Steiner Intervals in Strongly Chordal Graphs.

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Abstract

A Steiner tree for a set S of vertices in a connected graph G is a connected subgraph of G of smallest size that contains S. The Steiner interval $I(S)$ of S is the union of all vertices of G that belong to some Steiner tree for S. A graph is strongly chordal if it is chordal and has the property that every even cycle of length at least six has an odd chord. We develop an efficient algorithm for finding Steiner intervals of sets of vertices in strongly chordal graphs.

Keywords: Strongly chordal graphs, Steiner interval.

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1 Introduction

Terminology not given here can be found in [2] and [3]. The graphs we consider in this paper are not weighted. We begin with an overview of convexity notions in graphs, since Steiner intervals, the subject of this paper, also give rise to graph

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convexities. For a more extensive overview of other abstract convex structures see [17].

Let V be a finite set and $\mathcal{M}$ a collection of subsets of V. Then $\mathcal{M}$ is an *alignment* of V if and only if $\mathcal{M}$ is closed under taking intersections and contains both V and the empty set. If $\mathcal{M}$ is an alignment of V, then the elements of $\mathcal{M}$ are called *convex sets* and the pair $(V, \mathcal{M})$ is called an *aligned space*. If $S \subseteq V$, then the convex hull of S is the smallest convex set that contains S. Suppose $X \in \mathcal{M}$. Then, $x \in X$ is an *extreme point* for X if $X - \{x\} \in \mathcal{M}$. A *convex geometry* on a finite set is an aligned space with the additional property that every convex set is the convex hull of its extreme points. This property is referred to as the *Minkowski-Krein-Milman property* (MKM) property.

Let $G = (V, E)$ be a graph and u, v vertices of G. Then the *geodetic interval between* u and v, denoted by $I_g[u, v]$ is the union of all vertices that belong to some $u - v$ geodesic, i.e., a shortest $u - v$ path. A set S of vertices in a graph is g-convex if S contains the geodetic interval between every pair of vertices in S. It is easily seen that the vertex set V of a graph G together with the collection of all g-convex sets of G is an aligned space, which we will refer to as the *geodetic convexity*. The *geodetic closure* of a set S of vertices in a connected graph G, denoted by $I_g[S]$, is defined as $I_g[S] = \cup_{u,v \in S} I_g[u, v]$. It is clear that the convex hull, of a set T of vertices, with respect to the geodetic convexity can be obtained by repeatedly applying the geodetic closure operation until a set is obtained whose closure is the set itself. If the convex hull of T is obtained by iterating only once, then T is called a geodetic set for its hull.

If S is a set of vertices such that $I_g[S] = V(G)$, then S is called a *geodetic set* for G. The cardinality of a smallest *geodetic set* for G is called the *geodetic number* of G, and is denoted by $g(G)$. The problem of finding the geodetic number of a graph is NP-hard (see [1]). Apart from the geodetic interval several other interval notions have been defined and studied, see for example [7] and [5]. These naturally lead to other graph convexities for which corresponding extreme vertices of convex sets have been characterized. If $\mathcal{P}$ is a property possessed by an (extreme) vertex, then an ordering $v_1, v_2, \ldots, v_n$ of the vertices of a graph is a *$\mathcal{P}$-ordering* if v_i has property $\mathcal{P}$ in the subgraph induced by $\{v_i, v_{i+1}, \ldots, v_n\}$. Let $\mathcal{P}$ be a property that characterizes extreme vertices with respect to some graph convexity. Several papers in the literature deal with the problem of characterizing those classes of graphs for which every LexBFS ordering and MCS ordering of an arbitrary induced subgraph is a $\mathcal{P}$-ordering, see for example [5] and [10].

When finding the closure of a set S of vertices in G we connect all pairs of vertices of S by shortest paths. Another way of "optimally" connecting a set S of vertices in a connected graph G is by Steiner trees.

