Degree sequence conditions for super-edge-connected oriented graphs

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Abstract

If D is a digraph, δ its minimum degree and λ its edge-connectivity, then $\lambda \leq \delta$. A digraph D is called super-edge-connected or super-λ if every minimum edge-cut consists of edges adjacent to or from a vertex of minimum degree. Clearly, if D is super-λ, then $\lambda = \delta$. A digraph without any directed cycle of length 2 is called an oriented graph. Sufficient conditions for digraphs to be super-edge-connected were given by several authors. However, closely related results for oriented graphs have received little attention until recently. In this paper we will present some degree sequence conditions for oriented graphs as well as for oriented bipartite graphs to be super-edge-connected.

Keywords: oriented graph, edge-connectivity, super-edge-connectivity, degree sequence, oriented bipartite graph

1. Introduction and terminology

We consider finite digraphs without loops and multiple edges. A digraph without any directed cycle of length 2 is called an *oriented graph*. For a digraph D the vertex set is denoted by $V(D)$ and the edge set (or arc set) by $E(D)$. We define the *order* of D by $n = n(D) = |V(D)|$. If uv is an edge from u to v in a digraph D, then we write $u \to v$ and say u *dominates* v. If X and Y are two disjoint subsets of $V(D)$ such that every vertex of X dominates every vertex of Y, then we say that X *dominates* Y, denoted by $X \to Y$. For a vertex $v \in V(D)$ of a digraph D let $d^+(v) = d_D^+(v)$ its *out-degree* and $d^-(v) = d_D^-(v)$ its *in-degree*, respectively. The *degree* of a vertex v of a digraph D, denoted by $d(v) = d_D(v)$, is the minimum value of its out-degree and its in-degree. The *degree sequence* of D is defined as the nonincreasing sequence of the degrees of the vertices of D. The *minimum out-degree* and *minimum in-degree* of a digraph D are denoted by $\delta^+ = \delta^+(D)$ and $\delta^- = \delta^-(D)$ and $\delta = \delta(D) = \min\{\delta^+(D), \delta^-(D)\}$ is its *minimum degree*.

A digraph D is *strongly connected* or simply *strong* if for every pair u, v of vertices there exists a directed path from u to v in D. A digraph D is *k-edge-connected* if for any set S of at most $k-1$ edges the subdigraph $D-S$ is strong. The *edge-connectivity* $\lambda = \lambda(D)$ of a digraph D is defined as the largest value of k such that D is k-edge-connected. A digraph D is called *super-edge-connected* or *super-λ* if every minimum edge-cut is trivial, that means that every minimum edge-cut consists of edges adjacent to or from a vertex of minimum degree. Clearly, if D is super-λ, then $\lambda(D) = \delta(D)$.

For two disjoint vertex sets X and Y of a digraph D let (X, Y) be the set of edges from X to Y. For other graph theory terminology we follow Chartrand and Lesniak [3].

Sufficient conditions for digraphs to be super-edge-connected were given by several authors, for example by Balbuena and Carmona [1], Carmona and Fàbrega [2], Fiol [4, 5], Hellwig and Volkmann [6], Soneoka [7] and Volkmann [8]. However, closely related results for oriented graphs have received little attention until recently. In this paper we will present some degree and degree sequence conditions for oriented graphs and oriented bipartite graphs to be super-edge-connected. Examples will demonstrate that the received results are best possible in some sense.

2. Super-edge-connected oriented graphs

We start with a simple but useful observation.

Lemma 2.1 Let D be an oriented graph of edge-connectivity λ and minimum degree $\delta \geq 2$. If D is not super-λ, then there exist two disjoint sets $X, Y \subset V(D)$ with $X \cup Y = V(D)$ and $|(X,Y)| = \lambda$ such that $|X|, |Y| \geq 2\delta$.

