Prime Filters In A Pseudocomplemented Semilattice

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Abstract: In this paper we study the prime filters of a bounded pseudocomplemented semilattice. We extend some of the results of [3] to pseudocomplemented semilattice. It is observed that the set all prime filters $\mathcal{P}$ of a pseudocomlplemented semilattice S is a topology and it is T_0 and compact. We also obtain some necessary and sufficient conditions for the subspace of maximal filters to be normal.

Key words: Prime filters, maximal filters, stone topology, compact space, normal space.

1. Introduction and preliminaries

A semilattice is a partially ordered set in which any two elements have a greatest lower bound. Let S be a semilattice. A semiideal of S is a nonempty subset A of S such that $a \in A, b \leq a (b \in S) \Rightarrow b \in A$. An ideal of S is a semiideal A of S such that the join of any finite number of elements of A, whenever it exists, belongs to A. If $a \in S$, $\{x \in S | x \leq a\}$ is an ideal. It is called the principal ideal generated by a and is denoted by $(a]$. A filter of S is a nonempty subset F of S such that (i) $a \in F$, $b \geq a (b \in S) \Rightarrow b \in F$ and (ii) $a, b \in F \Rightarrow a \wedge b \in F$. The dual of a principal ideal is called a principal filter. The principal filter generated by a is denoted by $[a)$. A maximal ideal(filter) of S is a proper ideal (filter) which is not contained in any other proper ideal(filter). A prime semiideal(ideal) is a proper semiideal(ideal) A such that $a \wedge b \in A \Rightarrow a \in A$ or $b \in A$. A minimal prime semiideal(ideal) is a prime semiideal(ideal) which does not contain any other prime semiideal(ideal). Let $F(S)$ denote the set of filters of S. A prime filter of S is a filter A such that $B, C \in F(S)$, $B \cap C \subseteq A$, $B \cap C \neq \phi \Rightarrow B \subseteq A$ or $C \subseteq A$. If A is a prime filter of S and $A_1, \ldots, A_n \in F(S)$, $A_1 \cap \ldots \cap A_n \subseteq A$, $A_1 \cap \ldots \cap A_n \neq \phi \Rightarrow A_i \Rightarrow A$ for some $i \in \{1, \ldots, n\}$. A semilattice S with 0 is pseudocomplemented if for each $x \in S$, there exists a $x^* \in S$ such that $x \wedge y = 0$ if and only if $y \leq x^*$ ([6]). Pseudocomplemented semilattices have been studied by [8]. An element x of S is said to be normal if $a = a^{**}$.

A 0-distributive lattice is a lattice with 0 in which $a\wedge b=0=a\wedge c$ implies $a\wedge(b\vee c)=0$ ([1]). In [12], it has been proved that a lattice L bounded below is 0-distributive if and only if the ideal lattice $I(L)$ is pseudocomplemented. It is also observed that for an ideal lattice, the two notions of pseudocomplementedness and 0-distributivity are equivalent. A 0-distributive semilattice is a semilattice S with 0 such that $I(S)$, the lattice of ideals of S is 0-distributive ([2]). 0-distributive semilattice has been studied by [13]. It is observed that our concept of 0-distributivity is different from that of Varlet's sense ([2]). Most of the results of this paper are based on the fact that every maximal filter of a pseudocomplemented semilattice is prime. But it is observed that the maximal filter of a 0-distributive semilattice need not be prime ([2]). Hence we are motivated to consider the pseudocomplemented semilattice. Most of the results in this paper are the extension of the corresponding results of [3]. A complemented semilattice is a semilattice with 0 and 1 such that for any $a\in S$, there is a $b\in S$ such that $a\wedge b=0$ and 1 is the only upper bound of a and b ([6]).

For the topological concepts which have now become commonplace the reader is referred to [7] and [5]. For the lattice theoretic concepts the reader is referred to [4].

Let X be a topological space. X is called T_0 if distinct points of X have distinct closures. A point p of X is called a T_1 point if the closure of p contains no point other than p. A point p of X is called an anti-T_1 point if the closure of no other point other than p contains p (for closure we use the notation $Cl.$). X is called T_1 if every point of X is T_1. X is called T_2 if any two distinct points of X have disjoint neighbourhoods.