A subtree T of G is an *S-tree* if T contains all vertices of S. An S-tree of smallest possible size (i.e., number of edges) is called a *Steiner tree* for S and its size, denoted by $d(S)$ or $d_G(S)$, is the *Steiner distance* of S. The *Steiner interval* for S, denoted $I_G(S)$ or $I(S)$, is the collection of all vertices of G that lie on some Steiner tree for S. Thus if $S = \{u, v\}$, then $I(S) = I[u, v]$. Steiner intervals in graphs were introduced and studied in [12]. Since the problem of finding Steiner distances of sets of vertices in graphs is NP-hard, it is likely that the problem of finding Steiner intervals for a set of vertices is, in general, very difficult. Several classes of graphs, for which Steiner distances can be found efficiently, exist. In a survey by Winter [19] solutions to the Steiner problem for several of these classes are discussed. Among these classes is the class of 'strongly chordal' graphs that we are considering in this paper. It was shown that for this class of graphs the Steiner problem can be solved in polynomial time, see [18]. In the same article it is shown that the Steiner problem for chordal graphs is NP-hard.

Steiner intervals define another interesting class of graph convexities. For an integer $k \geq 2$, a set S of vertices in a connected graph is *k-Steiner convex* (kSC) if for any set T of k vertices of S the Steiner interval for T is a subset of S, see [13]. It is not difficult to see that the vertex set of a graph together with the kSC sets form an aligned space, which we call the *kS-convexity.* The extreme vertices with respect to the $3S$-convexity, called the *3-Steiner simplicial* ($3SS$) vertices have recently been characterized, see [13]. Their characterization imply the characterization of extreme vertices of a graph convexity studied in [5]. Moreover, those classes of graphs for which every LexBFS ordering and every MCS ordering of an arbitrary induced subgraph is a $3SS$-ordering have been characterized [14].

Steiner intervals of sets of vertices in a graph have been of interest in another graph problem. A set S of vertices of G is a *Steiner geodetic* set if $I(S) = V(G)$. The smallest cardinality of a Steiner geodetic set for G is called the *Steiner geodetic number* of G (also called the Steiner number in [4]), and is denoted by $sg(G)$. The problem of finding the Steiner interval of a set S of vertices in a graph G is of interest when studying relationships between $g(G)$ and $sg(G)$. It is incorrectly claimed in [4] that $g(G) \leq sg(G)$. A counterexample to this claim is given in [16] and in [15] it is shown that in general there is no relationship between

$g(G)$ and $sg(G)$ in the sense that $g(G)/sg(G)$ can be made as small or as large as we wish for specific choices of G. It is thus natural to ask for which graphs this inequality does indeed hold.

In [15] it is shown that for a distance hereditary graph G the inequality $g(G) \leq sg(G)$ holds. Moreover, it is shown that every distance hereditary graph has a unique Steiner geodetic set; and its cardinality is thus an upper bound for the geodetic number. The focus of this paper is on 'strongly chordal graphs' a subclass of the 'chordal graphs'. (For definitions see below.) In [9] it is shown that every Steiner geodetic set of an interval graph is also a geodetic set; thereby implying that $g(G) \leq sg(G)$ for this class of graphs. Moreover, the results of [15] show that this is also the case for the distance hereditary chordal graphs (i.e., the ptolemaic graphs). Both these classes are contained in the class of strongly chordal graphs. Moreover, it is also shown in [9] that this inequality does not extend to the class of chordal graphs. It is posed as an open problem in [9] whether this result holds for strongly chordal graphs. In order to better understand Steiner intervals in strongly chordal graphs, we develop an efficient algorithm for finding Steiner intervals for sets of vertices in this class of graphs.

A graph is *chordal* if every cycle of length at least four contains a *chord* (i.e., an edge that joins two vertices which are not adjacent on the cycle). If $C : v_1v_2\ldots v_{n-1}v_nv_1$ is a cycle of even length, then a chord v_iv_j is an *odd chord* if $i-j$ is odd. A graph G is *strongly chordal* if it is chordal and if every even cycle of length at least 6 has an odd chord. A polynomial algorithm for solving the Steiner problem in strongly chordal graphs was developed in [18]. In [6], several characterizations of strongly chordal graphs are given. A characterization most useful for our purposes hinges on the notion of a "simple" vertex. A vertex v in a graph G is *simple* if for every pair $u, w \in N(v)$, either $N[u] \subseteq N[w]$ or $N[w] \subseteq N[u]$. A graph G has a *simple elimination ordering* if its vertices can be ordered as $[v_1, v_2, \ldots, v_p]$ such that for each $i (1 \leq i \leq p), v_i$ is a simple vertex of $G_i =< \{v_i, v_{i+1}, \ldots, v_p\} >$, the subgraph induced by $\{v_i, v_{i+1}, \ldots, v_p\}$. A graph G is strongly chordal if and only if G has a simple elimination ordering (see [6]).

