Exact Wirelength of Hypercube Layout on k-Cube Necklace

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Abstract

Embeddings capablities play a vital role in evaluating interconnection networks. Wirelength is an important measure of an embedding. As far as the most versatile architecture hypercube is concerned, only approximate estimates of the wirelength of various embeddings are available. This paper presents an optimal embedding of hypercube into a new architecture called k-cube necklace which minimizes wirelength. In addition this paper gives exact formula of minimum wirelength of hypercube into k-cube necklace and thereby we solve completely the wirelength problem of hypercube into k-cube necklace.

Keywords: Fixed interconnection parallel architecture, hypercubes, cube necklace, embedding, wirelength.

1 Introduction and Terminology

A parallel algorithm or a massively parallel computer can be each modeled by a graph, in which the vertices of the graph represent the processes or processing elements, and the edges represent the communications among processes or processors. Thus, the problem of efficiently executing a parallel algorithm A on a parallel computer M can be often reduced to the problem of mapping the graph G, representing A, on the graph H, representing M, so that the mapping satisfies some predefined constraints. This is called graph embedding [13], which is defined more precisely as follows:

Let G and H be finite graphs with n vertices. $V(G)$ and $V(H)$ denote the vertex sets of G and H respectively. $E(G)$ and $E(H)$ denote the edge sets of G and H respectively. An embedding [4] f of G into H is defined as follows:

(*i*). f is a bijective map from $V(G) \to V(H)$

(*ii*). f is a one-to-one map from $E(G)$ to $\{P_f(f(u), f(v)) : P_f(f(u), f(v))$ is a path in H between $f(u)$ and $f(v)\}$.

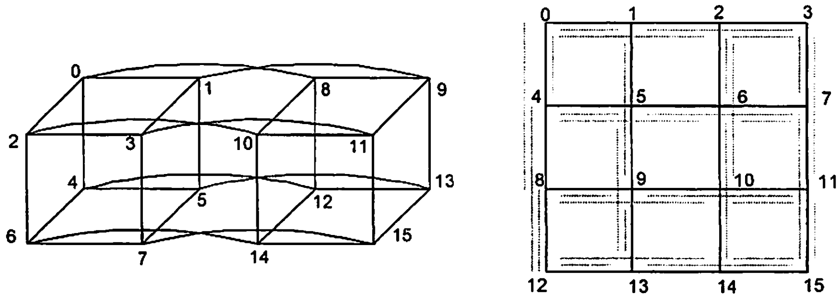

Figure 1: Embedding of a Hypercube into a Grid

See Figure 1. A set of edges of H is said to be an *edge cut* of H if the removal of these edges results in a disconnection of H.

The *congestion* of an embedding f of G into H is the maximum number of edges of the graph G that are embedded on any single edge of H. Let $EC_f(G, H(e))$ denote the number of edges (u, v) of G such that e is in the path $P_f(f(u), f(v))$ between $f(u)$ and $f(v)$ in H. In other words,

$$EC_f(G, H(e)) = |\{(u,v) \in E(G) : e \in P_f(f(u), f(v))\}|$$

where $P_f(f(u), f(v))$ denotes the path between $f(u)$ and $f(v)$ in H with respect to f.

The Edge Congestion Problem The *edge congestion* [16] of an embedding f of G into H is given by

$$EC_f(G, H) = \max EC_f(G, H(e))$$

where the maximum is taken over all edges e of H. Then, the *minimum edge congestion* of G into H is defined as

$$EC(G, H) = \min EC_f(G, H)$$

where the minimum is taken over all embeddings f of G into H. See Figure 2. The *edge congestion problem* of a graph G into H is to find an embedding of G into H that induces the minimum edge congestion $EC(G, H)$. ¤

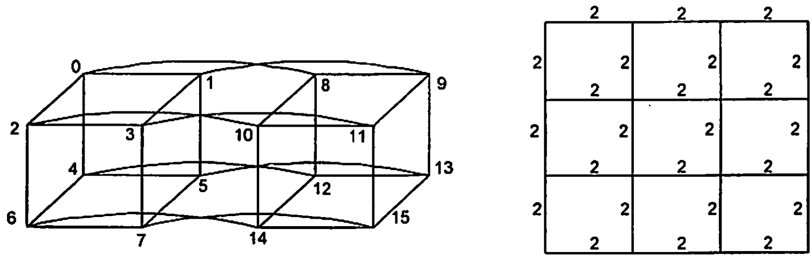

Figure 2: For the embedding mentioned in Figure 1, the edge congestions are marked on the respective edges of the Grid