A closed (open) subject of X is called a closed domain(open domain) if it is identical with the closure of its interior(interior of its closure). A closed(open) subset of X is called regular if it is an intersection(union) of closed domains(open domains) whose interiors(closures) contain(are contained in) it. X is called regular if every open subset of X is regular. The regularity of X may alternatively be expressed as follows. Given a nonempty closed subset C of X and a point of $p\notin C$, we can find closed subsets C_1, C_2 of X containing C, p respectively such that $p\notin C_1$, $C_1\cap C_2=\phi$ and $C_1\cup C_2=X$. A T_1 regular space is called a T_3 - space. A subset A of X is called compact, if every open cover of A has a finite subcover. A subset A of X is said to be dense if $Cl.\ A=X$. X is said to be connected if X is not the union of two disjoint nonempty open subsets of X. Otherwise, X is said to be disconnected.

Let A and B be any two disjoint subsets of X, we say A is weakly separable from B if there exists an open subset of X containing A and disjoint from B. Clearly A is weakly separable from B if and only if $A \cap Cl.B = \phi$. X is called π_0 if every nonempty open subset of X contains a nonempty closed subset. X is called normal if given any two disjoint closed subsets C_1, C_2 of X we can find subsets C_3, C_4 of X containing C_1, C_2 respectively such that $C_1 \cap C_4 = \phi = C_2 \cap C_3$ and $C_3 \cup C_4 = X$.

Throughout the remaining part of this paper S denotes a bounded pseudocomplemented semilattice.

In [11] it has been proved that every maximal filter of a pseudocomplemented lattice is prime.

Lemma 1.1: *Every maximal filter of S is prime.*

Proof: Let M be a maximal filter of S and let $x \in M$. Then there is a prime filter P in S such that $M \subseteq P$ and $x^* \notin P$ ([8]). It follows that $M = P$ and so M is prime.

Lemma 1.2: *Every Pseudocomplemented semilattice is 0-distributive.*

Proof: Let S be a pseudocomplemented semilattice. Then by Lemma 1.1, every maximal filter of S is prime. Hence S is 0-distributive ([2]).

0-distributive semilattice need not be pseudocomplemented.

$S(F)$ denotes the lattice of all filters of S and $\mathcal{P}$ the set of prime filters of S. For any filter A of S, $F(A)$ denote the set of all prime filters containing A and $F^1(A) = \mathcal{P} - F(A)$. Since every proper filter of S (not necessarily pseudocomplemented) is contained in a maximal filter, by Lemma 1.1 it follows that $F(A)$ is nonempty if $A \neq S$.

Theorem 1.3: *Let $\{A_i / i \in I\}$ be any family of filters of S and $A_1, \ldots, A_n$ be any finite number of filters of S. Then*

1. $F(\vee A_i) = \cap F(A_i)$
2. $F(A_1 \cap \ldots \cap A_n) = F(A_1) \cup \ldots \cup F(A_n)$
3. $F(S) = \phi$
4. $F([1)) = \mathcal{P}$

Proof:

1. Let B be a prime filter of S. Then $B \supseteq \vee A_i$ if and only if $B \supseteq A_i$ for all i. Hence 1.

2. If B is a prime filter such that $B \supseteq A_1 \cap \ldots \cap A_n$, then $B \supseteq A_j$ for some $j \in \{1, \ldots, n\}$. Hence $F(A_1 \cap \ldots \cap A_n) \subseteq F(A_1) \cup \ldots \cup F(A_n)$. The reverse inclusion is obvious.
3. and 4 are obvious.

   Hence the proof follows.

