G) \to \{0,1,2\}$ with $f(N[v]) \leq 2$ for all $v \in V(G)$. For a function $f : V(G) \to \{0,1,2\}$, the *Roman influence* of f denoted by $I_R(f)$ is defined to be $I_R(f) = (|V_1| + |V_2| + \sum_{v \in V_2} \deg(v))$. The *efficient Roman domination number* of G, denoted by $F_R(G)$ is defined to be the maximum of $I_R(f)$ such that f is a (2,2)-*packing*. That is $F_R(G) = \max\{I_R(f) : f is\ a(2,2) - packing\}$. A (2,2)-*packing* $F_R(G)$ with $F_R(G) = I_R(f)$ is called an $F_R(G)$ function. A graph G is said to be *efficiently Roman dominatable* if $F_R(G) = n$, and when $F_R(G) = n$, an $F_R(G)$-function is called an *efficient Roman dominating function*. In this paper, we focus our study on certain graphs which are efficiently Roman dominatable. We characterize the class of $2 \times m$ and $3 \times m$ gridgraphs, trees, unicyclic graphs and split graphs which are efficiently Roman dominatable.

Key words: Efficient Roman dominating function, Efficient Roman domination number.
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1 Introduction

We consider only finite simple undirected graphs $G = (V, E)$ of order $|V| = n$. For any vertex $v \in V$, the *open neighborhood* of v is the set $N(v) = \{u \in V/uv \in E\}$ and the *closed neighborhood* is the set $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, the open neighborhood is $N(S) = \cup_{v \in s} N(v)$ and the closed neighborhood is $N[S] = N(S) \cup S$. A set $S \subseteq V$ is a *dominating set* if $N[S] = V$, or equivalently, every vertex in $V \backslash S$ is adjacent to at least one vertex in S. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set in G, and a dominating set S of minimum cardinality is called a γ-set of G. A set S of vertices is called *2–packing* if for every pair of vertices $u, v \in S, N[u] \cap N[v] = \phi$.

A Roman dominating function on a graph $G = (V, E)$ is a function $f : V(G) \rightarrow \{0, 1, 2\}$ satisfying the condition that whenever $f(v) = 0$, there exists a $u \in N(v)$ for which $f(u) = 2$. Let V_0, V_1, V_2 be the ordered partition induced by f where $V_i = \{v \in V/f(v) = i\}$. The weight of a Roman dominating function is the value $f(V) = \sum_{v \in V(G)} f(v)$. The minimum weight of a Roman dominating function of G is called the *Roman domination number* of G and is denoted by $\gamma_R(G)$. Cockayne, Dreyer, Hedetniemi and Hedetniemi [3], represented Roman dominating functions by three sets: V_0 is the set of vertices which receive no legions, V_1 is the set of vertices which receive precisely one legion and V_2 is the set of vertices which receive two legions.

Emperor Constantine had the requirement that an army or legion could be sent from its home to defend a neighbouring location only if there was a second army which stay and protect the home. Thus there are two types of armies, stationary and traveling. Each vertex with no army must have a neighbouring vertex with a traveling army. Stationary armies then dominate their own vertices, and a vertex with two armies is dominated by its stationary army, and its open neighbourhood is dominated by the travelling army.

A variant of the domination number was suggested by an article in *Scientific American by Ian Stewart*, entitled "Defend the Roman Empire!" [17]. A few lesser known articles by ReVelle [11,12] in the Johns Hopkins Magazine suggested Roman domination a few years earlier. Since then, there have been several articles on Roman domination [3–5, 8–10, 13]