The Wirelength Problem The *wirelength* of an embedding f of G into H is given by

$$WL_f(G,H) = \sum_{(u,v)\in E(G)} d_H(f(u),f(v))$$

where $d_H(f(u),f(v))$ denotes the length of the path $P_f(f(u),f(v))$ in H. Then, the *minimum wirelength* of G into H is defined as

$$WL(G,H) = \min WL_f(G,H)$$

where the minimum is taken over all embeddings f of G into H. The *wirelength problem* [4, 5, 7, 13, 9] of a graph G into H is to find an embedding of G into H that induces the minimum wirelength $WL(G,H)$. ¤

2 Overview of the Paper

The wirelength of a graph embedding arises from VLSI designs, data structures and data representations, networks for parallel computer systems, biological models that deal with cloning and visual stimuli, parallel architecture, structural engineering and so on [16]. VLSI Layout Problem [1] is a part of grid embedding. Embedding problems have been considered for star networks into hypercubes [2], complete trees into hypercubes [3], hypercubes into grids [4], incomplete hypercube in books [10], grids in surfaces [8], cycles into faulty twisted cubes [12], complete graphs into hypercubes [11], ladders into hypercubes [6], hypercubes into complete binary trees [15], and binary trees into grids [13].

Even though there are numerous results and discussions on the wirelength problem, most of them deal with only approximate results and the estimation of lower bounds [4]. The embeddings discussed in this paper produce an optimal wirelength. We also derive a formula for the minimum wirelength of hypercubes into k-cube necklace. ¤

3 A Few Basic Results

Here onwards, for the sake of simplicity $EC_f(G, H(e))$ will be represented by $EC_f(e)$. For any set S of edges of H, $EC_f(S) = \sum_{e \in S} EC_f(e)$. The following lemma will be used throughout this paper. We apply this result to estimate the edge congestion and wirelength.

Partition Lemma [14] Let G be an r-regular graph and f be an embedding of G into H. Let S be an edge cut of H such that the removal of edges of S leaves H into 2 components H_1 and H_2 and let $G_1 = f^{-1}(H_1)$ *and* $G_2 = f^{-1}(H_2)$. *Also* S satisfies the following conditions:

(i) *For every edge* $(a, b) \in G_i$, $i = 1, 2, P_f(f(a), f(b))$ has no edges in S.

(ii) *For every edge* (a, b) in G with $a \in G_1$ and $b \in G_2$, $P_f(f(a), f(b))$ has exactly one edge in S.

(iii) G_1 *is a maximum subgraph on* k vertices where $k = |V(G_1)|$.

Then $EC_f(S)$ *is minimum.*

Lemma 1 *Let* $f : G \to H$ *be an embedding. Let* $\{S_1, S_2, ..., S_p\}$ *be a partition of* $E(H)$ *such that each* S_i *is an edge cut of* H. *Then*

$$WL_f(G, H) = \sum_{i=1} EC_f(S_i).$$

4 Wirelength Problem of Q^r into k-Cube Necklace

Definition 1 *For* $r \geq 1$ *let* Q^r *denote the graph of* r*-dimensional hypercube. The vertex set of* Q^r *is formed by the collection of all* r*-dimensional binary representations. Two vertices* $x, y \in V(Q^r)$ *are adjacent if and only if the corresponding binary representations differ exactly in one bit [3, 16].*

Definition 2 *Let* $P = v_1, v_2, ..., v_m$ *be a path. Let* H_i *be a graph such that* $P \cup H_i$ *has just* v_i *as a cut vertex,* $i = 1, 2, ..., m$. *Then the graph* $P \cup (\bigcup_{i=1}^{m} H_i)$ *is called a necklace. We refer to* P *as spine, the vertices and edges of* P *as spine vertices and spine edges respectively.*

If each H_i is isomorphic to a hypercube H^k then it is called *k-cube necklace*. A 1- cube necklace is a comb. When $k = 2$, we call it a *diamond necklace* and when $k = 3$ we call it a *cubic necklace.*

The main focus of this paper is to embed the guest graph Q^r into the host graph which is a k-cube necklace, $H^{r,k}, 1 \leq k < r$.

Theorem 1 *Let Q^r denote the graph of r-dimensional hypercube. For $i = 0, 1, ..., 2^r - 1$, $L_i = \{0, 1, ..., i\}$ induces a maximal subgraph of Q^r.*

In this section, we show that the lexicographic embedding solves the wirelength problem of Q^r into the k-cube necklace $H^{r,k}$.