2

Finding Steiner Intervals in Strongly Chordal Graphs.

For the remainder of the paper we assume that G is a connected strongly chordal graph and S a set of vertices of G. Moreover, we assume that $\sigma = [v_1, v_2, \ldots, v_p]$ is a simple elimination ordering of $V(G)$. We develop an algorithm for finding the Steiner interval for S in G. We observe first that we may assume that $v_1 \in S$. If v_1 is not in S, then $I_G(S) = I_{G-v_1}(S)$; for if T is an S-tree containing v_1, then $F =< V(T) \setminus \{v_1\} >$ is connected, since $< N_G(v_1) >$ is complete, and F contains S. So any spanning tree of F is an S-tree of smaller size than T. We may also assume that $N_G(v_1) \cap S = \emptyset$; otherwise, our problem reduces to finding $I_{G-v_1}(S \setminus \{v_1\})$, since in this case $I_G(S) = I_{G-v_1}(S \setminus \{v_1\}) \cup \{v_1\}$. For the remainder of this section we will thus assume that $v_1 \in S$ and that $N_G(v_1) \cap S = \emptyset$.

Lemma 2.1. *Let v_1 be a simple vertex of of G. Let v_1' be a neighbor of v_1 having maximum degree in G and let $S' = (S \setminus \{v_1\}) \cup \{v_1'\}$. Then*
(a) $I(S') \subseteq I(S)$
(b) $I(S) \subseteq I(S') \cup N_G[v_1]$.

Proof. (a) By the algorithm established in [18], there is a Steiner tree for S containing v_1'. Hence $S' \subseteq I(S)$. Suppose $v \in I(S') \setminus S'$. Let T be any Steiner tree for S' containing v. We show that the tree T' obtained from T by adding v_1 and the edge v_1v_1' is a Steiner tree for S. Suppose there exists an S-tree T_S such that $|E(T_S)| < |E(T')|$. Since v_1 is simple in G, $< N_G(v_1) >$ is complete. We may thus assume that v_1 is a leaf of T_S. If v_1 is not a leaf of T_S, then fix x_0 in $N_{T_S}(v_1)$ and delete all edges v_1x for $x \neq x_0$ and add xx_0. This produces a tree containing S where v_1 is a leaf. However, then $T_S - v_1$ is an S'-tree and $|E(T_S - v_1)| < |E(T_S)| \leq |E(T)|$. This contradicts the fact that T is a Steiner tree for S'. So $I(S') \subseteq I(S)$.
(b) Let $v \in I(S)$. If $v \in N_G[v_1] \cup S$, then $v \in I(S') \cup N_G[v_1]$. So assume $d_G(v_1, v) \geq 2$ and that $v \in I(S) \setminus S$. Let T be a Steiner tree for S containing v. Arguing as in the proof of (a), we may assume that v_1 is a leaf of T. Let x be the neighbor of v_1 in T. Then $v \neq x$. If $x = v_1'$, then we let $T_S = T$. If $x \neq v_1'$, then $N_G[x] \subseteq N_G[v_1]$ and we let T_S be the tree obtained from $T - x$ by adding v_1' and all the edges uv_1' where $ux \in E(T)$. Arguing as in (a) we can show that $T_S - v_1$ is a Steiner tree for S' that contains v. Hence $I(S) \subseteq I(S') \cup N_G[v_1]$. □

Let G, S and S' be as in the hypothesis of Lemma 2.1. Then $I(S)$ can be found by finding $I(S')$ and adding to this set all those vertices of $N_G[v_1]$ that belong to $I(S)$. So $I(S) = I(S') \cup (N_G[v_1] \cap I(S))$. Since $v_1, v_1' \in I(S)$, it suffices to determine which $v \in N(v_1) \setminus \{v_1'\}$ belong to $I(S)$. In what follows we establish a series of lemmas that will help us to prove the following theorem.