Bange, Barkauskas and *Slater* [1,2] introduced the following *efficiency measure* for a graph G. The *efficient domination number* of a graph, denoted by $F(G)$, is the maximum number of vertices that can be dominated by a set S that dominates each vertex at most once. A graph G of order $n = |V(G)|$ has an *efficient dominating set* if and only if $F(G) = n$. A vertex v of $\deg(v) = |N(v)|$ dominates $|N[v]| = 1 + deg(v)$ vertices. *Grinstead and Slater* [6] defined the *influence of a set of vertices* S to be $I(S) = \sum_{s \in S}(1+deg(v))$, the total amount of domination being done by S. Because S does not dominate any vertex more than once if and only if any two vertices in S are at a distance at least 3 (that is, S *is a* 2-packing), we have $F(G) = max\{I(S) : S \text{ is a 2-packing}\}$. A set S is an *efficient dominating set* if and only if $|N(v) \cap S| = 1$ for all vertices $v \in V(G)$, or equivalently, S is an efficient dominating set if and only if S is a 2-packing with $I(S) = n = F(G)$. A graph G has an efficient dominating set if and only if $F(G) = n$. The idea of efficiency was extended to Roman domination by Rubalcaba and Slater [15]. Following [14] a (j,k)-packing is a function $f : V(G) \to \{0, 1, 2, \dots, j\}$ with $f(N[v]) \leq k$ for all $v \in V(G)$. Thus a 2-packing is a (1,1)-packing, and in particular, a (2,2)-packing is a function $f : V(G) \to \{0, 1, 2\}$ with $f(N[v]) \leq 2$ for all $v \in V(G)$.

Each stationary army stationed at a vertex $\{u\}$ dominates only u. A traveling army stationed at a vertex v dominates only its neighbours, $N(v)$, and it has influence $\deg(v)$. Any vertex with a traveling army necessarily has a stationary army stationed at v Thus, for a function $f : V(G) \to \{0, 1, 2\}$ the Roman influence of f, denoted by $I_R(f)$ is defined to be $I_R(f) = (|V_1| + |V_2| + \sum_{v \in V_2} deg(v))$. The *efficient Roman domination number* of G, denoted by $F_R(G)$ is defined to be the maximum of $I_R(f)$ such that f is a (2,2)-packing. That is $F_R(G) = max\{I_R(f) : f \text{ is a (2,2)-packing}\}$. A (2,2)-packing f with $F_R(G) = I_R(f)$ is called an $F_R(G)$ function. Graph G is said to be efficiently *Roman dominatable*, if $F_R(G) = n$, and when $F_R(G) = n$, $F_R(G)$-function is called an *efficient Roman dominating function*.

A *star* $K_{1,n-1}$ has one vertex v of degree $n-1$ and $n-1$ vertices of degree one. A *split graph* is a graph $G = (V, E)$ whose vertices can be partitioned into two sets V' and V'', where the vertices in V' from a complete graph and vertices in V'' are independent.

In this paper, we characterize the class of $2 \times m$ and $3 \times m$ *grid graphs, trees, unicyclic graphs, and split graphs which are efficiently Roman dominatable.*

We need the following results for our further discussion.

Theorem 1.1 [15] *For every graph G of order $n = |V(G)|$, we have $F_R(G) \leq n \leq R_R(G)$.*

Theorem 1.2 [15] *For every graph* G, *we have* $F_R(G) = F(G)$
Theorem 1.3 [15] *If* G *is connected of order* $n \geq 3$ *and* $F_R(G) = n$, *then every efficient Roman dominating function* f *has* $V_1 = f^{-1}(1) = \phi$.

2 Main Results

First we characterize $2 \times m$ and $3 \times m$ grid graphs which are efficiently Roman dominatable.

Theorem 2.1 *Let* G *be a* $2 \times m$ *grid graph. Then* $F_R(G) = 2m$ *if and only if* m *is odd.*

