Lower Bounds on the p-Domination Number in Terms of Cycles and Matching Number

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Abstract

Let G be a simple graph, and let p be a positive integer. A subset $D \subseteq V(G)$ is a *p-dominating set* of the graph G, if every vertex $v \in V(G) - D$ is adjacent with at least p vertices of D. The *p-domination number* $\gamma_p(G)$ is the minimum cardinality among the p-dominating sets of G. Note that the 1-domination number $\gamma_1(G)$ is the usual *domination number* $\gamma(G)$.

In 1985, Fink and Jacobson showed that for every graph G with n vertices and m edges the inequality $\gamma_p(G) \geq n - m/p$ holds. In this paper we present a generalization of this theorem and analyze the 2-domination number γ_2 in cactus graphs G with respect on its relation to the matching number α_0 and the number of odd or rather even cycles in G. Further we show that $\gamma_2(G) \geq \alpha(G)$ for the cactus graphs G with at most one even cycle and characterize those which fulfill $\gamma_2(G) = \alpha(G)$ or rather $\gamma_2(G) = \alpha(G) + 1$.

Keywords: Domination; p-domination; Multiple domination; Cactus graph; Matching number

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1. Terminology and Introduction

We consider finite, undirected and simple graphs G with vertex set $V(G)$ and edge set $E(G)$. If multiple edges are allowed, we will specify the graph as a *multigraph*, otherwise we will call it only *graph*. The number of vertices

$|V(G)|$ of a graph G is called the *order* of G and is denoted by $n = n(G)$. The *size* $m = m(G)$ of a graph G is the number of edges $|E(G)|$. If R and S are subsets of the vertex set $V(G)$ of a graph G, then we denote by $(R,S)_G = (R,S)$ the set of edges in G with one end in R and the other in S. The number of edges in (R,S) is denoted with $m_G(R,S) = m(R,S)$.

The *open neighborhood* $N(v) = N_G(v)$ of a vertex v consists of the vertices adjacent to v and $d(v) = d_G(v) = |N(v)|$ is the *degree* of v. The *closed neighborhood* of a vertex v is defined by $N[v] = N_G[v] = N(v) \cup \{v\}$. A vertex of degree one is called a *leaf* and its neighbor is called a *support vertex*. An edge incident with a leaf is called a *pendant edge*. We denote with $L(G)$ the set of leaves of a graph G. For a subset $S \subseteq V(G)$, we define $N(S) = N_G(S) = \bigcup_{v \in S} N(v)$, $N[S] = N_G[S] = N(S) \cup S$, and $G[S]$ is the subgraph induced by S.

A *block* of a graph G is a maximal subgraph of G without cut vertices. If every block of a graph is complete, then we speak of a *block graph*. We write K_n for the *complete graph* of order n, and $K_{p,q}$ for the the *complete bipartite graph* with bipartition X, Y such that $|X| = p$ and $|Y| = q$. The *corona graph* $G \circ K_1$ of a graph G is the graph constructed from a copy of G, where for each vertex $v \in V(G)$, a new vertex v' and a pendant edge vv' are added.

The *subdivision graph* $S(G)$ of a graph G is the graph obtained from G by replacing each edge uv of G by a vertex w and edges uw and vw. In the case that G is the empty graph, we define $S(G) = G$. Let SS_t be the subdivision graph of the star $K_{1,t}$. A bipartite graph G is called *p-semiregular* if its vertex set can be bipartitioned in such a way that every vertex of one of the partite sets has degree p.

A set of pairwise not incident edges of a graph G is called *matching*. A matching M_0 with maximum number of edges is a *maximum matching* and the number $\alpha_0(G) = |M_0|$ is called the *matching number* of G. Let M be a matching of a graph G. A path is said to be M-alternating if its edges belong alternating to M and not to M. A vertex and an edge are said to *cover* each other if they are incident. A *vertex cover* in a graph G is a set of vertices that covers all edges of G. The minimum cardinality of a vertex cover in a graph G is called the *covering number* of G and is denoted by $\beta(G) = \beta$. A set of pairwise non-adjacent vertices of G is an *independent* set of G.