   Since $F^1(A) = \mathcal{P} - F(A)$ as a consequence of Theorem 1.3 we have

Theorem 1.4: Let $\{A_i / i \in I\}$ be any family of filters of S and $A_1, \ldots, A_n$ be any finite number of filters of S. Then

1. $F^1(\vee A_i) = \cup F^1(A_i)$
2. $F^1(A_1 \cap \ldots \cap A_n) = F^1(A_1) \cap \ldots \cap F^1(A_n)$
3. $F^1(S) = \mathcal{P}$
4. $F^1([1)) = \phi$

By Theorem1.4, it follows that $\{F^1(A) / A \in S(F)\}$ is a topology on $\mathcal{P}$. We shall denote this topology by T and the resulting topological space $(\mathcal{P}, T)$ also by $\mathcal{P}$ when there is no ambiguity. The sets $F(A)$ are precisely the closed subsets of $\mathcal{P}$. From Theorem 1.3 and Theorem 1.4, it follows that the mapping $A \to F^1(A)$ $(A \to F(A))$ is a homomorphism (dual homomorphism) of the lattice of open subsets (closed subsets) of $\mathcal{P}$ onto $S(F)$.

2. Properties

Theorem 2.1: If X is any subset of $\mathcal{P}$, $Cl.X = F(X_0)$ where X_0 is the intersection of the members of X.

Proof: Clearly $F(X_0)$ is a closed subset of $\mathcal{P}$ and $X \subseteq F(X_0)$. If $X \subseteq F(A)$ for some filter A, then $A \subseteq X_0$ and so $F(X_0) \subseteq F(A)$. Hence the result follows.

Theorem 2.2: *$\mathcal{P}$ is T_0 and compact.*

Proof: From Theorem 2.1, it follows that the closure of a single point is the set of all prime filters containing it. Clearly of any two distinct (prime) filters there is one which does not contain the other. Hence distinct points of $\mathcal{P}$ have distinct closures. Thus $\mathcal{P}$ is T_0.

Let $\mathcal{P} = \cup F^1(A_i)$. By Theorem 1.4, $\mathcal{P} = F^1(\vee A_i)$. Since every proper filter of S is contained in a maximal filter by Lemma 1.1 it follows that $\vee A_i = S$. Hence there exists a finite number of elements $a_{i1}, \ldots, a_{in}$ $(a_{ij} \in A_{ij})$ such that $0 = a_{i1} \wedge \ldots \wedge a_{in}$. Consequently $A_{i1} \vee \ldots \vee A_{in} = S$ and so $\mathcal{P} = F^1(A_{i1} \vee \ldots \vee A_{in})$ $= F^1(A_{i1}) \cup \ldots \cup F^1(A_{in})$. Thus $\mathcal{P}$ is compact.

Theorem 2.3: *The closure of the set of T_1 points of $\mathcal{P}$ is $F(D)$ where D is the filter consisting of the dense elements of S.*

Proof: By Lemma 1.1, every maximal filter of S is in $\mathcal{P}$. As an immediate consequence of Theorem 2.1, we have the T_1 points of $\mathcal{P}$ are precisely the maximal filters of S and the closure of the set of T_1 points of $\mathcal{P}$ is $F(D)$.

Theorem 2.4: $\mathcal{P}$ is π_0 if and only if $D=[1)$.

Proof: Let $D=[1)$ and $F^1(A)$ be any nonempty open subset of $\mathcal{P}$. Then $A\neq D$ and so $A\not\subseteq M$, for some maximal filter M of S. $\{M\}$ is a closed subset of $\mathcal{P}$ by Theorem 2.3 and clearly $F^1(A)\supseteq\{M\}$. Thus $\mathcal{P}$ is π_0.

Suppose $D\neq[1)$. Let $F(B)$ be any nonempty closed subset of $\mathcal{P}$. Then $B\neq S$ and so $B\subseteq M$ for some maximal filters M. by Lemma 1.1, $M\in\mathcal{P}$, so that $M\in F(B)$. But $M\notin F^1(D)$. Hence $F^1(D)\not\subseteq F(B)$ and thus $\mathcal{P}$ is not π_0.

