NOTE.
AN EXAMPLE OF AN $\mathbb{L}(n, d)$ LINEAR SPACE WITH MORE THAN $n^2 + n + 1$ LINES.

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ABSTRACT. An $\mathbb{L}(n, d)$ is a linear space with constant point degree $n + 1$, lines of size n and $n - d$, and with $v = n^2 - d$ points. Denote by $b = n^2 + n + z$ the number of lines of an $\mathbb{L}(n, d)$, then $z \geq 0$ and examples are known only if $z = 0, 1$ [7]. The linear spaces $\mathbb{L}(n, d)$ were introduced in [7] in relation with some classification problems of finite linear spaces. In this note, starting from the symmetric configuration 45_7 of Baker [1] we give an example of $\mathbb{L}(n, d)$, with $n = 7$, $d = 4$ and $z = 4$.

1. INTRODUCTION

A *(finite) linear space* $\mathbb{L}$ is a pair $(\mathcal{P}, \mathcal{L})$, where $\mathcal{P}$ is a (finite) set of *points*, and $\mathcal{L}$ is a family of subsets of $\mathcal{P}$, called *lines*, such that: *for any two distinct points there is exactly one line containing both, there are at least two lines, and every line has at least two points.*

The *degree* of a point $p \in \mathcal{P}$ is the number $[p]$ of lines containing it, and the *length* of a line $\ell \in \mathcal{L}$ is its size.

Let n and d two integers satisfying $1 \leq d \leq n - 2$, an $\mathbb{L}(n, d)$ is a finite linear space on $v = n^2 - d$ points with constant point degree $n + 1$, and which has only lines of length $n - d$ and n. For an $\mathbb{L}(n, d)$, the number z defined by $z(n - d) = d(d - 1)$ is an integer, and the number b of lines of $\mathbb{L}(n, d)$ is given by $n^2 + n + z$ [7].

Let α_n be a finite affine plane of order n, and let p_0 be a point of α. Deleting p_0 from α_n one obtains a finite $\mathbb{L}(n, 1)$ with $v = n^2 - 1$ points and $b = n^2 + n$ lines (*punctured affine plane of order n*) .

Another example of $\mathbb{L}(n, d)$ comes from projective geometry. Indeed, let π_n be a finite projective plane of square order n, and let $\mathcal{B}$ be a Baer-subplane of π_n. The linear space obtained from π_n by deleting $\mathcal{B}$ is an

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$\mathbb{L}(n,\sqrt{n})$ with $v = n^2 - \sqrt{n}$ points and $b = n^2+n+1$ lines (the *complement of a Baer–subplane in* π_n).

In [[7], Chapters 10 and 12] linear spaces $\mathbb{L}(n,d)$ are studied, and using their relation with symmetric designs some non–existence conditions are given. Finally, it is also given a characterization of such linear spaces.

In particular, it is proved that an $\mathbb{L}(n,d)$ with $z = 0$ is a punctured affine plane of order n, and an $\mathbb{L}(n,d)$ with $z = 1$ is the complement of a Baer–subplane in a projective plane of order n [7].

So far, the only known examples of $\mathbb{L}(n,d)$ are for $z = 0,1$. In this article an example with $z = 4$ is presented.

$\mathbb{L}(n,d)$ spaces are useful in some theorems on finite linear spaces. For example, in the classification of finite linear spaces on v points, with b lines, and with a point of degree n , $n \geq 2$, satisfying $n^2 - n + 2 \leq v \leq b \leq n^2 + n + 1$, every possible example of finite linear space with $v = n^2 - n + 2$ and $b = n^2 + n + 1$ is closely related to an $\mathbb{L}(n,d)$ with $z = 2$ (cf [7]).

Furthermore, they also appear in the problem of determining the maximum number of points for finite linear spaces with $n^2 + n + 2$ lines [[7], Chapter 11]. Indeed, in [7] the following result is proved.

Theorem 1.1 (Metsch, Thm. 11.1 [7] 1991). *Let* $\mathbb{L}$ *be a finite linear space, with* $n^2 + n + 2$ *lines for some integer* $n \geq 6$. *Denote by* v *its number of points, and by* e *the positive number with* $2n = e(e+1)$. *If every point has degree at most* $n + 1$ *then* $v \leq n^2 - e$ *with equality if and only if* $\mathbb{L}$ *is an* $\mathbb{L}(n,e)$.