Theorem 2.2. *Let G, S, S', v_1 and v_1' be as in Lemma 2.1. Let $v \in N(v_1) \setminus \{v_1'\}$. Then $v \in I(S)$ if and only if v is adjacent with at least one vertex from each component of $< I(S') \setminus \{v_1'\} >$.*

Proof. (necessity) Observe first that if T is a Steiner tree for S, then $|V(T) \cap N(v_1)| = 1$; otherwise, some Steiner tree for S contains at least two vertices of $N(v_1)$. Let $x, y \in V(T) \cap N(v_1)$. If $v_1' \in \{x, y\}$, say $v_1' = x$, then $< V(T) \setminus \{y\} >$ is a connected graph that contains S and has order less than the order of T. This is not possible since T is a Steiner tree for S. If $v_1' \notin \{x, y\}$, then $< (V(T) \setminus \{x, y\}) \cup \{v_1'\} >$ is a connected graph that contains S, since $N[x] \cup N[y] \subseteq N[v_1']$. Once again, this contradicts the fact that T is a Steiner tree for S. Hence every Steiner tree for S contains exactly one vertex of $N(v_1)$. Thus if $v \in I(S) \cap N(v_1)$ and T is a Steiner tree for S containing v, then v_1 is a leaf of T and $vv_1 \in E(G)$. So $T - v_1$ is a Steiner tree for $(S \setminus \{v_1\}) \cup \{v\}$. Since $N[v] \subseteq N[v_1']$, the tree T' obtained from $T - v_1$ by replacing v with v_1' must be a Steiner tree for S'. So the vertices of T' are in $I(S')$. Thus v is adjacent with at least one vertex from each component of $< I(S') \setminus \{v_1'\} >$, since v is adjacent with at least one vertex from each component of $T' - v$. □

Proof. (sufficiency) We establish a series of lemmas.

Lemma 2.3. *The number of components of $J =< I(S') \setminus \{v_1'\} >$ equals the number of components of $A =< I(S') \cap N(v_1') >$.*

Proof. Let $A_1, A_2, \ldots, A_m$ be the components of A and let $J_1, J_2, \ldots, J_n$ be the components of J. Since $I(S')$ is connected each component of J has a vertex adjacent with v_1'. Hence J has at most m components, so $n \leq m$. If A_i and $A_j (i \neq j)$ belong to the same component of J, then G has a cycle of length at least 4 without a chord, which is not possible. Thus $m \leq n$. The lemma now follows. □

Let A_i, J_i $(1 \leq i \leq m)$ be as in the proof of Lemma 2.3, and let $S_i' = (S \cap V(J_i)) \cup \{v_1'\}$. We may assume that A_i is contained in J_i for $1 \leq i \leq m$. Then a Steiner tree T for S' can be obtained by finding a Steiner tree T_i for S_i' for all i and taking the union of these trees T_i. Also if T is a Steiner tree for S', then the edges in $E(T) \cap (E(J_i) \cup \{v_1'x | x \in V(A_i)\})$ form a Steiner tree for S_i'. We will prove the sufficiency of Theorem 2.2 by showing that if $v \in N(v_1)$ is such that v is adjacent with some vertex of A_i for $1 \leq i \leq m$, then there is a Steiner tree T_i' for S_i' such that v_1' is a leaf of T_i' and such that v is adjacent with some vertex y_i of $V(T_i') \cap V(A_i)$. A Steiner tree for S containing v can be obtained from the union of the trees $T_i' - v_1'$, $1 \leq i \leq m$ by adding the vertices v and v_1 and the edges $v_1 v$ and vy_i, for $1 \leq i \leq m$.

For the remainder of this section, we assume that A is connected. Let $H_0 =< N_G(v_1) >$, $H_1 = A$ and H_2 the subgraph induced by all vertices of $I(S')$ at distance 2 from v_1' in $< I(S') >$. So if $x \in V(H_2)$, then $d_G(x, v_1') = 2$, $x \in I(S')$ and x is adjacent with a vertex of H_1.