Proof. Since D is not super-λ, there exist two disjoint subsets $X, Y \subset V(D)$ with $X \cup Y = V(D)$ such that $|(X,Y)| = \lambda = \delta$ and $|X|, |Y| \geq 2$. By reason of symmetry we only show that $|X| \geq 2\delta$. If we suppose to the contrary that $|X| \leq 2\delta - 1$, then we deduce that

$$|X|\delta \leq |X|\delta^+ \leq \sum_{x \in X} d^+(x) \leq \frac{|X|(|X|-1)}{2} + \lambda \leq |X|(\delta - 1) + \delta.$$

This implies that $|X| \le \delta$ and thus

$$|X|\delta \le |X|\delta^+ \le \sum_{x\in X} d^+(x) \le \frac{|X|(|X|-1)}{2} + \lambda \le \frac{\delta(|X|-1)}{2} + \delta,$$

and hence we obtain the contradiction $|X| \le 1$. □

Corollary 2.2 (Fiol [4] 1992) If D is an oriented graph of order n and minimum degree $\delta \ge \lceil (n+1)/4 \rceil$, then D is super-λ.

Let D be an arbitrary digraph of order n, edge-connectivity λ and degree sequence $d_1 \ge d_2 \ge \ldots \ge d_n = \delta \ge 1$. If $\delta \ge \lfloor n/2 \rfloor + 1$ or if $\delta \le \lfloor n/2 \rfloor$ and

$$\sum_{i=1}^{k} (d_i + d_{n+i-\delta}) \ge k(n-2) + 2\delta + 1$$

for some integer k with $1 \le k \le \delta$, then Volkmann [8] has proved that D is super-λ. The following weaker condition leads to super-edge-connected oriented graphs.

Theorem 2.3 Let D be an oriented graph of order n, edge-connectivity λ and degree sequence $d_1 \ge d_2 \ge \ldots \ge d_n = \delta \ge 1$. If $\delta \ge \lceil (n+1)/4 \rceil$ or if $\delta \le \lceil (n+1)/4 \rceil - 1$ and

$$\sum_{i=1}^{k} (d_i + d_{n+i-2\delta}) \ge k(n-k-1) + 2\delta + 1$$

for some integer k with $1 \le k \le 2\delta$, then D is super-λ.

Proof. Suppose to the contrary that D is not super-λ. Then, by Lemma 2.1, there exist two disjoint sets $X, Y \subset V(D)$ with $X \cup Y = V(D)$ and $|(X,Y)| = \delta$ such that $|X|, |Y| \ge 2\delta$. This implies that $\delta \le \lceil (n+1)/4 \rceil - 1$.

Now let $S \subset X$ and $T \subset Y$ be two k-sets with $1 \le k \le 2\delta$. Since there are exactly δ edges from X to Y, we deduce that

$$\begin{aligned}\sum_{v\in S} d^+(v) &\le \frac{|S|(|S|-1)}{2} + |S|(|X| - |S|) + \delta \\ &= k\left(|X| - \frac{k}{2} - \frac{1}{2}\right) + \delta.\end{aligned}$$

Similarly we obtain

$$\sum_{v\in T} d^-(v) \le k\left(|Y| - \frac{k}{2} - \frac{1}{2}\right) + \delta$$

and thus in total

$$\sum_{v \in S \cup T} d(v) \leq k\,(n-k-1)+2\delta. \tag{1}$$

Now we choose S and T to contain the k vertices in X and in Y of highest degree, respectively. Then $S \cup T$ contains the k vertices of highest degree but not the $2\delta - k$ vertices of lowest degree in D. This implies that

$$\sum_{v \in S \cup T} d(v) \geq \sum_{i=1}^{k} (d_i + d_{n+i-2\delta}).$$

Combining the last inequality together with (1), we obtain a contradiction to our hypothesis. □

The following family of examples will demonstrate that the degree sequence condition in Theorem 2.3 is best possible in the sense that

$$\sum_{i=1}^{k} (d_i + d_{n+i-2\delta}) \geq k(n-k-1)+2\delta$$

for some k with $1 \leq k \leq 2\delta$ does not guarantee that the oriented graph is super-λ.

Example 2.4 Let $p \geq 2$ be an integer, and let T_1' and T_2' be two p-regular tournaments of order $2p+1$ with vertex sets $V(T_1) = \{x_1, x_2, \ldots, x_{2p+1}\}$ and $V(T_2) = \{y_1, y_2, \ldots, y_{2p+1}\}$ such that, without loss of generality,

$$\{x_1, x_2, \ldots, x_p\} \rightarrow x_{2p+1}$$

and $y_{2p+1} \rightarrow \{y_1, y_2, \ldots, y_p\}$. If $T_1 = T_1' - x_{2p+1}$ and $T_2 = T_2' - y_{2p+1}$, then let T be the oriented graph consisting of the disjoint union of T_1 and T_2 together with the edge sets