Lexicographic Embedding

The lexicographic embedding of Q^r with labeling 0 to $2^r - 1$ into k-cube necklace $H^{r,k}$ is an assignment of labeling of nodes of $H^{r,k}, 1 \leq k < r$ as follows: For $0 \leq i \leq 2^{r-k} - 1$

(i) The 2^{r-k} spine vertices on the Spine are labeled as $i2^k$, from left to right..

(ii) The vertices of hypercube incident with spine vertices labeled $i2^3$, is labeled $(i \times 2^k + 1, i \times 2^k + 2, ..., (i \times 2^k + (2^k - 1)), 0 \leq i \leq 2^{r-k} - 1$.

This embedding is denoted by f. See Figure 3.

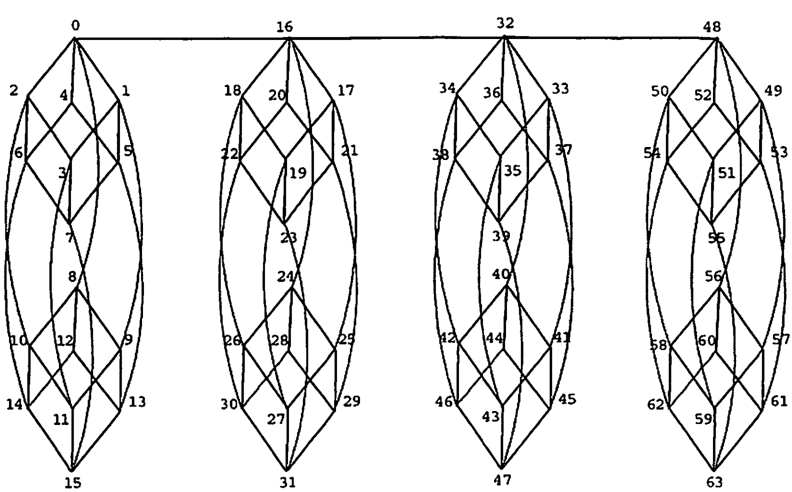

Figure 3: Lexicographic embedding of Q^6 into $H^{6,4}$

We bigin with $k = 3$.

Lemma 2 *There exists an embedding f of Q^r into $H^{r,3}$ which induces minimum wirelength $WL(Q^r, H^{r,3})$.*

Proof. Let A_i, $0 \leq i \leq 2^{r-3} - 1$ be an edge cut of the hypercube in cubic necklace $H^{r,3}$ consisting of 2^{r-3} vertices on the spine such that A_i disconnects each $H^{r,3}$ into two components $H^{r,3}_{i_1}$ and $H^{r,3}_{i_2}$ where the vertices of $H^{r,3}_{i_1}$are labeled $\{2^3 i, 2^3 i + 1, 2^3 i + 4, 2^3 i + 5\}$, $0 \leq i \leq 2^{r-3} - 1$. See Figure 4. Let $G_{i_1} = f^{-1}(H^{r,3}_{i_1})$ and $G_{i_2} = f^{-1}(H^{r,3}_{i_2})$. Then, G_{i_1} is a subcube induced by the vertices $\{2^3 i, 2^3 i + 1, 2^3 i + 4, 2^3 i + 5\}$, $0 \leq i \leq 2^{r-3} - 1$. Hence by Partition Lemma, $EC_f(A_i)$ is minimum.

Consider an edge cut B_j, $0 \leq j \leq 2^{r-3} - 1$. The edge cut B_j disconnects $H^{r,3}$ into two components $H^{r,3}_{j_1}$ and $H^{r,3}_{j_2}$ where the vertices of $H^{r,3}_{j_1}$are labeled $\{2^3 j + 1, 2^3 j + 3, 2^3 j + 5, 2^3 j + 7\}$, $0 \leq j \leq 2^{r-3} - 1$. See Figure 5. Let $G_{j_1} = f^{-1}(H^{r,3}_{j_1})$ and $G_{j_2} = f^{-1}(H^{r,3}_{j_2})$. Then, G_{j_1} is a subcube induced by thevertices $\{2^3 j + 1, 2^3 j + 3, 2^3 j + 5, 2^3 j + 7\}$, $0 \leq j \leq 2^{r-3} - 1$ and by Partition Lemma, $EC_f(B_j)$ is minimum.