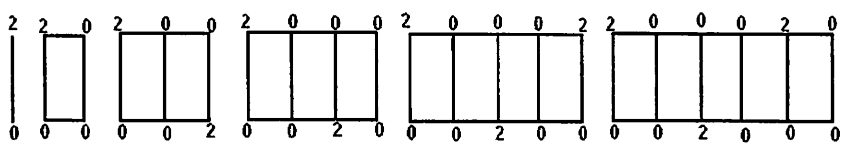

Fig. 2.1. Construction for $2 \times m$ grid graphs, $1 \leq m \leq 6$

Proof. Let G be a $2 \times$ m grid graph where m is odd. Let the vertices of G be denoted as $v_{1,1}, v_{1,2}, v_{1,3}, \ldots v_{1,m}$, $v_{2,1}, v_{2,2}, v_{2,3}, \ldots v_{2,m}$. Define a function $f : V(G) \to \{0, 1, 2\}$ as follows. For each i, such that $1 + 4i \leq m$, let $f(v_{1,1+4i}) = 2$ and for each j, such that $3 + 4j \leq m$, let $f(v_{2,3+4j}) = 2$. For all the remaining vertices u, let $f(u) = 0$. Then it is easy to see that f is a (2,2)-packing with $F_R(G) = I_R(G) = 2m$. (Refer Fig 2.1)

Conversely let $F_R(G) = 2m$. Then we claim that m is odd. Suppose not. Since $F_R(G) = 2m$, there exists a (2,2)-packing f with $F_R(G) = 2m$. Hence every vertex in V_0 is adjacent to exactly one vertex in V_2. Now without loss of generality we assume that $v_{1,1} \in V_2$ and $v_{2,1} \in V_0$. Then clearly $v_{1,m}, V_{2,m} \in V_0$. But $v_{1,m}$ is not adjacent to any member of V_2, a contradiction. Therefore m is odd. □

Theorem 2.2 *Let* G *be a* $3 \times m$ *grid graph. Then* $F_R(G) = 3m$ *if and only if* $m \leq 2$.

Proof. Let the vertices of G be $v_{i,1}, v_{i,2}, \ldots, v_{i,m}$, $i = 1, 2, 3$. If $m = 2$, then define $f : V(G) \to \{0, 1, 2\}$ with $f(v_{1,1}) = f(v_{3,2}) = 2$ and $f(w) = 0, w \neq v_{1,1}, v_{3,2}$. Then clearly f is a (2,2)-packing with $F_R(G) = I_R(G) = 3m$. (Refer Fig. 2.2).

Conversely suppose $F_R(G) = 3m$. Then there exists a (2,2)-packing f : $V(G) \to \{0, 1, 2\}$ such that every member of V_0 is adjacent to exactly one

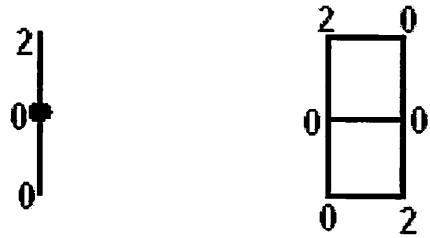

Fig. 2.2. Construction for 3×1 and 3×2 grid graphs

member of V_2 and no two members of V_2 are adjacent. By Theorem 1.3 $V_1 = \phi$. We claim $m \leq 2$. Suppose $m \geq 3$. Then one of the following cases arise. (i) $v_{1,1} \in V_2$ (ii) $v_{2,1} \in V_2$ (iii) $v_{2,2} \in V_2$.

Case (i): $v_{1,1} \in V_2$.

Since f is a (2,2)-packing $v_{1,2}, v_{1,3}, v_{2,1}, v_{2,2}, v_{3,1} \in V_0$. Now $v_{2,2}, v_{1,3}, v_{3,1}$ must be adjacent to a member of V_2. Hence both $v_{2,3}, v_{3,2} \in V_2$, which is a contradiction since $v_{3,3}$ is adjacent to two members of V_2. (Refer Fig 2.3).

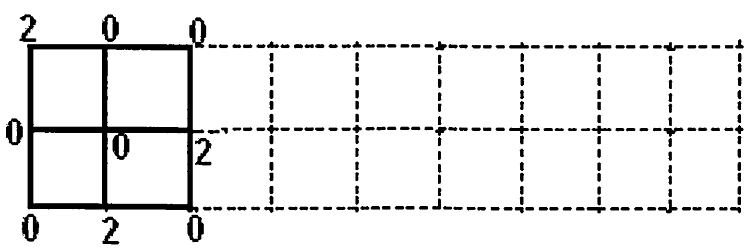

Fig. 2.3.

Case (ii): $v_{2,1} \in V_2$.