Theorem 2.5: *$\mathcal{P}$ is normal if and only if $[a)\cap[a)^*=[1)$ for every normal element $a\in S$.*

Proof: Suppose $[a)\cap[a)^*=[1)$ for every normal element $a\in S$. Let $F(A)$ and $F(B)$ be any two distinct closed subsets of $\mathcal{P}$. Then by Theorem 1.3, $F(A\vee B)=\phi$. Hence $A\vee B=S$. Hence there exist $a\in A$ and $b\in B$ such that $a\wedge b=0$. It follows that $b\leq a^*$ and so $a^*\in B$. Let $C_1=F([a^{**}))$ and $C_2=F([a^*))$. Then $F(A)\cap C_2=F(A)\cap F([a^*))\subseteq F([a))\cap F([a^*)) =F([a)\vee[a^*))=F([a\wedge a^*)) =F([0))=F(S)=\phi$ and $F(B)\cap C_1=F(B)\cap F([a^{**}))\subseteq F([a^*))\cap F([a^{**})) =F([a^*)\vee[a^{**}))=F([a^*\wedge a^{**})) =F([0))=F(S)=\phi$. Since $a\leq a^{**}$, $a^{**}\in A$ and so $C_1\supseteq F(A)$. As $a^*\in B$, $C_2\supseteq F(B)$. Also $C_1\cup C_2=F([a^{**}))\cup F([a^*))=F([a^{**})\cap [a^*))=F([a^{**}\vee a^*))=F([1))=\mathcal{P}$. Hence $\mathcal{P}$ is normal.

Conversely suppose $\mathcal{P}$ is normal and x be any normal element of S. Set $F_1=F([x))$ and $F_2=F([x^*))$. Then $F_1\cap F_2=\phi$. As $\mathcal{P}$ is normal there exist closed subsets $F_3=F(A)$, $F_4=F(B)$ containing F_1, F_2 respectively $F_1\cap F_4=\phi$, $F_2\cap F_3=\phi$ and $F_3\cup F_4=\mathcal{P}$. From $F_1\cap F_4=\phi$, it follows that $[x)\vee B=S$ and so $x\wedge b=0$ for some $b\in B$. Hence $b\leq x^*$. By similar arguments $a\leq x^{**}=x$ for some $a\in A$. Since $F_3\cup F_4=\mathcal{P}$ it follows that $A\cap B=[1)$. Hence $[a)\cap[b)=[1)$ and so $a\vee b=1$. Now $1=a\vee b\leq x^*\vee x^{**}$. It follows that $[x)\cap[x^{**})=[1)$.

Let us denote the set of all maximal filters of S by $\mathcal{M}$. By Theorem 2.3, the subspace $\mathcal{M}$ is T_1.

Theorem 2.6: *The subspace $\mathcal{M}$ is the smallest of the subspaces X of $\mathcal{P}$ such that X is not weakly separable from any point outside it.*

Proof: Let $A \in \mathcal{P}$ and $A \in \mathcal{M}$. Clearly $Cl\{A\}=F(A)$. Also $A \subseteq M$ for some $M \in \mathcal{M}$ and hence $M \cap Cl\{A\} \neq \phi$. Thus $\mathcal{M}$ is not weakly separable from A.

Let Y be any subspaces of $\mathcal{P}$ such that $M \not\subseteq Y$. Then there exists $M \in \mathcal{M}$ such that $M \notin Y$. By Theorem 2.3, $Cl.\{M\}=\{M\}$ and so $Y \cap Cl.\{M\}=\phi$. Thus Y is weekly separable from M.

Theorem 2.7: Let X be any subset of $\mathcal{P}$ containing $\mathcal{M}$. Then X is compact. In particular $\mathcal{M}$ is compact.

Proof: Let $X \subseteq F^1(A_i)$. Then $X \subseteq F^1(\vee A_i)$ and so $\mathcal{M} \subseteq F^1(\vee A_i)$. It follows that $\vee A_i=S$. Hence there exists a finite number of elements $a_{i1}, \ldots, a_{in} (a_{ij} \in A_{ij})$ such that $0=a_{i1} \wedge \ldots \wedge a_{in}$. Consequently $A_{i1} \vee \ldots \vee A_{in}=S$ and so $X \subseteq F^1(A_{i1} \vee \ldots \vee A_{in})=F^1(A_{i1}) \cup \ldots \cup F^1(A_{in})$. Thus X is compact.