In the same way $EC_f(C_k)$ is minimum for $k = 0, 1, ..., 2^{r-3} - 1$. Also, D_l disconnects $H^{r,3}$ into two components such that D_{l_1} and D_{l_2} such that $f^{-1}(D_{l_1})$ is a subcube induced by the vertices $\{0, 1, ..., (l2^3 - 1)\}$ $l = 1, 2, .., 2^{r-3} - 1$. By Theorem 1, $f^{-1}(D_{l_1})$ is a subgraph of Q^r and hence by Partition Lemma $EC_f(D_l)$ is minimum for $l = 1, 2, ..., 2^{r-3} - 1$. See Figure 6. ■

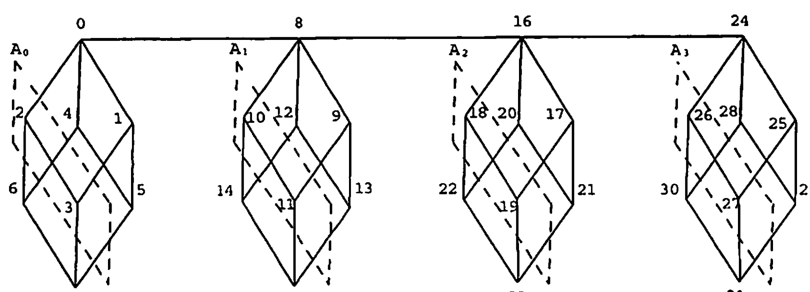

Figure 4: Each A_i is an edge cut of the cubic necklace such that A_i disconnects $H^{5,2}$ into two components $H^{5,3}_{i_1}$and $H^{5,3}_{i_2}$

4.1 Derivation of Wirelength of $WL(Q^r, H^{r,3})$

Until now we have demonstrated that the embedding f of Q^r into $H^{r,3}$ provides minimum edge congestion and minimum wirelength. We proceed to obtain the exact wirelength.

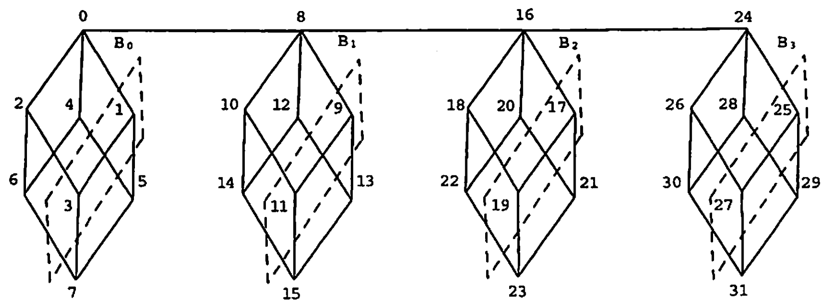

Figure 5: Each B_j is an edge cut of the cubic necklace such that B_j disconnects $H^{5,2}$ into two components $H^{5,3}_{j_1}$and $H^{5,3}_{j_2}$

Theorem 2 [14] $WL(Q^r, P_{2^r}) = 2^{2r-1} - 2^{r-1}$.

We now derive a similar expression for $WL(Q^r, H^{r,3})$.

Theorem 3 *The exact wirelength of* Q^r *into* $H^{r,3}$ *is* $WL(Q^r, H^{r,3}) = 2^{2r-4} - 2^{r-1} + 3 \times 2^{r-1}(r-2)$.

Proof. The embedding f has a nice symmetric property. The sum of the edge congestions on edges of cut A_i, cut B_j, and cut C_k are the same and is equal to $4r - 8$. By Theorem 2, the sum of edge congestion on the spine $= 2^3 WL(Q^{r-3}, P_{2^{r-3}})$.Hence the total wirelength

$$\begin{aligned} &= 2^{2r-4} - 2^{r-1} + 2^{r-3} \times 3(4r-8) \\ &= 2^{2r-4} - 2^{r-1} + 3 \times 2^{r-1}(r-2) \quad \blacksquare \end{aligned}$$

This proves the theorem.

Proceeding in the same lines as in Lemma 2 we obtain the following results for $k > 3$.

Lemma 3 *There exists an embedding* f *of* Q^r *into* $H^{r,k}$ *which induces minimum wirelength* $WL(Q^r, H^{r,k})$.