$$S_1 = \{x_1y_1, x_2y_2, \ldots, x_py_p\}, \quad S_2 = \{y_{p+1}x_{p+1}, y_{p+2}x_{p+2}, \ldots, y_{2p}x_{2p}\}$$

and arbitrary further edges from T_2 to T_1. Then T is of order $4p$ with $\delta(T) = d_T(v) = p$ for all vertices $v \in V(T)$. It follows that

$$\sum_{i=1}^{2\delta} (d_i + d_{n+i-2\delta}) = 4p^2 = 2\delta(T)(n(T) - 2\delta(T) - 1) + 2\delta(T).$$

However, since the set S_1 is a non-trivial minimum edge-cut of the oriented graph T, we notice that T is not super-λ.

Corollary 2.5 Let D be an oriented graph of even order n, minimum degree $\delta \geq 2$ and edge-connectivity λ. If there are $\lfloor n/2 \rfloor$ disjoint pairs of vertices (v_i, w_i) with

$$d(v_i) + d(w_i) \geq n - 2\delta \text{ for } i = 1, 2, \ldots, 2\delta - 1$$

and

$$d(v_i) + d(w_i) \geq n - 2\delta + 1 \text{ for } i = 2\delta, 2\delta + 1, \ldots, \lfloor n/2 \rfloor,$$

then D is super-λ.

Proof. If $\delta \geq \lceil (n-1)/4 \rceil$, then D is super-λ by Corollary 2.2. If $\delta \leq \lceil (n-1)/4 \rceil - 1$, then from the $\lfloor n/2 \rfloor$ pairs of vertices choose 2δ pairs $(v'_1, w'_1), (v'_2, w'_2), \ldots, (v'_{2\delta}, w'_{2\delta})$ containing the 2δ vertices of lowest degree of v_i and w_i. This leads to

$$\begin{aligned} \sum_{i=1}^{2\delta}(d_i + d_{n+i-2\delta}) &\geq \sum_{i=1}^{2\delta}(d(v'_i) + d(w'_i)) \\ &\geq 2\delta(n - 2\delta) + 1 \\ &= 2\delta(n - 2\delta - 1) + 2\delta + 1 \end{aligned}$$

Now Theorem 2.3 with $k = 2\delta$ yields the desired result. □.

Example 2.4 shows that Corollary 2.5 is best possible in the sense that the conditions

$$d(v_i) + d(w_i) \geq n - 2\delta \text{ for } i = 1, 2, \ldots, 2\delta$$

and

$$d(v_i) + d(w_i) \geq n - 2\delta + 1 \text{ for } i = 2\delta + 1, 2\delta + 2, \ldots, \lfloor n/2 \rfloor$$

do not guarantee that the oriented graph D is super-λ.

Corollary 2.6 Let D be an oriented graph of odd order n, minimum degree $\delta \geq 2$ and edge-connectivity λ. If there are $\lfloor n/2 \rfloor$ disjoint pairs of vertices (v_i, w_i) with

$$d(v_i) + d(w_i) \geq n - 2\delta + 1 \text{ for } i = 1, 2, \ldots, 2\delta - 3$$

and

$$d(v_i) + d(w_i) \geq n - 2\delta + 2 \text{ for } i = 2\delta - 2, 2\delta - 1, \ldots, \lfloor n/2 \rfloor,$$

then D is super-λ.

Proof. If $\delta \geq \lceil (n-1)/4 \rceil$, then we are done by Corollary 2.2. If $\delta \leq \lceil (n-1)/4 \rceil - 1$, then from the $\lfloor n/2 \rfloor$ pairs of vertices choose $2\delta - 1$ pairs $(v_1', w_1'), (v_2', w_2'), \ldots, (v_{2\delta-1}', w_{2\delta-1}')$ containing the $2\delta - 1$ vertices of lowest degree of v_i and w_i. This leads to

$$\begin{aligned}\sum_{i=1}^{2\delta-1} (d_i + d_{n+i-2\delta}) &\geq \sum_{i=1}^{2\delta-1} (d(v_i') + d(w_i')) \\ &\geq (2\delta-1)(n-2\delta+1)+2 \\ &= (2\delta-1)(n-(2\delta-1)-1)+2\delta+1,\end{aligned}$$

and Theorem 2.3 with $k = 2\delta - 1$ yields the desired result. □.