Theorem 2.8: *The first two of the following statements concerning S are equivalent and each of these is implied by the third.*

1. *$\mathcal{P}$ is a T_1 - space*
2. $\mathcal{P}=\mathcal{M}$
3. *Every prime ideal of L is minimal prime.*

Proof: By Theorem 2.3. $1 \Leftrightarrow 2$.

$3 \Leftrightarrow 2$. Suppose 3 holds let $A \in \mathcal{P}$. Clearly $S-A$ is a prime ideal and is therefore a minimal prime ideal by 3. Let B be any minimal prime semiideal contained in $S-A$. $S-B$ is a proper filter and so $S-B \subseteq M$ for some maximal filter M. By Lemma 1.2, S is 0-distributive.It follows that $S-M$ is a minimal prime ideal ([2] Theorem 2.3 4). Also $S-M \subseteq B \subseteq S-A$. Hence $S-M=S-A$ and so $M=A$. Thus $A \in \mathcal{M}$. Hence $\mathcal{P} \subseteq \mathcal{M}$. The reverse inclusion follows by Lemma 1.1.

Theorem 2.9: *If S is Complemented $\mathcal{P}$ is T_2.*

Proof: Let X and Y be distinct points of $\mathcal{P}$. Then $X \not\supseteq Y$ or $Y \not\supseteq X$. Without loss of generality, we may assume $X \not\supseteq Y$. Let $a \in Y-X$, $U=F^1([a))$ and $V=F^1([a^1))$ where a^1 is a complement of a. Clearly U and V are neighborhoods of X and Y respectively. Also $U \cap V=F^1([a)) \cap F^1([a^1))=F^1([a \vee a^1))=F^1([1))=\phi$. Thus $\mathcal{P}$ is T_2.

Theorem 2.10 *Let S be complemented. Then $\mathcal{P}$ is connected if and only if S is the two element chain.*

Proof: Suppose S is the two-element chain. Then $\mathcal{P}=[1)$ and so $\mathcal{P}$ is connected. Suppose $S\neq\{0,1\}$. Let $a\in S$ such that $a\neq 0$, 1 and a^1 be a complement of a. Then $F^1([a)) \cup F^1([a^1))=F^1([a)\vee[a^1))=F^1([a\wedge a^1))=F^1(S)=\mathcal{P}$ and $F^1([a))\cap F^1([a^1)]=F^1([a)\cap[a^1))=F^1([a\vee a^1)=F^1([1))=\phi$. Since $F^1([a))=F([a^1))$ and $F^1([a^1))=F([a))$ and S is 0-distributive(Lemma 1.1), $F^1([a))$ and $F^1([a^1))$ are nonempty ([2]). It follows that $\mathcal{P}$ is disconnected.

Theorem 2.11: *Let S be complemented. Then $\mathcal{M}$ is closed in $\mathcal{P}$.*

Proof: By Theorem 2.1, $Cl.\mathcal{M}=F(D)$. Let $A\in Cl.\mathcal{M}$. We shall prove that $A\in\mathcal{M}$. Let $a\in S$ and $a\notin A$. As $a\vee a^1=1$ (where a^1 is the complement of a) we have $(a\vee a^1)^*=0$ and so $a\vee a^1\in D\subseteq A$. Then $[a)\cap[a^1)\subseteq A$. As A is prime and $[a)\nsubseteq A$ it follows that $[a^1)\subseteq A$ and so $a^1\in A$. Consequently $0=a\wedge a^1\in A\vee[a)$ and so $A\vee[a)=S$. Hence $A\in\mathcal{M}$. It follows that $Cl\ \mathcal{M}=\mathcal{M}$. Hence the result follows.

Corrollary 2.12: *Let S be complemented. Then the subspace $\mathcal{M}$ is compact.*

Lemma 2.13: *The set $U(a)=\{M\in\mathcal{M}/a\in M\}$ is a closed subset of $\mathcal{M}$.*

Proof: Clearly $U(a)=F([a))\cap\mathcal{M}$. Hence $U(a)$ is a closed subset of $\mathcal{M}$.