Now $v_{1,1}, v_{1,2}, v_{2,2}, v_{2,3}, v_{3,1} v_{3,2} \in V_0$. Hence $v_{1,2}, v_{2,3}, v_{3,2} \in V_0$ must be adjacent to a member of V_2. Hence $v_{1,3}, v_{3,3} \in V_2$. But $v_{2,3} \in V_0$ is adjacent to two members of V_2, a contradiction. Similarly we deal with case *(iii)* and get a contradiction. Hence $m \leq 2$. (Refer Fig. 2.4). □

We now characterize trees which are efficiently Roman dominatable. For this purpose, we use the following notation.

Notation 2.3 For the star $K_{1,n}$, we call the vertex of degree n, the head vertex and the vertices of degree one, the end vertices. For K_2, we will consider one vertex to be the head vertex and the other as the end vertex.

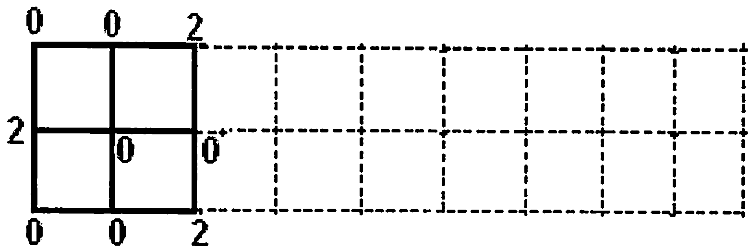

Fig. 2.4.

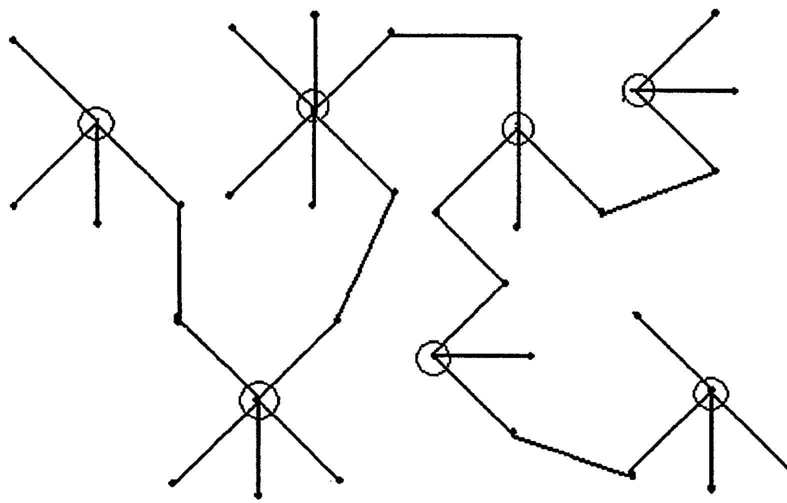

Fig. 2.5. A graph T* with 7 headvertices

Definition 2.4 We define a graph T^* to be the union of stars $K_{1,n_i}, 1 \leq i \leq k$ and a collection $\mathscr{E}$ of edges subject to the following conditions.

(i). If $e = vw \in \mathscr{E}$, then e is an edge joining $v \in V(K_{1,n_i})$ and $w \in V(K_{1,n_j}), i \neq j$ where v and w are end vertices.

(ii). For any pair of vertices $v \in V(K_{1,n_i})$ and $w \in V(K_{1,n_j}), i \neq j$, there exists a unique path joining v and w. (Refer Fig. 2.5).

Remark 2.5 Clearly T^* is a tree.

Theorem 2.6 *Let T be a tree of order n. Then $F_R(T) = n$ if and only if $T \cong T^*$*

Proof. Suppose T is of the given type. Then define $f : V(T) \to \{0,1,2\}$ with $f(v) = 0$ if v is an end vertex of a star K_{1,n_i} and $f(v) = 2$, if v

is a head vertex of a star K_{1,n_i} for some i. It is easy to see that f is a (2,2)-packing with $F_R(T) = I_R(f) = n$.