Suppose H_2 has components $F_1, F_2, \ldots, F_k$. For $1 \leq i \leq k$ let $H_{F_i} = \{x \in V(H_1) | zx \in E(G) \text{ for some } z \in V(F_i)\}$.

Lemma 2.4. *Let $z \in V(H_2)$ and suppose $x, y \in V(H_1)$ are such that $xz, yz \in E(G)$. Then $xy \in E(G)$.*

Proof. Since $z \in V(H_2)$, $d(v_1', z) = 2$. So $v_1'z \notin E(G)$. Since $v_1'xzyv_1'$ is a 4-cycle and G is strongly chordal, we now have $xy \in E(G)$. □

Lemma 2.5. *If $x, y \in H_{F_i}$, then $xy \in E(G)$ and x and y have a common neighbor in F_i.*

Proof. Let $x, y \in H_{F_i}$. By definition $N(x; F_i) = N(x) \cap V(F_i)$ and $N(y; F_i) = N(y) \cap V(F_i)$ are both nonempty. If $N(x; F_i) \cap N(y; F_i) \neq \emptyset$, then the result follows from Lemma 2.4. If $N(x; F_i) \cap N(y; F_i) = \emptyset$, let $P : x_1 x_2 \ldots x_l$ be a shortest path in F_i from $N(x; F_i)$ to $N(y; F_i)$; such a path exists since F_i is connected. By our choice of P, $l \geq 2$, $xx_1, yx_l \in E(G)$ and $xx_t \notin E(G)$ for $2 \leq t \leq l$ and $yx_s \notin E(G)$ for $1 \leq s < l$. So P followed by $x_l y v_1' x x_1$ if $xy \notin E(G)$ (or $x_l y x x_1$ if $xy \in E(G)$ is a cycle of length at least 5 (or 4, respectively) without a chord which is not possible. Hence x and y must have a common neighbor in F_i. The lemma now follows from Lemma 2.4. □

Thus H_{F_i} induces a complete graph in G. For the remainder of the paper let H_{01} be the subgraph of G induced by the vertices in H_0 and H_1.

Lemma 2.6. *Let $v_i \in H_{F_i}$ be such that $deg_{H_{01}}(v_i) \geq deg_{H_{01}}(y)$ for all $y \in H_{F_i}$. Then $N_{H_{01}}[y] \subseteq N_{H_{01}}[v_i]$ for all $y \in H_{F_i}$.*

Proof. Assume, to the contrary, that there is a $y \in H_{F_i}$ and $h_1 \in N_{H_{01}}[y] \setminus N_{H_{01}}[v_i]$. Since $deg_{H_{01}}(v_i) \geq deg_{H_{01}}(y)$, there is an $h_2 \in N_{H_{01}}[v_i] \setminus N_{H_{01}}[y]$. By Lemma 2.5, $v_iy \in E(G)$, and v_i and y share a common neighbor z which lies in F_i. By assumption h_1v_i and h_2y are not edges of G, and since $d_G(v_1', z) = 2$, $v_1'z \notin E(G)$. So $v_1'h_1yzv_ih_2v_1'$ is a 6-cycle without an odd chord (see Figure 1). This is not possible in a strongly chordal graph. The lemma now follows. □

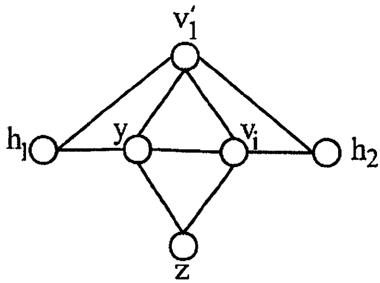

Figure 1: A 6-cycle without odd chords

Lemma 2.7. *Let H' be the subgraph of H_1 induced by those vertices y such that $N_G(y) \cap V(H_2) = \emptyset$. Then $V(H') \subseteq S$.*