3. Super-edge-connected oriented bipartite graphs

Lemma 3.1 Let D be an oriented bipartite graph of edge-connectivity λ and minimum degree $\delta \geq 2$. If D is not super-λ, then there exist two disjoint sets $X, Y \subset V(D)$ with $X \cup Y = V(D)$ and $|(X,Y)| = \lambda$ such that $|X|, |Y| \geq 4\delta - 1$.

Proof. Since D is not super-λ, there exist two disjoint subsets $X, Y \subset V(D)$ with $X \cup Y = V(D)$ and $|(X,Y)| = \lambda$ such that $|X|, |Y| \geq 2$. We only show that $|X| \geq 4\delta - 1$. If we suppose to the contrary that $|X| \leq 4\delta - 2$, then we obtain

$$\begin{aligned}|X|\delta &\leq |X|\delta^+ \leq \sum_{x \in X} d^+(x) \\ &\leq \frac{|X|^2}{4} + \delta \leq \frac{|X|(4\delta-2)}{4} + \delta.\end{aligned}$$

It follows that $|X| \leq 2\delta$ and thus

$$\begin{aligned}|X|\delta &\leq |X|\delta^+ \leq \sum_{x \in X} d^+(x) \\ &\leq \frac{|X|^2}{4} + \delta \leq \frac{2\delta|X|}{4} + \delta.\end{aligned}$$

This leads to $|X| \leq 2$, a contradiction to Lemma 2.1. □

Corollary 3.2 If D is an oriented bipartite graph, then D is super-λ when

$$\delta(D) \geq \left\lceil \frac{n(G)+3}{8} \right\rceil.$$

Theorem 3.3 Let D be an oriented bipartite graph of order n, edge-connectivity λ and degree sequence $d_1 \geq d_2 \geq \ldots \geq d_n = \delta \geq 2$. If $\delta \geq \lceil (n+3)/8 \rceil$ or if $\delta \leq \lceil (n+3)/8 \rceil - 1$ and

$$\sum_{i=1}^{4\delta-1} (d_i + d_{n+i+1-4\delta}) \geq 2\delta(n-4\delta) + 2\delta + 1,$$

then D is super-λ.

Proof. Suppose to the contrary that D is not super-λ. According to Lemma 3.1, there exist two disjoint sets $X, Y \subset V(D)$ with $X \cup Y = V(D)$ and $|(X,Y)| = \lambda$ such that $|X|, |Y| \geq 4\delta - 1$. This implies that $\delta \leq \lceil (n+3)/8 \rceil - 1$.

If V' and V'' is a bipartition of D, then we define by $X' = X \cap V'$, $X'' = X \cap V''$, $Y' = Y \cap V'$ and $Y'' = Y \cap V''$. We assume, without loss of generality, that $|X''| \geq |X'|$. Now we distinguish two cases.

Case 1. Assume that $|X'| \geq 2\delta - 1$. Since $|X| \geq 4\delta - 1$ and $|X''| \geq |X'|$, there exist subsets $S' \subseteq X'$ and $S'' \subseteq X''$ such that $|S'| = 2\delta - 1$ and $|S''| = 2\delta$. If $S = S' \cup S''$, then it follows that

$$\begin{aligned} \sum_{v \in S} d^+(v) &\leq 2\delta(2\delta-1) + 2\delta(|X'| - 2\delta + 1) + (2\delta - 1)(|X''| - 2\delta) + \delta \\ &= 2\delta(|X| - 2\delta) + \delta - |X''| + 2\delta \\ &\leq 2\delta(|X| - 2\delta) + \delta. \end{aligned} \tag{2}$$

Case 2. Assume that $|X'| = 2\delta - t$ for $2 \leq t \leq 2\delta$. Now let $S' = X'$ and $S'' \subseteq X''$ such that $|S''| = 2\delta + t - 1$. If $S = S' \cup S''$, then we conclude that

$$\begin{aligned} \sum_{v \in S} d^+(v) &\leq (2\delta - t)(2\delta + t - 1) + (2\delta - t)(|X''| - 2\delta - t + 1) + \delta \\ &= (2\delta - t)|X''| + \delta \\ &= 2\delta|X''| - t|X''| + \delta + 2\delta|X'| - 2\delta|X'| \\ &= 2\delta(|X| - 2\delta) + \delta + 2t\delta - t|X''| \\ &\leq 2\delta(|X| - 2\delta) + \delta. \end{aligned}$$

Hence we see that inequality (2) is also valid in this case.