We denote $\mathcal{M}-U(a)$ by $U^1(a)$.

Definition 2.14: Two ideals A and B of a semilattice S (not necessarily pseudocomplemented) are said to be weakly comaximal if $(A\vee B)\cap D\neq\phi$.

Definition 2.15: Two ideals A and B of a semilattice S (not necessarily pseudocomplemented) are said to be comaximal if $A\vee B=S$.

Lemma 2.16: *Let N be any non dense minimal prime ideal of S. Then $N=(a)^*$ for some $a\in N^*$.*

Proof: Since N is non dense $a\neq 0$ for some $a\in N^*$. Let $b\in N$. Then $a\wedge b=0$ and so $b\in(a)^*$. Let $c\in(a)^*$. Then $c\wedge a=0$. Clearly $a\in S-N$ and so $c\in N$. It follows that $N=(a)^*$.

The following theorem extends the result of [9].

Theorem 2.17: *The set complement of a prime filter of* S *is a prime ideal and every minimal prime ideal of* S *is non-dense and* $(a)^*\wedge(b)^*=(a\wedge b)^*$ *for all* a, b *in* S. *Then the following statements are equivalent.*

1. *Each* $A\in F(D)$ *is contained in a unique maximal filter.*
2. *For* $M_1\neq M_2$ *in* $\mathcal{M}$ *there exist* $a_1\in S-M_1$ *and* $a_2\in S-M_2$ *such that* $a_1\vee a_2\in D$.
3. $\mathcal{M}$ *is* T_2.
4. $\mathcal{M}$ *is normal.*
5. *Any two distinct minimal prime ideals of L are weakly comaximal.*

Proof: 1⇒2. Suppose 1 holds. Let $M_1, M_2\in\mathcal{M}$ and $M_1\neq M_2$. Since S is 0-distributive (Lemma 1.2), $S-M_1$ and $S-M_2$ are minimal prime ideals ([2]). By Lemma 2.16, $S-M_1=(x)^*$ and $S-M_2=(y)^*$ for some $x\in(S-M_1)^*$ and $y\in(S-M_2)^*$ such that $x\neq 0$ and $y\neq 0$. Clearly $(S-M_1)\vee(S-M_2)=(x)^*\vee(y)^*=(x\wedge y)^*$. We claim $((S-M_1)\vee(S-M_2))\cap D\neq\phi$. Suppose $((S-M_1)\vee(S-M_2))\cap D=\phi$. Hence $(x\wedge y)^*\cap D=\phi$. Since S is 0-distributive (Lemma 1.2), for each a in S and each proper filter B containing a, there is a prime ideal N containing $(a)^*$ and contained in B([2], Theorem 2.4 6). Now $S-N$ is a prime filter containing B and disjoint from $(a)^*$. Hence there exists $A\in F(D)$ such that $((S-M_1)\vee(S-M_2))\cap A=\phi$ and consequently $A\subseteq M_1\cap M_2$ which is a contradiction to 1. Hence $((S-M_1)\vee(S-M_2))\cap D\neq\phi$. Let $t\in((S-M_1)\vee(S-M_2))\cap D$. Then t is an upper bound of a_1 and a_2 for some $a_1\in S-M_1$ and $a_2\in S-M_2$. Since $t\in D$, we have $[a_1)\cap[a_2)\subseteq D$.

2⇒3. Suppose 2 holds. Let $M_1, M_2\in\mathcal{M}$ such that $M_1\neq M_2$. By 2 there exists $a_1\in S-M_1$ and $a_2\in S-M_2$ such that $[a_1)\cap[a_2)\subseteq D=\cap\mathcal{M}$. It follows that $U^1(a_1)$, $U^1(a_2)$ are neighborhoods of M_1, M_2 respectively and $U^1(a_1)\cap U^1(a_2)=\phi$.

3⇒4. Suppose 3 holds. Then $\mathcal{M}$ is T_2. By Theorem 2.7, $\mathcal{M}$ is compact. It follows that $\mathcal{M}$ is normal.

