Minimally k-equitable Labeling of Butterfly and Benes Networks

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Abstract

A labeling of the vertices of a graph with distinct natural numbers induces a natural labeling of its edges: the label of an edge (x, y) is the absolute value of the difference of the labels of x and y. We say that a labeling of the vertices of a graph of order n is minimally k-equitable if the vertices are labeled with $1, 2...n$ and in the induced labeling of its edges, every label either occurs exactly k times or does not occur at all. In this paper, we prove that Butterfly and Benes networks are minimally 2^r- equitable where r is the dimension of the networks.

Keywords: Labeling, minimally k-equitable, Butterfly and Benes networks.
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1 Introduction

The area of graph theory has experienced a fast development during the last 60 years. Among all the different kinds of problems that appear while studying graph theory, one that has been growing strong during the last three decades is the area that studies *labelings of graphs*. This is not only due to its mathematical importance but also because of the wide range of applications arising from this area. For instance, we can find labelings of graphs showing up in x-rays, crystallography, coding theory, radar, astronomy, circuit design, and communication network addressing [5, 6].

A graph labeling is an assignment of integers to the vertices or edges, or both, subject to certain conditions. Graph labeling was first introduced in the 1960s. Over the past three decades more than 600 papers have spawned a bewildering array of graph labeling methods. Despite the unabated procession of papers, there are few general results on graph labeling. Indeed, the papers focus on particular classes of graphs and methods and feature ad hoc arguments.

2 An overview of the paper

In 1990, Cahit [7] proposed the idea of k-equitable labeling. For any graph $G(V, E)$ and any positive integer k, assign vertex labels from $\{0, 1...k-1\}$ so that when the edge labels are induced by the absolute value of the difference of the vertex labels, the number of vertices labeled with i and number of vertices labeled with j differ by at most one and the number of edges labeled with i and number of edges labeled with j differ by at most one. A graph with such an assignment is called *k-equitable*.

Bloom defined a labeling of graph to be k-equitable if in the induced labeling of edges, every label occurs exactly k times. Furthermore a k-equitable labeling of a graph of order n is said to be minimal if the vertices are labeled with $1, 2...n$. A graph is *minimally k-equitable* if it has a minimal k-equitable labeling.

Wojciechowski [12, 13] proved that C_n is minimally k-equitable if and only if k is a proper divisor of n. Barrientos and Hevia [2] proved that if G is k-equitable of size $q = kw$ then $\delta(G) \leq \omega$ and $\Delta(G) \leq 2\omega$. Barrientos, Dejter, and Hevia [1] have shown that a forest of even size is 2-equitable. They also prove that for $k = 3$ or 4, a forest of size kw is k-equitable if and only if its maximum degree is at most $2w$ and that if 3 divides $mn + 1$, then the double star $S_{m,n}$ is 3-equitable if and only if $q/3 \leq m \leq \lfloor (q-1)/2 \rfloor$. They discuss the k-equitability of forests for $k \geq 5$ and characterize all caterpillars of diameter 2 that are k-equitable for all possible values of k. Barrientos proves that the one-point union of a cycle and a path and the disjoint union of a cycle and a path are k-equitable for all k that divide the size of the graph. Barrientos and Hevia [2] have shown the following: $C_n \times K_2$ is 2-equitable when n is even; books $B_n (n \geq 3)$ are 2-equitable when n is odd; the vertex union of k-equitable graphs is k-equitable; and wheels W_n are 2-equitable when $n \not\equiv 3(\bmod 4)$. They conjecture that W_n is 2-equitable when $n \equiv 3(\bmod 4)$ except when $n = 3$.

Bhat-Nayak and M.Acharya [3, 4] have proved the following: the crowns $C_{2n} \odot K_1$ are minimally 4-equitable, the crowns $C_{3n} \odot K_1$ are minimally 3-equitable. More complete results on minimally k-equitable graphs can be seen in the survey paper by Gallian [8].

In this paper we prove that Butterfly and Benes networks are minimally 2^r-equitable where r is the dimension of the networks.

3 Terminology

Networks are represented as undirected graphs whose nodes represent processors and whose edges represent inter-processor communication links. A multistage network consists of a series of switch stages and interconnection patterns, which allows N inputs to be connected to N outputs. A multistage network uses dynamic interconnection to allow communication paths to be established as needed for the transfer of information between I/O nodes. The Butterfly and Benes networks are important multistage interconnection networks, which possess attractive topological for communication networks. They have been used in parallel computing systems such as IBM SP1/SP2, MIT Transit Project, and NEC Cenju-3, and used as well in the internal structures of optical couplers, e.g., star couplers [9, 10].