Similarly we can choose $T \subseteq Y$ with $|T| = 4\delta - 1$ such that

$$\sum_{v \in T} d^-(v) \leq 2\delta(|Y| - 2\delta) + \delta. \tag{3}$$

Adding (2) and (3), we obtain

$$\sum_{v \in S \cup T} d(v) \leq 2\delta(n - 4\delta) + 2\delta. \tag{4}$$

Now we choose S and T to contain the $4\delta - 1$ vertices in X and in Y of highest degree, respectively. Then $S \cup T$ contains the $4\delta - 1$ vertices of highest degree. This implies that

$$\sum_{v \in S \cup T} d(v) \geq \sum_{i=1}^{4\delta-1} (d_i + d_{n+i+1-4\delta}).$$

Combining the last inequality with (4), we obtain a contradiction to our hypothesis. □

The next examples demonstrate that the degree sequence condition in Theorem 3.3 is best possible in the sense that

$$\sum_{i=1}^{4\delta-1} (d_i + d_{n+i+1-4\delta}) \geq 2\delta(n - 4\delta) + 2\delta$$

does not guarantee that the oriented graph is super-λ.

Example 3.4 Let $p \geq 2$ be an integer, and let B_1' and B_2' be two p-regular bipartite tournaments of order $4p$ with bipartition

$$X' = \{x_1, x_2, \ldots, x_{2p}\}, \quad X'' = \{y_{2p+1}, y_{2p+2}, \ldots, y_{4p}\}$$

and

$$Y'' = \{y_1, y_2, \ldots, y_{2p}\}, \quad Y' = \{x_{2p+1}, x_{2p+2}, \ldots, x_{4p}\}$$

such that, without loss of generality, $\{x_1, x_2, \ldots, x_p\} \to y_{4p}$ and $x_{4p} \to \{y_1, y_2, \ldots, y_p\}$. If $B_1 = B_1' - y_{4p}$ and $B_2 = B_2' - x_{4p}$, then let B be the bipartite tournament consisting of the disjoint union of B_1 and B_2 such that $X' \cup (Y' - x_{4p})$ and $Y'' \cup (X'' - y_{4p})$ are the partite sets of B together with the edge set

$$S = \{x_1y_1, x_2y_2, \ldots, x_py_p\}$$

and all further possible edges from B_2 to B_1. Then B is of order $8p - 2$ with $\delta(B) = d_B(v) = p$ for all vertices $v \in V(B)$. It follows that

$$\begin{aligned} \sum_{i=1}^{4\delta-1} (d_i + d_{n+i+1-4\delta}) &= p(8p-2) \\ &= 2\delta(T)(n(T) - 4\delta(T)) + 2\delta(T). \end{aligned}$$

Since the set S is a non-trivial minimum edge-cut of the bipartite tournament B, we note that B is not super-λ.

Corollary 3.5 Let D be an oriented bipartite graph of even order n, minimum degree $\delta \geq 2$ and edge-connectivity λ. If there are $\lfloor n/2 \rfloor$ disjoint pairs of vertices (v_i, w_i) with

$$d(v_i) + d(w_i) \geq \frac{2\delta n + 1}{4\delta - 1} - 2\delta$$

for $i = 1, 2, \ldots, \lfloor n/2 \rfloor$, then D is super-λ.

Proof. If $\delta \geq \lceil (n+3)/8 \rceil$, then we are done by Corollary 3.2. If $\delta \leq \lceil (n+3)/8 \rceil - 1$, then from the $\lfloor n/2 \rfloor$ pairs of vertices choose $4\delta - 1$ pairs

$$(v_1', w_1'), (v_2', w_2'), \ldots, (v_{4\delta-1}', w_{4\delta-1}')$$

containing the $4\delta - 1$ vertices of lowest degree of v_i and w_i. This leads to

$$\begin{aligned}\sum_{i=1}^{4\delta-1} (d_i + d_{n+i+1-4\delta}) &\geq \sum_{i=1}^{4\delta-1} (d(v_i') + d(w_i')) \\ &\geq (4\delta - 1)\left(\frac{2\delta n + 1}{4\delta - 1} - 2\delta\right) \\ &= 2\delta(n - 4\delta) + 2\delta + 1.\end{aligned}$$

and thus D is super-λ in view of Theorem 3.3. □

Example 3.4 shows that Corollary 3.5 is best possible in the sense that the existence of $\lfloor n/2 \rfloor$ disjoint pairs of vertices (v_i, w_i) with

$$d(v_i) + d(w_i) \geq \frac{2\delta n}{4\delta - 1} - 2\delta$$

for $i = 1, 2, \ldots, \lfloor n/2 \rfloor$ does not guarantee super-edge-connectivity of an oriented bipartite graph.