Let p be a positive integer. A subset $D \subseteq V(G)$ is a *p-dominating set* of the graph G, if $|N(v) \cap D| \geq p$ for every $v \in V(G) - D$. The *p-domination number* $\gamma_p(G)$ is the minimum cardinality among the p-dominating sets of G. Note that the 1-domination number $\gamma_1(G)$ is the usual *domination number* $\gamma(G)$. A p-dominating set of minimum cardinality of a graph G is called a *$\gamma_p(G)$-set*.

In [1], [2], Fink and Jacobson introduced the concept of p-domination. For a comprehensive treatment of domination in graphs, see the monographs by Haynes, Hedetniemi and Slater [3], [4].

Recently, Volkmann showed in [6] that, if T is a nontrivial tree, then $\gamma_2(T) \geq \beta(T)+1$, and he characterized all such trees with $\gamma_2(T) = \beta(T)+1$. This implies that

$$\gamma_2(T) \geq \beta(T) + 1 \geq \alpha_0(T) + 1.$$

Applying the well-known identity $\beta(G) = \alpha_0(G)$ of König [5] for every bipartite graph G, we observe that, for a nontrivial tree T, $\gamma_2(T) = \beta(T) + 1$ if and only if $\gamma_2(T) = \alpha_0(T) + 1$. As an extension of the inequality $\gamma_2(T) \geq \alpha_0(T) + 1$ for nontrivial trees T, we show in this paper $\gamma_2(G) \geq \alpha_0(G) + 1$ for all connected cactus graphs G without cycles of even length and for all cactus graphs G of odd order and one even cycle.

2. Generalization of a Theorem of Fink and Jacobson

In [1], Fink and Jacobson presented the following theorem.

Theorem 2.1 (Fink, Jacobson [1], 1985) If G is a graph with n vertices and m edges, then

$$\gamma_p(G) \geq n - \frac{m}{p}$$

for each $p \geq 1$. Furthermore, if $m \neq 0$, then $\gamma_p(G) = n - \frac{m}{p}$ if and only if G is a p-semiregular graph.

We will now give a generalization of this theorem introducing a new parameter μ_o, which represents the minimum number of edges that can be removed from a graph G such that the remaining graph is bipartite.

Theorem 2.2 Let G be a graph of order n and size m. If $\mu_o = \mu_o(G)$ is the minimum number of edges that can be removed from G such that the remaining graph is bipartite, then

$$\gamma_p(G) \geq n - \frac{m - \mu_o}{p}.$$

Additionally, if $m \neq 0$, then $\gamma_p(G) = \lceil n - \frac{m-\mu_o}{p} \rceil$ if and only if G contains a p-semiregular factor H with $m(H) = m - \mu_o - r$, where r is an integer such that $0 \leq r \leq p - 1$ and $m - \mu_o - r \equiv 0 \pmod{p}$.

Proof. Let $V = V(G)$ and let D be a $\gamma_p(G)$-set. Let $K = (D, D) \cup (V - D, V - D)$. Since $G - K$ contains no odd cycles, it follows that $|K| \geq \mu_o$. As every vertex in $V - D$ has at least p neighbors in D, it follows

$$\begin{aligned} m &= m(D, V - D) + |K| \geq p\,|V - D| + |K| \\ &\geq p\,|V - D| + \mu_o = p(n - \gamma_p(G)) + \mu_o, \end{aligned}$$

and consequently we obtain

$$\gamma_p(G) \geq n - \frac{m - \mu_o}{p}.$$

Now assume that $m \neq 0$. Suppose first that $\gamma_p(G) = n - \frac{m-\mu_o-r}{p}$ for an integer r with $0 \leq r \leq p-1$. Since $m - \mu_o > 0$, it follows that $\gamma_p(G) \leq n-1$ and thus $V - D$ can never be empty. Let H be the p-semiregular factor of G such that the vertex sets D and $V - D$ are both independent sets and every vertex in $V - D$ has exactly degree p. Since D is still a p-dominating set of H, we obtain

$$\gamma_p(H) = \gamma_p(G) = n - \frac{m - \mu_o - r}{p} = n - \frac{(n - \gamma_p(G))p}{p} = n - \frac{m(H)}{p}.$$

It follows that $m(H) = m - \mu_o - r$.