3.1 The Butterfly Network

The set of nodes V of an r-dimensional Butterfly corresponds to the set of pairs $[w, i]$, where i is the dimension or level of a node $(0 \leq i \leq r)$ and w is an r-bit binary number that denotes the row of the node. Two nodes $[w, i]$ and $[w', i']$ are linked by an edge if and only if $i' = i + 1$ and either

1. w and w' are identical, or
2. w and w' differ in precisely the ith bit.

An r-dimensional Butterfly is denoted by $BF(r)$. The r-dimensional butterfly has $(r+1)2^r$ nodes and $r2^{r+1}$ edges.

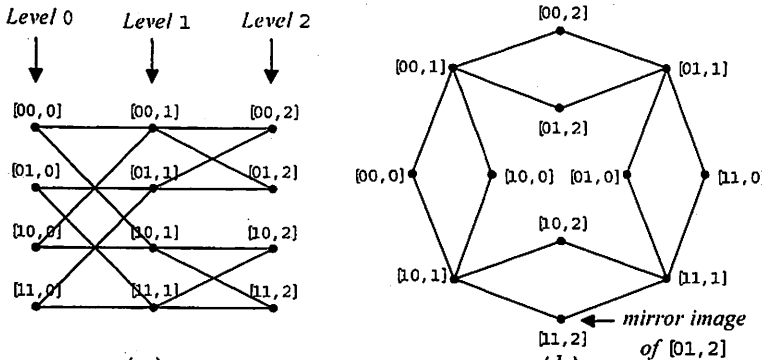

Figure 1: Binary labeling of a 2-dimensional butterfly (a) Normal form and (b) Diamond form

Efficient representations for Butterfly and Benes networks have been obtained by Manuel et al. [11]. The butterfly in Figure $1(a)$ is drawn in *normal representation*; an alternative representation, called the *diamond representation*, is given in Figure $1(b)$. Note that by diamond we mean a cycle of length 4.

Two nodes $[w,i]$ and $[w',i]$ are said to be *mirror images* of each other if w and w' differ precisely in the first bit. The removal of the level 0 vertices $v_1, v_2, \ldots, v_{2^r}$ of $BF(r)$ gives two subgraphs H_1 and H_2 of $BF(r)$, each isomorphic to $BF(r-1)$. Since $\{v_1, v_2, \ldots, v_{2^r}\}$ is a vertex-cut of $BF(r)$, the vertices are called *binding vertices* of $BF(r)$.

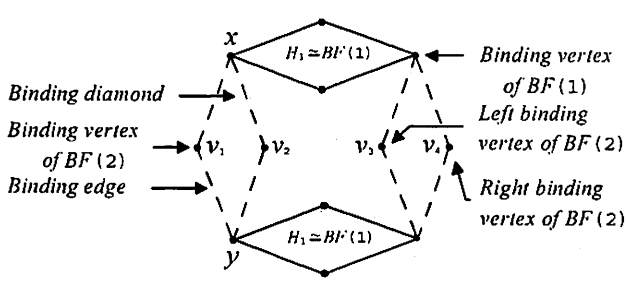

Figure 2: A 2-dimensional butterfly

A 4-cycle xv_1yv_2x in $BF(r)$ where $x \in V(H_1), y \in V(H_2)$ and v_1, v_2 are binding vertices of $BF(r)$ is called a *binding diamond*. The edges of a binding diamond are called *binding edges*. There are exactly two binding vertices of $BF(r)$ adjacent to a binding vertex of $H_1 \equiv BF(r-1)$. One is called the left binding vertex and the other is called the right binding vertex. See Figure 2.

3.2 The Benes Network

The Benes network is very similar to the butterfly network, in terms of both its computational power and its network structure. As butterfly is known for FFT, Benes is known for permutation routing.

The Benes network consists of back-to-back butterflies. An r-dimensional Benes network has $2r+1$ levels, each level with 2^r nodes. The level zero to level r vertices in the network form an r-dimensional butterfly. The middle level of the Benes network is shared by these butterflies. An r-dimensional Benes is denoted by $B(r)$. The r-dimensional Benes has $(2r+1)2^r$ nodes and $r2^{r+2}$ edges. See Figure 3.