Proof. Let $y \in V(H')$. Let T be a Steiner tree for S' containing y. We may assume that v_1' is in T for if the unique vertex v of H_0 that belongs to T is not v_1', then we can replace v with v_1', i.e., we delete v and add v_1' as well as the edges $v_1'z$ for all z such that $vz \in E(T)$. We show that there exists a Steiner tree T' for S' in which $deg_{T'}(y) = 1$.
Case 1 $v_1'y \in E(T)$. Since v_1' is the only vertex of T in H_0, $N_T(y) \subseteq V(H_1) \cup \{v_1'\}$. So if $yx \in E(T)$ for $x \in N_T(y) \setminus \{v_1'\}$, we can delete yx and add $v_1'x$ to obtain a new tree containing S' where the degree of y is less than in T. We continue in this manner until v_1' is the only neighbor of y and let T' be the resulting tree. Thus $y \in S$; otherwise, $T' - y$ is a tree that contains S' and has fewer edges than T. This contradicts the fact that T is a Steiner tree for S'.
Case 2 $yv_1' \notin E(T)$. Then there is a $v_1' - y$ path in T. Moreover, this path is unique; otherwise, T has a cycle. For all $x \in N_T(y) \setminus \{v_1'\}$ such that x does not lie on the $v_1' - y$ path in T, remove the edge yx and replace it with $v_1'x$ to obtain a tree T'. Then $|E(T')| = |E(T)|$. Also $deg_{T'}(y) = 1$, since there is a unique $v_1' - y$ path in T. As in Case 1, $y \in S'$. □

Let B be the intersection graph for the collection of sets $H_{F_1}, H_{F_2}, \ldots, H_{F_k}$, i.e., the vertices of B are $H_{F_1}, \ldots, H_{F_k}$ and $H_{F_i}H_{F_j} \in E(B)$ if and only if $H_{F_i} \cap H_{F_j} \neq \emptyset$. Suppose B has components $C_1, \ldots, C_t$.

Lemma 2.8. *For $1 \leq s \leq t$, C_s is a complete graph. Moreover, if $V_s = \cup\{H_{F_i} | H_{F_i} \in V(C_s)\}$, then V_s contains a vertex y_s that is adjacent with every other vertex of V_s.*

Proof. Suppose C_s is not complete. Then there exist vertices H_{F_i} and H_{F_j} such that $d_{C_s}(H_{F_i}, H_{F_j}) = 2$. Let $H_{F_i}H_{F_l}H_{F_j}$ be a path of length 2 in C_s. Let $v_i \in H_{F_i}$ be such that $deg_{H_{01}}(v_i) \geq deg_{H_{01}}(y)$ for all $y \in H_{F_i}$. Since both H_{F_i} and H_{F_l} induce complete graphs, it follows from Lemma 2.6, that v_i is adjacent with every vertex of H_{F_i} and H_{F_l}. So $v_i \in H_{F_i} \cap H_{F_l}$. Also if $v_l \in H_{F_l}$ is such that $deg_{H_{01}}(v_l) \geq deg_{H_{01}}(y)$ for all $y \in H_{F_l}$, then, by Lemma 2.6, $N_{H_{01}}[y] \subseteq N_{H_{01}}[v_l]$ for all $y \in H_{F_l}$. So $N_{H_{01}}[v_i] \subseteq N_{H_{01}}[v_l]$, and $N_{H_{01}}[v_l] \subseteq N_{H_{01}}[v_i]$. Thus $N_{H_{01}}[v_l] = N_{H_{01}}[v_i]$. Similarly if $v_j \in H_{F_j}$ is such that $deg_{H_{01}}(v_j) \geq deg_{H_{01}}(y)$ for all $y \in H_{F_j}$, then $N_{H_{01}}[v_j] = N_{H_{01}}[v_l] = N_{H_{01}}[v_i]$. So $H_{F_i} \cap H_{F_j}$ is nonempty. But then $H_{F_i}H_{F_j} \in E(B)$, contrary to our assumption. The first part of the lemma now follows. For the second part of the lemma let H_{F_i} be a vertex of C_s and $v_i \in H_{F_i}$ be such that $deg_{H_{01}} v_i \geq deg_{H_{01}} y$ for all $y \in H_{F_i}$. Then $v_i \in H_{F_j}$ for all j such that $H_{F_j} \in C_s$ and $N_{H_{01}}[y] \subseteq N_{H_{01}}[v_i]$ for all $y \in V_s$. The second part of the lemma follows with $y_s = v_i$. □

Lemma 2.9. *If y_s is chosen as in the Proof of Lemma 2.8, then y_s is adjacent with every $v \in N_G(v_1)$ such that $vx \in E(G)$ for some $x \in V_s$.*