4⇒1Suppose there exists $A\in F(D)$ such that $A\subseteq M_1\in\mathcal{M}$, $A\subseteq M_2\in\mathcal{M}$ and $M_1\neq M_2$. Clearly $\{M_1\}$ and $\{M_2\}$ are disjoint closed subsets of $\mathcal{M}$. Let $F^1(A_1)$ be any neighborhood of $\{M_1\}$ and $F^1(A_2)$ be any neighborhood of $\{M_2\}$. Then $A\in F^1(A_1)\cap F^1(A_2)$ and so $F^1(A_1)\cap F^1(A_2)\neq\phi$. Thus $\mathcal{M}$ is not normal.

2⇒5. Suppose 2 holds. Let N_1 and N_2 be any two distinct minimal prime ideals of S. Clearly $S-N_1$ and $S-N_2$ are proper filters. Hence $S-N_1\subseteq M_1$ and $S-N_2\subseteq M_2$ for some maximal filter M_1 and M_2 of S. Since S is 0-distributive (Lemma 1.2), $S-M_1$ and $S-M_2$ are minimal prime ideals ([2]). Also $S-M_1\subseteq N_1$

and $S-M_2 \subseteq N_2$. It follows that $S-M_1=N_1$ and $S-M_2=N_2$. Also N_1 and N_2 are minimal prime semi ideals. Hence $S-N_1$ and $S-N_2$ are distinct maximal filters. By 2 there exist $a_1 \in N_1$ and $a_2 \in N_2$ such that $[a_1) \cap [a_2) \subseteq D$. It follows that $(N_1 \vee N_2) \cap D \neq \phi$.

5⇒2. Suppose 5 holds. Let $M_1, M_2 \in \mathcal{M}$ and $M_1 \neq M_2$. Since S is 0-distributive (Lemma 1.2), $S-M_1$ and $S-M_2$ are minimal prime ideals ([2]). By 5, $((S-M_1) \vee (S-M_2)) \cap D \neq \phi$. Hence there exist $a_1 \in S-M_1$ and $a_2 \in S-M_2$ such that $[a_1) \cap [a_2) \in D$.

The following theorem generalizes some results of [10].

Theorem 2.18: *The set complement of a prime ideal in S is a prime filter. Then the following statements are equivalent.*

1. *Every prime filter of S is contained in a unique maximal filter.*
2. *Every prime ideal of S contains a unique minimal prime ideal.*
3. *Any two distinct minimal prime ideals of S are comaximal.*
4. *For any two maximal filters M_1 and M_2 of S there exist $a_1 \in S-M_1$ and $a_2 \in S-M_2$ such that 1 is the only upper bound of a_1 and a_2.*
5. *$\mathcal{M}$ is T_2.*
6. *$\mathcal{M}$ is normal.*

Proof: 1⇒2. Suppose there is a prime filter Q such $Q \subseteq M_1$ and $Q \subseteq M_2$ for some distinct maximal filters M_1 and M_2. Since S is 0-distributive (Lemma 1.2), $S-M_1$ and $S-M_2$ are distinct minimal prime ideals ([2]). Clearly $S-Q$ is a prime ideal and $S-M_1 \subseteq S-Q$ and $S-M_2 \subseteq S-Q$. It follows that 1⇒2.

2⇒1. Let Q be any prime ideal of S. It follows that $Q \supseteq N_1$ for some minimal prime ideal N_1([2]). Suppose $Q \supseteq N$ for some minimal prime ideal $N_1 \neq N$. By our hypothesis $S-Q$ is a prime filter, Now $S-Q \subseteq S-N$ and $S-Q \subseteq S-N_1$. It is easily seen that $S-N$ and $S-N_1$ are maximal filters. Thus 2⇒1.

3⇒4. Suppose 3 holds. Let M_1 and M_2 be distinct maximal filters of S. Since S is 0-distributive (Lemma1.2), $S-M_1$ and $S-M_2$ are distinct minimal prime ideals ([2]). By 3, $(S-M_1) \vee (S-M_2)=S$. Hence there exist $a_1 \in S-M_1$ and $a_2 \in S-M_2$ such that $a_1 \vee a_2=1$.