Conversely suppose $F_R(T) = n$. Let f be a (2,2) *packing* with $I_R(f) = n$. Note that since $I_R(f) = n$, for any $v \in V_0$, there exists exactly one vertex $z \in N(v)$ with $f(z) = 2$. Further by Theorem 1.3, V_1 is empty. Let $V_2 = \{u_1, u_2, \ldots u_k\}, k < n$. Since f is a (2,2)-packing, V_2 is independent.

Let K_{1,n_i} be the star with u_i as the head vertex, $1 \leq i \leq k$. Now since f is a (2,2)-packing, $N(u_i) \cap N(u_j) = \phi$ for every i, j, $1 \leq i \leq k$, $1 \leq j \leq k$, $i \neq j$. Let $e = vw$ be an edge in $G[V \backslash V_2]$. Now there exists no i such that v, w are both end vertices of K_{1,n_i}. Otherwise $u_i vw\ u_i$ form a cycle in T, a contradiction. Hence v and w are the end vertices of K_{1,n_i} and K_{1,n_j}, respectively for some i, j, $i \neq j$. Let $v \in V(K_{1,n_i})$ and $w \in V(K_{1,n_j})$, $1 \leq i \leq k, 1 \leq j \leq k$ and $i \neq j$. Since T is a tree, there exists a unique path joining $v \in V(K_{1,n_i})$ and $w \in V(K_{1,n_j}), i \neq j$. Hence $T \cong T^*$. □

We now proceed to characterize the class of unicyclic graphs which are *efficiently Roman dominatable.*

Theorem 2.7 *Let G be a unicyclic graph of order n. Let C be the cycle in G. Then $F_R(G) = n$ if and only if one of the following holds.*

- *(i) There exists an edge $e = vw$ in C such that $G - e \cong T^*$ where either both v and w are end vertices of K_{1,n_i} in T^* for some i or v is an end vertex of K_{1,n_i} in T^* and w is and end vertex of K_{1,n_j} in T^* for some i and $j, i \neq j$.*
- *(ii) There exists a vertex w in C such that the components $T_1, T_2, \ldots T_s$ of $G - w$ are isomorphic to T^* where G is obtained by joining w to a head vertex of one component T_i and to end vertices of stars in the other components $T_j (j \neq i)$.*

Proof. Suppose the graph is of type (i). Let $e = vw$ be an edge in C. Define $f : V(G) \to \{0, 1, 2\}$ such that $f(v) = 2$ if v is the head of a star in T^*, $f(v) = 0$ if v is an end vertex of a star in T^*. If the graph G is of type (ii), let $1 \leq T_i \leq s$ be the components of $G - w$. Define $f : V(G) \to \{0, 1, 2\}$ such that, $f(w) = 0$, $f(z) = 2$ if z is the head vertex of a star in T_i, $1 \leq i \leq k$ and $f(z) = 0$ if z is an end vertex of some star in T_i, $1 \leq i \leq k$. Clearly f is a (2,2)-packing with $F_R(G) = I_R(G) = n$.

Conversely suppose $F_R(G) = n$. Let f be a (2,2)-packing with $I_R(f) = n$. As in the proof of Theorem 2.6, we define V_2 and K_{1,n_i}, $1 \leq i \leq k$. Since G is unicyclic there exists at most one star K_{1,n_i}, such that $N(u_i)$ is not independent.

Case (i): $N(u_i)$ is not independent for some i, $1 \leq i \leq k$.
Let $N(u_r)$ be not independent. In this case there exists neighbours w_1 and w_2 of u_r such that w_1 and w_2 are adjacent. Let $e = w_1w_2$. Suppose $e' \neq e$ is an edge in $G[V \backslash V_2]$. Then clearly e' is an edge joining an end vertex of a star K_{1,n_i} and an end vertex of a star K_{1,n_j}, $i \neq j$. Now between a vertex of K_{1,n_i} and a vertex of K_{1,n_j}, there exists a unique path. Otherwise G will have more than one cycle which is a contradiction. Hence in this case $G \cong T^* \cup e$.

Case (ii): $N(u_i)$ is independent for all i.