Proof. Let $s = (s_1, s_2, ..., s_k)$ be a k-tuple and let t be an integer. We define $s+t$ as follows: $s+t = (s_1+t, s_2+t, ..., s_k+t)$. Let $T_i = \{0, 1, \ldots, 2^{i-1}-1\}$. For $1 \le i \le k-1$, $0 \le j \le 2^{r-k}-1$, define C^j_{i+1} as $C^j_{i+1} = \{T_i + j2^k, T_i + 2^i + j2^k, T_i + 2 \times 2^i + j2^k \ ..., (T_i + 2^i(2^{k-i}-1) + j2^k)\}$. The k-edge cuts of $H^{r,k}$ are of the form A^j_i, $1 \le i \le k$. Each edge cut A^j_i divides $H^{r,k}$ into two components $A^j_{i_1}$ and $A^j_{i_2}$ where the vertices of $A^j_{i_1}$ are $A^j_{i_1} = \{C^j_{i+1}\}$, for $1 \le i \le k-1, 0 \le j \le 2^{r-k}-1$, and for $i = k$, $A^j_{i_1} = \{0, 1, ..., 2^{k-1}-1+j2^k\}$. Proceeding as in Lemma 2, we notice that the inverse images $A^j_{i_1}$ and $A^j_{i_2}$

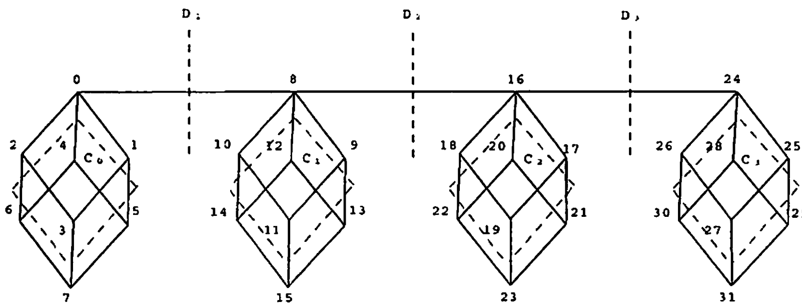

Figure 6: Each C_k is an edge cut of the cubic necklace such that C_k disconnects $H^{5,2}$ into two components $H^{5,3}_{k_1}$and $H^{5,3}_{k_2}$ and each D_l is a cut dege such that D_l disconnects $H^{5,2}$ into two components $H^{5,3}_{l_1}$and $H^{5,3}_{l_2}$

induces maximum subgraph in Q^r. Then by Partition Lemma, $EC_f(A_i^j)$ is minimun for $1 \leq i \leq k-1$, $0 \leq j \leq 2^{r-k}-1$.

Also, each B_l disconnects G into B_{l_1} and B_{l_2} such that $f^{-1}(B_{l_1})$ is a subcube induced by $\{0, 1, ..., l2^k - 1\}, l = 1, 2, ..., 2^{r-k}-1, 2 \leq k \leq r-1$. By Theorem 1, $f^{-1}(B_{l_1})$ is a maximum subgraph of Q^r and hence by Partition Lemma, $EC_f(B_l)$ is minimum for $l = 1, 2, ..., 2^{r-k} - 1$. ■

4.2 Derivation of Wirelength of $WL(Q^r, H^{r,k})$

Until now we have demonstrated that the embedding f of Q^r into $H^{r,k}$ provides minimum edge congestion and minimum wirelength. We proceed to obtain the exact wirelength.

Theorem 4 *The exact wirelength of Q^r into $H^{r,k}$ is* $WL(Q^r, H^{r,k}) = 2^{2r-k-1} - 2^{r-1} + 2^{r-1}(r-k+1)k$, $1 \leq k < r$.

Proof. By Theorem 2, $2^k WL(Q^{r-3}, P_{2^{r-3}})$ gives the sum of edge congestion on the spine. The edge congestion of each of the k cuts is the same and is equal to $2^{k-1}(r-k+1)$, $1 \leq k < r$. Hence the total wirelength

$$\begin{aligned} &= 2^k(2^{2(r-k)-1} - 2^{r-k-1}) + 2^{r-k} \times 2^{k-1} \times k(r-k+1) \\ &= 2^{2r-k-1} - 2^{r-1} + 2^{r-1} \times k(r-k+1) \\ &= 2^{2r-k-1} - 2^{r-1} + 2^{r-1}(r-k+1)k \end{aligned}$$

■

This proves the theorem.

5 Conclusion

We solve the wirelength problem of hypercube into k-cube necklace.The Partition Lemma yields a new technique to estimate the lower bound of wirelength. This problem deals with necklaces which contains all possible hypercubes of dimension one less than that of the guest hypercube. Using this technique it will be interesting to find whether it is possible to solve wirelength problem for architectures such as butterfly, torus, star and pancake architectures. ¤

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