The next result is a variation of Theorem 3.3 and its proof is similar.

Theorem 3.6 Let D be an oriented bipartite graph of order n, edge-connectivity λ and degree sequence $d_1 \geq d_2 \geq \ldots \geq d_n = \delta \geq 2$. If $\delta \geq \lceil (n+3)/8 \rceil$ or if $\delta \leq \lceil (n+3)/8 \rceil - 1$ and

$$\sum_{i=1}^{2k} (d_i + d_{n+i+1-4\delta}) \geq k(n - 2k) + 2\delta + 1$$

for some integer k with $1 \leq k \leq 2\delta - 1$, then D is super-λ.

Proof. Suppose to the contrary that D is not super-λ. According to Lemma 3.1, there exist two disjoint sets $X, Y \subset V(D)$ with $X \cup Y = V(D)$ and $|(X,Y)| = \lambda$ such that $|X|, |Y| \geq 4\delta - 1$. This implies that $\delta \leq \lceil (n+3)/8 \rceil - 1$.

If V' and V'' is a bipartition of D, then we define by $X' = X \cap V'$, $X'' = X \cap V''$, $Y' = Y \cap V'$ and $Y'' = Y \cap V''$.

Case 1. Assume that $|X'|, |X''| \geq k$. Let $S' \subseteq X'$ and $S'' \subseteq X''$ such that $|S'| = |S''| = k$ and $S = S' \cup S''$. It follows that

$$\sum_{v \in S} d^+(v) \leq k^2 + k(|X'| - k + |X''| - k) + \delta = k(|X| - k) + \delta. \quad (5)$$

Case 2. Assume that $|X'| = k - t$ for $1 \leq t \leq k$. Now let $S' = X'$ and $S'' \subseteq X''$ such that $|S''| = k + t$ and $S = S' \cup S''$. It follows that

$$\begin{aligned} \sum_{v \in S} d^+(v) &\leq (k+t)(k-t) + (k-t)(|X''| - k - t) + \delta \\ &= k(|X| - k) + \delta + kt - t|X''| \\ &\leq k(|X| - k) + \delta. \end{aligned}$$

Hence we see that inequality (5) is also valid in this case.

Similarly we can choose $T \subseteq Y$ with $|T| = 2k$ such that

$$\sum_{v \in T} d^-(v) \leq k(|Y| - k) + \delta. \quad (6)$$

Adding (5) and (6), we obtain

$$\sum_{v \in S \cup T} d(v) \leq k(n - 2k) + 2\delta. \quad (7)$$

Now we choose S and T to contain the $2k$ vertices in X and in Y of highest degree, respectively. Then $S \cup T$ contains the $2k$ vertices of highest degree but not the $4\delta - 1 - 2k$ vertices of lowest degree in D. This implies that

$$\sum_{v \in S \cup T} d(v) \geq \sum_{i=1}^{2k} (d_i + d_{n+i+1-4\delta}).$$

Combining the last inequality with (7), we obtain a contradiction to our hypothesis. □

Corollary 3.7 Let D be an oriented bipartite graph of order n, minimum degree $\delta \geq 2$ and edge-connectivity λ. Then D is super-λ when there are $\lfloor n/2 \rfloor$ disjoint pairs of vertices (v_i, w_i) for $i = 1, 2, \ldots, \lfloor n/2 \rfloor$ with

$$d(v_i) + d(w_i) \geq \frac{n+4}{2} - 2\delta.$$

Proof. If $\delta \geq \lceil (n+3)/8 \rceil$, then D we are done by Corollary 3.2. If $\delta \leq \lceil (n+3)/8 \rceil - 1$, then from the $\lfloor n/2 \rfloor$ pairs of vertices choose $4\delta - 2$ pairs $(v'_1, w'_1), (v'_2, w'_2), \dots, (v'_{4\delta-2}, w'_{4\delta-2})$ containing the $4\delta - 2$ vertices of lowest degree of v_i and w_i. Thus we obtain