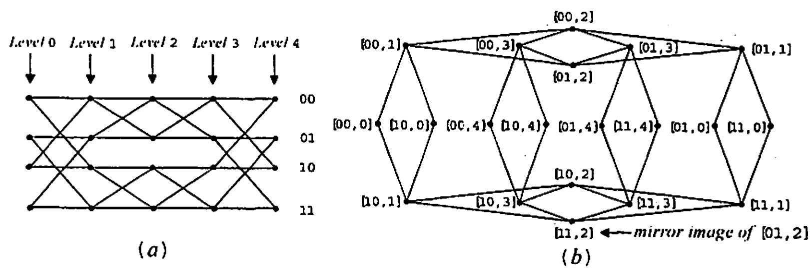

Figure 3: Binary labeling of a 2-dimensional Benes network (a) Normal form and (b) Diamond form

The removal of the level 0 vertices $v_1, v_2, ..., v_{2^r}$ and the level $2r$ vertices $v_{2^r+1}, v_{2^r+2}, ..., v_{2^{r+1}}$ of $B(r)$ gives two subgraphs H_1 and H_2 of $B(r)$, each isomorphic to $B(r-1)$. As in Butterfly networks, we may define binding vertices, binding edges and binding diamonds for a Benes network. See Figure 4.

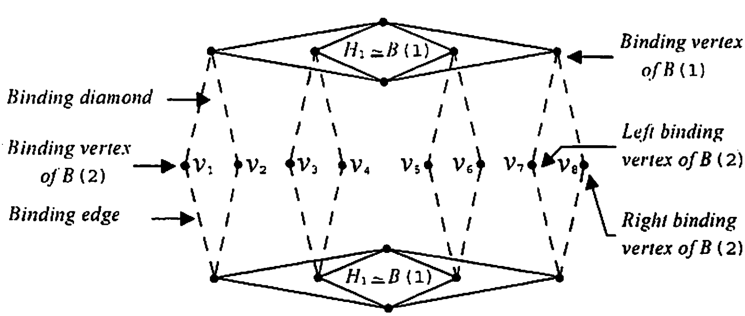

Figure 4: A 2-dimensional Benes network

4 $BF(r)$ and $B(r)$ are minimally 2^r- equitable

Theorem 1 *The r-dimensional butterfly $BF(r)$ is minimally 2^r- equitable.*

Proof. We give an inductive labeling of the vertices of $BF(r)$. The 1-dimensional butterfly is labeled as shown in Figure 5.

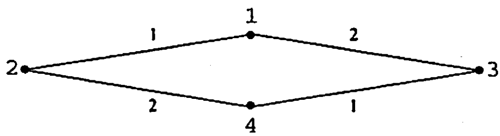

Figure 5: Labeling of a 1-dimensional Butterfly

For $t \geq 2$, the mirror image of vertex labeled i in $BF(t-1)$ is labeled as $i + (t+2)2^{t-1}$, $1 \leq i \leq t2^{t-1}$. The left child of the binding vertex of $BF(t-1)$ with label i is labeled as $i + (t+1)2^{t-2}$and the right child is labeled as $i + (t+3)2^{t-2}$. Figure 6 illustrates the labeling of $BF(3)$.

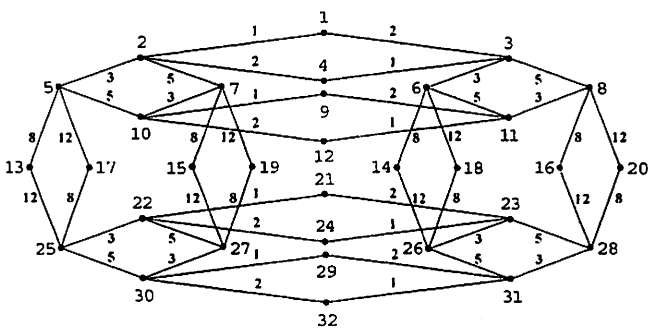

Figure 6: Labeling of $BF(3)$

We prove by induction that the labels on the edges of $BF(r)$ induce a minimally 2^r- equitable labeling. By verification, $BF(1)$ is minimally 2- equitable. Assume that $BF(t-1)$ is minimally 2^{t-1}- equitable. Consider $BF(t)$. Since H_1 and H_2 are isomorphic to $BF(t-1)$, H_1 and H_2 are minimally 2^{t-1}- equitable. Therefore each edge label is repeated $2(2^{t-1})$ times.