Conversely, assume that G has a p-semiregular factor H with $m(H) = m - \mu_o - r$ for an integer r such that $0 \leq r \leq p - 1$ and $m - \mu_o - r \equiv 0$ ($\bmod p$). Let S be the partition set in H of vertices of degree p. Then $|S| = m(H)/p$ and $V - S$ is a p-dominating set of H. This implies

$$\gamma_p(H) \leq |V - S| = n - |S| = n - \frac{m(H)}{p}$$

and thus, together with Theorem 2.1, $V - S$ is a $\gamma_p(H)$-set. Since $V - S$ is also a p-dominating set of G, we obtain

$$\gamma_p(G) \leq |V - S| = n - \frac{m(H)}{p} = n - \frac{m - \mu_o - r}{p}$$

and so $\gamma_p(G) = n - \frac{m-\mu_o-r}{p} = \lceil n - \frac{m-\mu_o}{p} \rceil$ follows. □

Observation 2.3 Let G be a connected graph and let T be a spanning tree of G with partition sets A and B. Since T contains no cycles, it is obvious that $\mu_o(G) \leq m_G(A, A) + m_G(B, B)$. Let now K be a set of edges of G such that $|K| = \mu_o(G)$ and $G - K$ is bipartite and let A' and B' be the partition sets of $G - K$. Then $G - K$ is connected and every edge

$e \in K$ belongs either to (A', A') or to (B', B'), otherwise it would contradict the minimality of $\mu_o(G)$. Conversely, every edge $e \in (A', A') \cup (B', B')$ belongs to K. This shows that $\mu_o(G) = |K| = m_G(A', A') + m_G(B', B')$. It follows that there is a spanning tree with bipartition sets A' and B' and $\mu_o(G) = m_G(A', A') + m_G(B', B')$ and consequently

$$\mu_o(G) = \min\{m_G(A, A) + m_G(B, B) \mid A, B \text{ partition sets of a spanning tree of } G\}.$$

Lemma 2.4 Let G be the subdivision graph of a connected multigraph H and $n = n(G)$. Then $\gamma_2(G) = \alpha_0(G)$ when n is even and $\alpha_0(G) \leq \gamma_2(G) \leq \alpha_0(G) + 1$ when n is odd.

Proof. It is evident that the sets $A := V(H)$ and $B := V(G) \setminus V(H)$ form a bipartition of G where all vertices in B are of degree 2, that is, G is a 2-semiregular simple graph. Since A is a 2-dominating set,

$$\gamma_2(G) \leq |A| = n(G) - \frac{m(G)}{2}$$

holds and thus Theorem 2.1 implies that A is a $\gamma_2(G)$-set. Then it is clear that $\alpha_0(G) \leq \gamma_2(G)$.

Let M be a maximum matching of G and suppose that $\gamma_2(G) > \alpha_0(G)$. It follows that there has to be a vertex $u \in A$ such that $u \notin V(M)$. If n is even, this implies that there is another vertex $v \neq u$ such that $v \notin V(M)$. If n is odd, assume that there is another vertex $v \neq u$ such that $v \notin V(M)$. Let x be the first vertex in a path P from u to v in G such that $x \notin V(M)$ and let P_{ux} be the part of the path P from u to x. It follows that $x \in B$, otherwise would P_{ux} be of even length and either x should be in $V(M)$ or there would be a vertex before x in P_{ux} that does not belong to $V(M)$ (remember that every vertex in B has degree 2). But then $(M \setminus E(P_{ux})) \cup (E(P_{ux}) \setminus M)$ is a matching in G with one more edge than M and we obtain a contradiction. It follows that $\gamma_2(G) = \alpha_0(G)$ when n is even, and that $\gamma_2(G) \leq \alpha_0(G) + 1$ when n is odd. □

Corollary 2.5 Let G be the subdivision graph of a connected multigraph. If G has odd order and $\gamma_2(G) = \alpha_0(G) + 1$, then G contains an almost perfect matching.

Proof. Since $\gamma_2(G) = \alpha_0(G) + 1$, following the proof of Lemma 2.4, this implies that $A = (V(M) \cap A) \cup \{u\}$ and $B = V(M) \cap B$ and the proof is complete. □

3. Cactus Graphs

Since for cactus graphs G the parameter $\mu_o(G)$ equals the number of odd cycles $\nu_o(G)$ in G, we obtain the following corollary from Theorem 2.2.