$$\begin{aligned}
\sum_{i=1}^{4\delta-2} (d_i + d_{n+i+1-4\delta}) &\geq \sum_{i=1}^{4\delta-2} (d(v'_i) + d(w'_i)) \\
&\geq (4\delta - 2)\left(\frac{n+4}{2} - 2\delta\right) \\
&= (2\delta - 1)(n - 4\delta + 4) \\
&= (2\delta - 1)(n - 2(2\delta - 1)) + 4\delta - 2 \\
&\geq (2\delta - 1)(n - 2(2\delta - 1)) + 2\delta + 1
\end{aligned}$$

and Theorem 3.6 with $k = 2\delta - 1$ leads to the desired result. □

In the case that n is even and $\delta \leq \lceil (n+3)/8 \rceil - 1$, we observe that

$$\left\lceil \frac{2\delta n + 1}{4\delta - 1} \right\rceil \geq \frac{n+4}{2}.$$

This implies that Corollary 3.7 is better than Corollary 3.5 if n is even. Since Corollary 3.5 is best possible, this is also true for Corollary 3.7 in the case that n is even.

If the order n of an oriented bipartite graph is odd, then we can relax the condition in Corollary 3.7 slightly.

Corollary 3.8 Let D be an oriented bipartite graph of odd order n, minimum degree $\delta \geq 2$ and edge-connectivity λ. If there are $\lfloor n/2 \rfloor$ disjoint pairs of vertices (v_i, w_i) with

$$d(v_i) + d(w_i) \geq \frac{n+3}{2} - 2\delta \text{ for } i = 1, 2, \dots, 4\delta - 4$$

and

$$d(v_i) + d(w_i) \geq \frac{n+5}{2} - 2\delta \text{ for } i = 4\delta - 3, 4\delta - 2, \dots, \lfloor n/2 \rfloor,$$

then D is super-λ.

Proof. If $\delta \geq \lceil (n+3)/8 \rceil$, then we are done by Corollary 3.2. If $\delta \leq \lceil (n+3)/8 \rceil - 1$, then from the $\lfloor n/2 \rfloor$ pairs of vertices choose $4\delta - 2$ pairs

$(v_1', w_1'), (v_2', w_2'), \ldots, (v_{4\delta-2}', w_{4\delta-2}')$ containing the $4\delta - 2$ vertices of lowest degree of v_i and w_i. Thus we obtain

$$\begin{aligned}\sum_{i=1}^{4\delta-2}(d_i + d_{n+i+1-4\delta}) &\geq \sum_{i=1}^{4\delta-2}(d(v_i') + d(w_i')) \\ &\geq (4\delta-2)\left(\frac{n+3}{2} - 2\delta\right) + 2 \\ &= (2\delta-1)(n-2(2\delta-1)) + 2\delta + 1\end{aligned}$$

and Theorem 3.6 with $k = 2\delta - 1$ leads to the desired result □.

References

[1] C. Balbuena and A. Carmona, On the connectivity and superconnectivity of bipartite digraphs and graphs, *Ars Combin.* **61** (2001) 3-21.

[2] A. Carmona and J. Fàbrega, On the superconnectivity and the conditional diameter of graphs and digraphs, *Networks* **34** (1999), 197-205.

[3] G. Chartrand and L. Lesniak, Graphs and digraphs, 3rd ed., Chapman & Hall, 1996.

[4] M.A. Fiol, On super-edge-connected digraphs and bipartite digraphs, *J. Graph Theory* **16** (1992), 545-555.

[5] M.A. Fiol, The superconnectivity of large digraphs and graphs, *Discrete Math.* **124** (1994), 67-78.

[6] A. Hellwig and L. Volkmann, Neighborhood and degree conditions for super-ege-connected bipartite digraphs, *Result. Math.* **45** (2004), 45-58.

[7] T. Soneoka, Super edge-connectivity of dense digraphs and graphs, *Discrete Appl. Math.* **37/38** (1992), 511-523.

[8] L. Volkmann, Degree sequence conditions for super-edge-connected graphs and digraphs, *Ars Combin.* **67** (2003), 237-249.