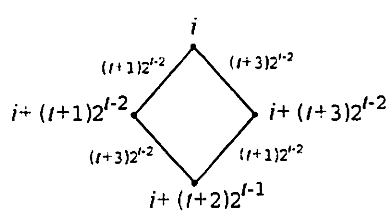

Figure 7: Labeling of a binding diamond of $BF(t)$

We need only to consider the binding diamonds of $BF(t)$. The labeling of when t is even and $i + 2^t$ when t is odd. The right child of the binding vertex of $B(t)$ with label i is labeled as $i + (6t + 3)2^{t-2}$ when t is even and $i + 2^{t+1}$ when t is odd. Figure 9 illustrates the labeling of $B(3)$.

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We need only to consider the binding diamonds of $B(t)$. The labeling of a binding diamond of $B(t)$ when t is even as shown in Figure 10(a). The labels $(6t-1)2^{t-2}, (6t+3)2^{t-2}, (2t+1)2^{t-2}, (2t+5)2^{t-2}$ are repeated only once in each binding diamond. $B(t)$ has 2^t binding diamonds and hence each edge label is repeated 2^t times when t is even. Next the labeling of a binding diamond of $B(t)$ when t is odd as shown in Figure 10(b). The labels $2^t, 2^{t+1}, (2t+1)2^{t-1}, (2t-1)2^{t-1}$ are repeated only once in each binding diamond. $B(t)$ has 2^t binding diamonds and hence each edge label is repeated 2^t times in $B(t)$ when t is odd.

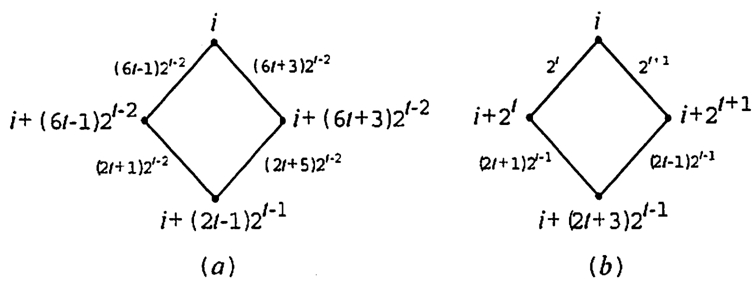

Figure 10: Labeling of a binding diamond of $B(t)$ (a) t even and (b) t odd

It remains to show that the labels $\{(6t-1)2^{t-2}, (6t+3)2^{t-2}, (2t+1)2^{t-2}, (2t+5)2^{t-2}\}$ in the binding diamond of $B(t)$ when t is even and the labels $\{2^t, 2^{t+1}, (2t+1)2^{t-1}, (2t-1)2^{t-1}\}$ in the binding diamond of $B(t)$ when t is odd are pair wise distinct. Let $(6t-1)2^{t-2}$ and 2^{t+1} be the labels of the binding edges of $B(t)$ and $B(t+1)$ respectively. Suppose $(6t-1)2^{t-2} = 2^{t+1}$. Then $6t = 9$, which is a contradiction. All other pairs can be proved similarly. Hence $B(t)$ is minimally 2^t- equitable. □

5 Conclusion

In this paper we have proved that Butterfly and Benes networks are minimally 2^r-equitable where r is the dimension of the networks. This problem is under investigation for certain other parallel architecture like hexagonal and honeycomb networks.

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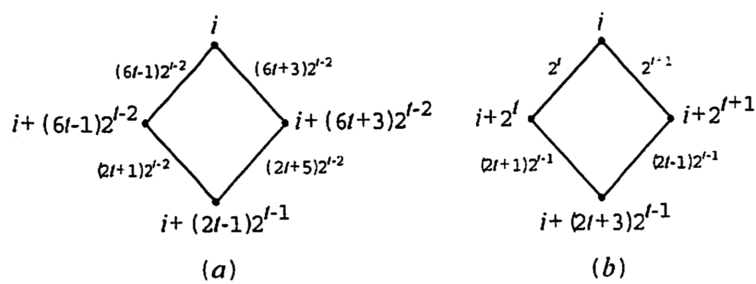

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