Corollary 3.1 Let G be a connected cactus graph of order n, size m and ν_o cycles of odd length. If $p \geq 1$ is an integer, then

$$\gamma_p(G) \geq n - \frac{m - \nu_o}{p}$$

and, if $m \neq 0$, then $\gamma_p(G) = \lceil n - \frac{m-\nu_o}{p} \rceil$ if and only if G has a p-semiregular factor H with $m(H) = m - \nu_o - r$ for an integer r with $0 \leq r \leq p-1$ and $m - \nu_o - r \equiv 0 \pmod{p}$.

Corollary 3.2 Let G be a connected cactus graph of order n, size m and ν_e cycles of even length. If $p \geq 1$ is an integer, then

$$\gamma_p(G) \geq \frac{(p-1)n - \nu_e + 1}{p}.$$

Proof. This follows directly from Corollary 3.1 and the well known identity $m = n + \nu_e + \nu_o - 1$ for cactus graphs. □

Theorem 3.3 If G is a connected cactus graph of order n with ν_e cycles of even length, then

(1) $$\gamma_2(G) \geq \alpha_0(G) + 1 - \left\lceil \frac{\nu_e}{2} \right\rceil,$$

and if n and ν_e are both odd, then

(2) $$\gamma_2(G) \geq \alpha_0(G) + 2 - \left\lceil \frac{\nu_e}{2} \right\rceil.$$

Proof. Corollary 3.2 implies

(3) $$\gamma_2(G) \geq \frac{n + 1 - \nu_e}{2}.$$

Using the fact that $\alpha_0(G) \leq \frac{n}{2}$, it follows from (3) that

$$\gamma_2(G) \geq \alpha_0(G) + \frac{1 - \nu_e}{2}$$

and thus (1). If n is odd, then $\alpha_0(G) \leq \frac{n-1}{2}$, and we deduce from (3) that

$$\gamma_2(G) \geq \alpha_0(G) + \frac{2 - \nu_e}{2}.$$

This leads to (2) when ν_e is odd, and the proof is complete. □

In the following figures we present examples which show that Theorem 3.3 is best possible.

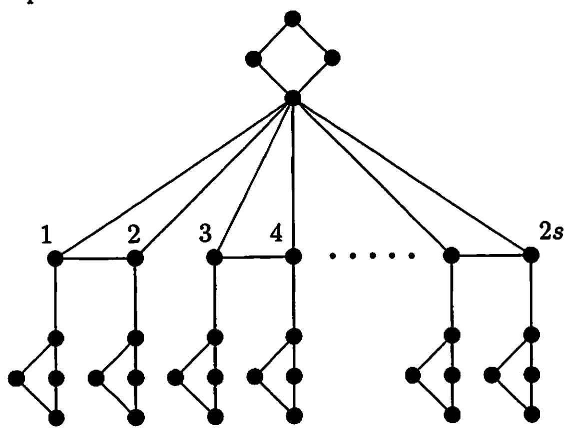

Figure 1

The cactus graph in Figure 1 is of even order $n = 10s + 4$ with an odd number $\nu_e = 2s + 1$ of cycles of even length such that $\gamma_2 = 4s + 2$ and $\alpha_0 = 5s + 2$ and therefore equality in (1).

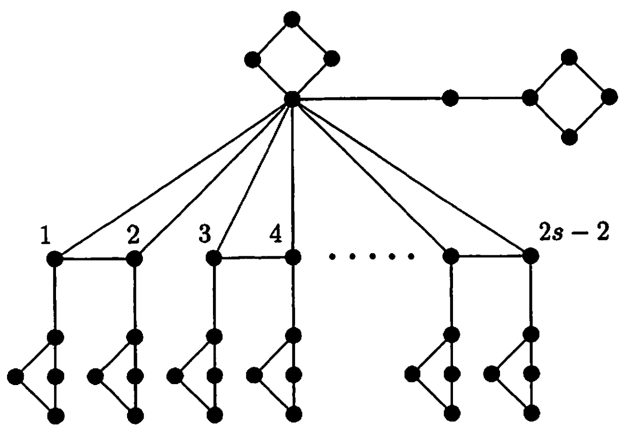

Figure 2

The cactus graph in Figure 2 is of odd order $n = 10s - 1$ with an even number $\nu_e = 2s$ of cycles of even length such that $\gamma_2 = 4s$ and $\alpha_0 = 5s - 1$ and therefore equality in (1).