As in case (i) if e' is an edge in $G[V \backslash V_2]$ then e' joins an end vertex of K_{1,n_i} and an end vertex of K_{1,n_j}. Since G is unicyclic there exists a cycle say C. Let $e_1 = vw$ be an edge in C. If $v, w \in V_0$, then clearly $G - e_1$ is isomorphic to T^*. Hence $G \cong T^* \cup e_1$. Then the graph is of type (i). If $v \in V_2$ and $w \in V_0$ then there exists a vertex z_1 in C which is adjacent to w such that $z_1 \in V_0$. Let the components of $G - w$ be $T_1, T_2, \ldots T_s$ with T_1 containing v and z_1. Let $z_2, z_3 \ldots z_s$ be the neighbours of w such that $z_i \notin V(C)$ and $z_i \in V(T_i)$, $1 \leq i \leq s$. Clearly $z_i \in V_0$, $1 \leq i \leq s$. Hence z_i is an end vertex of a star in T_i. Hence the components of $G - w$ are isomorphic to T^* where G is obtained by joining w to v, a head vertex in T_1 and to z_i end vertices of stars in T_i. Hence the theorem. □

Theorem 2.8 *Let G be a split graph of order n with bipartition (X_1, X_2) with X_1 independent and $G[X_2]$ complete. Then $F_R(G) = n$ if and only if one of the following conditions hold.*

(i). *There exists a vertex u in X_2 such that $N[u] = X_1 \cup X_2$.*

(ii). *Every vertex in X_2 has exactly one neighbour in X_1.*

Proof. Suppose the graph is of the given type. If (i) holds then define $f : V(G) \to \{0, 1, 2\}$ with $f(u) = 2$ and $f(v) = 0$ for every $v \neq u$, $v \in V(G)$. It is easy to see that f is a (2,2)-packing with $F_R(G) = I_R(f) = n$. If (ii) holds then define $f : V(G) \to \{0, 1, 2\}$ with $f(w) = 2$ for every $w \in X_1$ and $f(w) = 0$ for every $w \in X_2$. Since $\bigcup_{w \in X_1} N(w) = X_2$, $I_R(f) = n$. Hence f is a (2,2)-*packing* with $F_R(G) = I_R(f) = n$.

Conversely, let G be a split graph with bipartition (X_1, X_2) with X_1 independent and $G[X_2]$ complete with $F_R(G) = n$. Let f be a (2,2)-packing with $I_R(f) = n$. By Theorem. 1.3, $V_1 = \phi$. Since $I_R(f) = n$ for any $v \in V_0$, there exists exactly one vertex $z \in N(v)$ with $f(z) = 2$. Since f is a (2,2)-packing, V_2 is independent. If there exists a vertex u in X_2 such that $N[u] = X_1 \cup X_2$ we are through. Otherwise we claim that $f(v) \neq 2$ for every v in X_2. Suppose $f(v) = 2$ for some v in X_2. Then $f(w) = 0$

for every $w \in X_2$, $w \neq v$. Let $x \in X_1 \backslash N(v)$. Since f is a (2,2)-packing, $f(x) = 0$. Therefore x is not adjacent to any member of V_2, a contradiction. Therefore $f(v) \neq 2$ for every $v \in X_2$. Hence $f(v) = 2$ for every $v \in X_1$. Now we claim $N(u) \cap N(v) \neq \phi$, for every $u, v \in X_1$. Let $w \in N(u) \cap N(v)$. Then w is dominated by both u and v, a contradiction. Hence our claim. Finally we claim $\bigcup_{w \in X_1} N(w) = X_2$. Suppose not. Then there exists a $x \in X_2 \backslash (\cup_{w \in X_1} N(w))$. Now x is not adjacent to any vertex in X_1, which in turn is not adjacent to any member of V_2, a contradiction. Hence G is of type (ii). □

Remark 2.9 In view of Theorem 1.2, the class of $2 \times m$ and $3 \times m$ grid graphs, trees, unicyclic graphs and split graphs with $F(G) = n$ have also been characterized.

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