4⇒3. Let N_1 and N_2 be distinct minimal prime ideals of S. By our hypothesis $S-N_1$ is a prime filter and so $S-N_1 \subseteq M$ for some maximal filter M. Also M is prime(Lemma 1.1). Clearly $S-M$ is a prime ideal contained in N_1. By the minimality of N_1, $S-M=N_1$. Hence N_1 is a minimal prime semiideal([2]). By similar arguments N_2 is a minimal prime semiideal. Then $S-N_1$ and $S-N_2$

are maximal filters of S. By 4, there exist $a_1 \in S-(S-N_1)=N_1$ and $a_2 \in S-(S-N_2)=N_2$ with $a_1 \vee a_2=1$. Hence $1 \in N_1 \vee N_2$ and so N_1 and N_2 are comaximal.

4⇒1. Let A be a prime filter of S and let $A \subseteq M_1$ and $A \subseteq M_2$ such that $M_1 \neq M_2$. Then there exist $a_1 \in S-M_1$ and $a_2 \in S-M_2$ such that $a_1 \vee a_2=1$. Clearly $[a_1) \not\subseteq A$ and $[a_2) \not\subseteq A$ and $[a_1) \cap [a_2)=[1) \subseteq A$. This contradicts the primality of A. Hence the result.

1⇒3. Suppose 1 holds. Let N_1 and N_2 be any two distinct minimal prime ideals of S such that $N_1 \vee N_2 \neq S$. Clearly $S-(N_1 \vee N_2)$ is a prime filter. By 1, $S-(N_1 \vee N_2)$ is contained in a unique maximal filter M. It is easily seen that $S-N_1$ and $S-N_2$ are maximal filters. Also $S-(N_1 \vee N_2) \subseteq S-N_1$ and $S-(N_1 \vee N_2) \subseteq S-N_2$. It follows that 1⇒3.

3⇒5. Suppose 3 holds. Let $M_1, M_2 \in \mathcal{M}$ and $M_1 \neq M_2$. Hence $S-M_1$ and $S-M_2$ are distinct minimal prime ideals ([2]). By 3, $(S-M_1) \vee (S-M_2)=S$. Hence there exist $a_1 \in S-M_1$ and $a_2 \in S-M_2$ such that $a_1 \vee a_2=1$. It follows that $U^1(a_1)$ and $U^1(a_2)$ are neighborhoods of M_1 and M_2 respectively and that $U^1(a_1) \cap U^1(a_2)=\phi$. Thus $\mathcal{M}$ is T_2.

5⇒6. Suppose 5 holds. Then $\mathcal{M}$ is T_2. By Theorem 2.7, $\mathcal{M}$ is compact. It follows that $\mathcal{M}$ is normal.

6⇒1. Suppose there is a prime filter Q of S such that $Q \subseteq M_1$ and $Q \subseteq M_2$ for some $M_1, M_2 \in \mathcal{M}$ with $M_1 \neq M_2$. Clearly $\{M_1\}$ and $\{M_2\}$ are disjoint closed subsets of $\mathcal{M}$. Let $F^1(A)$ be any neighborhoods of $\{M_1\}$ and $F^1(B)$ any neighborhood of $\{M_2\}$. Clearly $Q \in F^1(A) \cap F^1(B)$ and so $F^1(A) \cap F^1(B) \neq \phi$. It follows that 6⇒1. Hence the proof.

3. Conclusion

We have elaborated the stone topology on the set of prime filters of a pseudocomplemented semilattice. Most of the results of [3] are based on the fact that every maximal filter of a 0-distributive lattice is prime. So the extension needs a class of semilattices satisfying the above mentioned condition. We have restricted ourselves to pseudocomplemented semilattice since every maximal filter of it is prime. In [14], 0-distributive poset is defined as follows. A 0-distributive poset is a poset P with 0 such that $I(P)$, the lattice of ideals of P is 0-distributive. The results of this paper can be extended to the class of 0-distributive posets in which every maximal filter is prime.

4. References

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