Proof. Suppose $vx \in E(G)$ for some $x \in V_s$, $x \neq y_s$. Since, by Lemma 2.6, $N_{H_{01}}[x] \subseteq N_{H_{01}}[y_s]$, we have $vy_s \in E(G)$. □

Lemma 2.10. *Let H' be as in Lemma 2.7 and suppose $U \subseteq V(H_1)$ is such that $V(H') \cup \{y_i | 1 \leq i \leq t\} \subseteq U$. Then $< U >$ is connected.*

Proof. Let $C_1, C_2, \ldots, C_t$ be as described prior to Lemma 2.8. Suppose $B_1, B_2, \ldots, B_r$ are the components of H'. Let H_1' be the graph with vertex set $\{C_1, C_2, \ldots, C_t\} \cup \{B_1, B_2, \ldots, B_r\}$ and edge set $\{C_iC_j |$some vertex of V_i is ad- jacent with some vertex of V_j, $1 \leq i < j \leq t\}$ $\cup\{C_iB_j |$some vertex of V_i is adja- cent with some vertex of B_j for $1 \leq i \leq t$ and $1 \leq j \leq r\}$.

Since H_1 is connected, so is H'_1. Moreover, by Lemma 2.6, if $C_iB_j \in E(H'_1)$, then y_i is adjacent with every vertex of B_j that has a neighbor in V_i. Also if $C_iC_j \in E(H'_1)$, then $y_iy_j \in E(G)$. Since y_i is adjacent with every vertex of V_i $(1 \leq i \leq t)$ and as $y_i \in V_i$ the graph $C'_i =< U \cap V_i >$ is connected. Since $C_iC_j \in E(H'_1)$ if and only if there is an edge from C'_i to C'_j for $1 \leq i < j \leq t$ and as $C_iB_j \in E(H'_1)$ for $1 \leq i \leq t$ and $1 \leq j \leq r$ if and only if a vertex of C'_i (namely y_i) is adjacent with a vertex of B_j, the subgraph induced by $U = (\cup_{i=1}^{t} V(C'_i)) \cup (\cup_{j=1}^{r} V(B_j))$ is connected. □

Using arguments similar to those used in Lemma 2.3, we can show that $H'' =< I(S') \setminus (V(H') \cup \{v'_1\}) >$ has t components $H''_1, H''_2, \ldots, H''_t$, such that H''_i $(1 \leq i \leq t)$ contains the vertices of exactly one V_s $(1 \leq s \leq t)$. We may assume that the components H''_i and sets V_s have been indexed in such a way that $V_s \subseteq V(H''_s)$ for $1 \leq s \leq t$. Let $S''_i = S' \cap (V(H''_i) \cup \{v'_1\})$. Then S''_i must contain at least two vertices of S'. Recall if T is a Steiner tree for S', then any spanning tree of $< (V(H''_i) \cup \{v'_1\}) \cap V(T) >$ is a Steiner tree for S''_i. Moreover, a Steiner tree for S' can be obtained by finding a Steiner tree T''_i for S''_i in $< V(H''_i) \cup \{v'_1\} >$ for $1 \leq i \leq t$, and then identifying these T''_i's in v'_1 and finally adding the vertices of H' and all edges of the type v'_1x where $x \in V(H')$.

Let y_i be as in the proof of Lemma 2.8. Suppose T_{y_i} is a Steiner tree for S' that contains y_i and let T''_i be a Steiner tree for S''_i in $< (V(H''_i) \cup \{v'_1\}) \cap V(T_{y_i}) >$ for $1 \leq i \leq t$. Then y_i is in T''_i. Let $U = \cup_{i=1}^{t}(V(H_1) \cap V(T''_i))) \cup V(H')$. By Lemma 2.10, $< U >$ is connected. By the above, there is a Steiner tree for S' with vertex set $\mathcal{T} = V(H') \cup (\cup_{i=1}^{t} V(T''_i))$. Since $U \subseteq \mathcal{T} \setminus \{v'_1\}$ and as $< U >$ is connected, $< \mathcal{T} \setminus \{v'_1\} >$ is connected. Let T be any spanning tree for $< \mathcal{T} \setminus \{v'_1\} >$ and let T' be T together with v'_1 and any edge v'_1x where x is in U. Then T' is an S'-tree and T' is a Steiner tree for S' since each T''_i is a Steiner tree for S''_i in $< V(H''_i) \cup \{v'_1\} >$. If $v \in N(v_1) \setminus \{v'_1\}$ is adjacent with some vertex of H_1, then v is either adjacent with a vertex of H' or with a vertex of some V_s $(1 \leq s \leq t)$. In either case v is adjacent with a vertex of U. In the former case this follows from the fact that $V(H') \subseteq U$ and in the latter case this follows from Lemma 2.9. So T together with v and v_1 and the edges v_1v and vx for some x in U is a Steiner tree for S. This completes the proof of Theorem 2.2. □