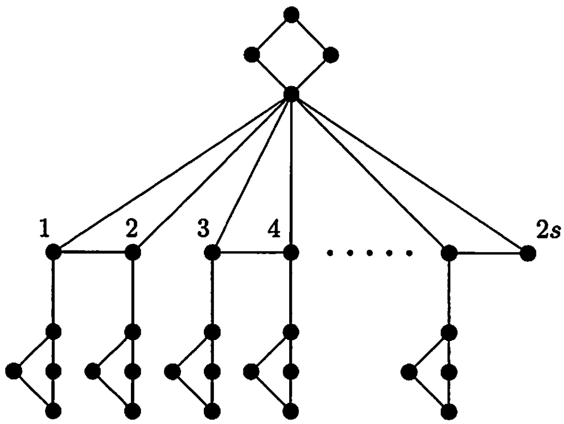

Figure 3

The cactus graph in Figure 3 is of even order $n = 10s$ with an even number $\nu_e = 2s$ of cycles of even length such that $\gamma_2 = 4s + 1$ and $\alpha_0 = 5s$ and therefore equality in (1).

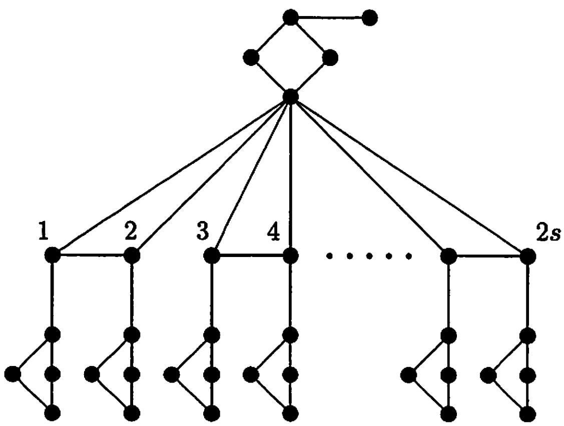

Figure 4

The cactus graph in Figure 4 is of odd order $n = 10s + 5$ with an odd number $\nu_e = 2s + 1$ of cycles of even length such that $\gamma_2 = 4s + 3$ and $\alpha_0 = 5s + 2$ and therefore equality in (2).

Theorem 3.4 Let G be a connected cactus graph of order n and size m and let $\nu_e = \nu_e(G)$ and $\nu_o = \nu_o(G)$. Then the following holds.

1) If n is odd and $\nu_e \leq 1$, then $\gamma_2(G) \geq \alpha_0(G)+1$ and $\gamma_2(G) = \alpha_0(G)+1$ if and only if G has a 2-semiregular factor H with $m(H) = m-\nu_o-\nu_e$.

2) If n is even and $\nu_e = 1$, then $\gamma_2(G) \geq \alpha_0(G)$ and $\gamma_2(G) = \alpha_0(G)$ if and only if G has a 2-semiregular factor H with $m(H) = m - \nu_o$.

3) If n is even and $\nu_e = 0$, then $\gamma_2(G) \geq \alpha_0(G)+1$ and $\gamma_2(G) = \alpha_0(G)+1$ if and only if G has a 2-semiregular factor H with $m(H) = m-\nu_o-1$.

Proof. 1) Let n be odd and $\nu_e \leq 1$. From Theorem 3.3 follows directly $\gamma_2(G) \geq \alpha_0(G) + 1$. Assume now that $\gamma_2(G) = \alpha_0(G) + 1$. Then Corollary 3.2 leads to

$$\alpha_0(G) + 1 \;=\; \gamma_2(G) \;\geq\; \left\lceil \frac{n - \nu_e + 1}{2} \right\rceil \;=\; \frac{n+1}{2} \;\geq\; \alpha_0(G) + 1.$$

This implies $\gamma_2(G) = \lceil \frac{n-\nu_e+1}{2} \rceil$, which is the same as $n - \frac{m-\nu_o}{2}$, if $\nu_e = 0$, and the same as $n - \frac{m-\nu_o-1}{2}$, if $\nu_e = 1$. It follows that $\gamma_2(G) = n - \frac{m-\nu_o-\nu_e}{2}$ and so, by Theorem 2.2, G contains a 2-semiregular factor H with $m(H) = m - \nu_o - \nu_e$.

Conversely, assume that G contains a 2-semiregular factor H such that $m(H) = m - \nu_o - \nu_e$. It is easy to see that H is the subdivision graph of a particular multigraph. Since H is bipartite, at least ν_o edges from $E(G) \setminus E(H)$ belong to pairwise different odd cycles of G and hence, as $\nu_e \leq 1$, H consists of at most two components, in such a case is one of them odd and the other one even. Thus, Lemma 2.4 leads to $\alpha_0(H) \leq \gamma_2(H) \leq \alpha_0(H) + 1$. According to (1) and (2), we obtain

$$\alpha_0(G) + 1 \;\geq\; \alpha_0(H) + 1 \;\geq\; \gamma_2(H) \;\geq\; \gamma_2(G) \geq \alpha_0(G) + 1$$

and hence $\gamma_2(G) = \alpha_0(G) + 1$.

2) Let n be even and $\nu_e = 1$. From Theorem 3.3 follows directly that $\gamma_2(G) \geq \alpha_0(G)$. Suppose that $\gamma_2(G) = \alpha_0(G)$. Then, again Corollary 3.2 yields

$$\alpha_0(G) = \gamma_2(G) \geq \left\lceil \frac{n + 1 - \nu_e}{2} \right\rceil = \frac{n}{2} \geq \alpha_0(G).$$

This leads to the fact that $\gamma_2(G) = n - \frac{m-\nu_o}{2}$. Hence, applying Theorem 2.2, G has a 2-semiregular factor H with $m(H) = m - \nu_o$.

Conversely, assume that G has a 2-semiregular factor H with $m(H) = m - \nu_o$. As above, H is the subdivision graph of a particular multigraph.

Since H is a bipartite cactus graph, the set $E(G) \setminus E(H)$ contains exactly one edge of every odd cycle in G and thus H is connected. It follows with Theorem 3.3 and Lemma 2.4 that

$$\alpha_0(G) \geq \alpha_0(H) = \gamma_2(H) \geq \gamma_2(G) \geq \alpha_0(G),$$

which implies $\alpha_0(G) = \gamma_2(G)$.

3) Let n be even and $\nu_e = 0$. Theorem 3.3 implies that $\gamma_2(G) \geq \alpha_0(G) + 1$. Assume first that $\gamma_2(G) = \alpha_0(G) + 1$. Then, applying once more Corollary 3.2, it follows

$$\alpha_0(G) + 1 \; = \; \gamma_2(G) \; \geq \; \left\lceil \frac{n+1-\nu_e}{2} \right\rceil \; = \; \frac{n+2}{2} \; \geq \; \alpha_0(G) + 1,$$

which implies that $\gamma_2(G) = n + \frac{\nu_o - m + 1}{2}$ and thus G has a 2-semiregular factor with $m(H) = m - \nu_o - 1$.

Suppose now that G has a 2-semiregular factor with $m(H) = m - \nu_o - 1$. With the same arguments as above, H consists of exactly two components. If $\gamma_2(H) = \alpha_0(H)$, then it follows, together with (1),

$$\alpha_0(G) \geq \alpha_0(H) = \gamma_2(H) \geq \gamma_2(G) \geq \alpha_0(G) + 1,$$

which is a contradiction. Therefore, regarding Lemma 2.4, H has two odd components H_1 and H_2 with $\gamma_2(H_i) = \alpha_0(H_i) + 1$ for at least one $i \in \{1, 2\}$. If, say, $\gamma_2(H_1) = \alpha_0(H_1)$, then $\gamma_2(H_2) = \alpha_0(H_2) + 1$ and thus, together with (1),