We now describe an algorithm that determines, for a strongly chordal graph G and set S of vertices of G, the Steiner interval for S. Let $\sigma = [v_1, v_2, \ldots, v_p]$ be a simple elimination ordering of $V(G)$.

Algorithm To find the Steiner interval $I(S)$ of S. (Step 2 finds a Steiner tree for S and Step 3 finds all other vertices that belong to some Steiner tree for S.)

1. $I(S) \leftarrow S$;
   For $j = 1, 2, \ldots, p$
   $CHECK(v_j) \leftarrow \emptyset$;
   $i \leftarrow 1$;
   $G_1 \leftarrow G$.

2. While $i < p$ do

   (a) If $v_i \notin I(S)$
   $G_{i+1} \leftarrow G_i - v_i$;
   $i \leftarrow i + 1$; and repeat Step 2

   (b) If $v_i \in I(S)$ and $N_{G_i}(v_i) \cap I(S) \neq \emptyset$
   $G_{i+1} \leftarrow G_i - v_i$;
   $i \leftarrow i + 1$; and repeat Step 2

   (c) If $v_i \in I(S)$ and $N_{G_i}(v_i) \cap I(S) = \emptyset$ let $v_i' \in N_{G_i}(v_i)$ be such that $deg_{G_i} v_i' \geq deg_{G_i} v$ for all $v \in N_{G_i}(v_i)$;
   $I(S) \leftarrow I(S) \cup \{v_i'\}$;
   $CHECK(v_i) \leftarrow N_{G_i}(v_i) \setminus \{v_i'\}$;
   $G_{i+1} \leftarrow G_i - v_i$;
   $i \leftarrow i + 1$; and repeat Step 2

   otherwise go to Step 3.

3. While $i \geq 1$ do

   (a) If $CHECK(v_i) = \emptyset$
   $i \leftarrow i - 1$; and repeat Step 3

   (b) i. If $CHECK(v_i) \neq \emptyset$, then for each $x \in CHECK(v_i)$ if x is adjacent with a vertex from each component of $< N_{G_i}(v_i') \cap I(S) >$, then $ADD(x) = T$; otherwise, $ADD(x) = F$.

   ii. $I(S) \leftarrow I(S) \cup \{x \in CHECK(v_i) | ADD(X) = T\}$

   iii. $i \leftarrow i - 1$; and repeat Step 3 otherwise go to Step 4.

4. Output $I(S)$ and stop.

The correctness of this algorithm can be proven by induction on the order of the graph and using Theorem 2.2. In particular after passing through Step 3 for a given i, $I(S) \cap V(G_i)$ is the Steiner interval for $S \cap V(G_i)$.

The algorithm can be executed in $O(|V||E|)$ time for a connected strongly chordal graph $G = (V, E)$. As pointed out in [18] Step 2 has complexity $O(|V|^2)$. Since the components of the subgraphs in Step 3 can be found in $O(max\{|V|, |E|\})$ time, Step 3 can be performed in $O(|V||E|)$ time. So the algorithm has complexity $O(|V||E|)$.

3 Closing Remarks

In this paper we developed an efficient algorithm for finding the Steiner interval of a set of vertices in a strongly chordal graph. The problem of determining a relationship between the geodetic and Steiner geodetic number of these graphs remains open as does the problem of determining a procedure for finding a Steiner geodetic set in a strongly chordal graph.

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