$$\begin{aligned} \alpha_0(G) + 1 \;&\geq\; \alpha_0(H) + 1 \;=\; \alpha_0(H_1) + \alpha_0(H_2) + 1 \\ &=\; \gamma_2(H_1) + \gamma_2(H_2) \;=\; \gamma_2(H) \\ &\geq\; \gamma_2(G) \;\geq\; \alpha_0(G) + 1, \end{aligned}$$

which means that $\gamma_2(G) = \alpha_0(G) + 1$. Let now $\gamma_2(H_i) = \alpha_0(H_i) + 1$ for $i = 1, 2$. Let uv be an edge in G such that $H + uv$ is connected (there has to be such an edge since there are ν_o different edges in $E(G) \setminus E(H)$ which belong to pairwise different odd cycles of G). Let $u \in V(H_1)$ and $v \in V(H_2)$. Let M_1' be a maximum matching in H_1 and suppose that $u \in V(M_1')$. Let D and $V(H_1) \setminus D$ be a bipartition of H_1 such that D is a $\gamma_2(H_1)$-set and $V(H_1) \setminus D$ consists of vertices of degree 2. Since $\gamma_2(H_1) = \alpha_0(H_1) + 1$, there is a vertex $x \in D$ such that $x \notin V(M_1')$. As H_1 is a connected bipartite graph and since M_1' is an almost perfect matching in H_1 (see Corollary 2.5) and every vertex in $V(H_1) \setminus D$ has degree 2 in H_1, it follows that there is an M_1'-alternating path P from x to v. Then $M_1 = (M_1' \setminus E(P)) \cup (E(P) \setminus M_1')$ is also a maximum matching of H_1 with $u \notin V(M_1)$. Analogously, there is a maximum matching M_2 of H_2 such that $v \notin V(M_2)$. It follows that $M = M_1 \cup M_2 \cup \{uv\}$ is a matching in G and so, with (1),

$$\alpha_0(G)+1 \;\geq\; \alpha_0(H)+2 \;=\; \gamma_2(H) \;\geq\; \gamma_2(G) \;\geq\; \alpha_0(G)+1,$$

which implies that $\gamma_2(G) = \alpha_0(G) + 1$. □

Corollary 3.5 Let G be a connected unicyclic graph of order n.

1) If n is odd, then $\gamma_2(G) \geq \alpha_0(G) + 1$ and $\gamma_2(G) = \alpha_0(G) + 1$ if and only if there is an edge $e \in E(G)$ such that $G - e$ is the subdivision graph of a unicyclic multigraph.

2) If n and the unique cycle of G are both even, then $\gamma_2(G) \geq \alpha_0(G)$ and $\gamma_2(G) = \alpha_0(G)$ if and only if G is the subdivision graph of a unicyclic multigraph.

3) If n is even and the unique cycle of G is odd, then $\gamma_2(G) \geq \alpha_0(G)+1$ and $\gamma_2(G) = \alpha_0(G) + 1$ if and only if there are two edges $e, f \in E(G)$ such that $G - \{e, f\}$ is the subdivision graph of a unicyclic multigraph.

Proof. Since a 2-semiregular unicyclic graph is a subdivision graph of a unicyclic multigraph and vice versa, the statements 1) - 3) follow directly from Theorem 3.4. □

References

[1] J.F. Fink and M.S. Jacobson, n-domination in graphs. *Graph Theory with Applications to Algorithms and Computer Science.* John Wiley and Sons. New York (1985), 282-300.

[2] J.F. Fink and M.S. Jacobson, On n-domination, n-dependence and forbidden subgraphs. *Graph Theory with Applications to Algorithms and Computer Science.* John Wiley and Sons. New York (1985), 301-311.

[3] T.W. Haynes, S.T. Hedetniemi, and P.J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, New York (1998).

[4] T.W. Haynes, S.T. Hedetniemi, and P.J. Slater (eds.), *Domination in Graphs: Advanced Topics*, Marcel Dekker, New York (1998).

[5] D. König, Graphen und Matrizen, Mat. Fiz. Lapok **38** (1931), 116-119 (Hungarian).

[6] L. Volkmann, Some remarks on lower bounds on the p-domination number in trees, *J. Combin. Math. Combin. Comput.* **61** (2007), 159-167.