In this Note, starting from the symmetric configuration 45_7 (*elliptic semiplane of order* 6) described in [1], we give an example of $\mathbb{L}(7,4)$ with $z = 4$.

2. $\mathbb{L}(n,d)$ SPACES AND SYMMETRIC CONFIGURATIONS

Let n and κ be two positive integers, a *symmetric configuration* n_κ is a pair $(\mathcal{P},\mathcal{L})$, where $\mathcal{P}$ is a set of *points* of size n, and $\mathcal{L}$ is family of subsets of points, called *lines* such that *any two distinct points belong to at most one line*[1], *every point belongs to* κ *lines, and every line has size* κ.

From double counting it follows that for a symmetric configuration the size of $\mathcal{P}$ is equal to the size of $\mathcal{L}$.

Furthermore, $n \geq \kappa(\kappa-1)^2 + 1$. For $\kappa = 3$ and $\kappa = 4$ the necessary existence condition $n \geq \kappa(\kappa-1)^2 + 1$ is also sufficient, while per $\kappa \geq 5$ gaps start to appear in the existence spectrum (see e.g. [3]). For $\kappa = 5$ a symmetric configuration exists if and only if $v = 21$ or $v \geq 23$ [3] For $\kappa = 6$, a symmetric configuration exists if and only if $v = 31$ (the projective plane of order 5) or $v \geq 34$ [5].

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[1]In other words, a symmetric configuration is a regular semilinear space.

In 1978 Baker [1] constructed a 45_7 configuration, via its incidence matrix. The incidence matrix N of a symmetric configuration is a zero–one square matrix of order v whose rows are indexed by the points and columns by the lines and with the (i, j)–entry one precisely when the point p_i is on the line L_j.

Assuming the points and lines are enumerated by *parallel* classes[2] of size m, N is a block matrix of order v/m, each block of order m. Following Baker notation, we give the incidence matrix N of the configuration 45_7 he constructed, and which we will denote with $\mathcal{B}$.

A blank entry is the zero matrix of order 3, I is the identity matrix of order 3, and

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, C = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix};$$

$$N = \begin{pmatrix}
 & I & I & I & I & I & I & I & & & & & & & \\
I & & A & B & & C & & & & A & C & & B & & \\
I & & & A & B & & C & & & & A & C & & B & \\
I & & & & A & B & & C & & & & A & C & & B \\
I & C & & & & A & B & & B & & & & A & C & \\
I & & C & & & & A & B & & B & & & & A & C \\
I & B & & C & & & & A & C & & B & & & & A \\
I & A & B & & C & & & & A & C & & B & & & \\
 & & A & C & & B & & & A & & & C & & C & C \\
 & & & A & C & & B & & C & A & & & C & & C \\
 & & & & A & C & & B & C & C & A & & & C & \\
 & B & & & & A & C & & & C & C & A & & & A \\
 & & B & & & & A & C & C & & C & C & A & & \\
 & C & & B & & & & A & & C & & C & C & A & \\
 & A & C & & B & & & & & & C & & C & C & A
\end{pmatrix}.$$

Thus, we may group the points of $\mathcal{B}$ into 15 triples of consecutive indexed points three by three non-collinear (clearly, they correspond to the parallel classes of points of $\mathcal{B}$):

$$\ell_1 = \{p_1, p_2, p_3\}, \ell_2 = \{p_4, p_5, p_6\}, \dots, \ell_{14} = \{p_{40}, p_{41}, p_{42}\}, \ell_{15} = \{p_{43}, p_{44}, p_{45}\}.$$

Let $\mathcal{P}$ and $\mathcal{L}$ be the set of points and lines of $\mathcal{B}$ respectively.

Consider, the pair $(\mathcal{P}^*, \mathcal{L}^*)$, where

[2]Two points (lines) are *parallel* if they are not collinear (intersecting).

$$\mathcal{P}^* = \mathcal{P}$$
$$\mathcal{L}^* = \mathcal{L} \cup \{\ell_i \mid i = 1, \ldots 15\}.$$

Clearly, $(\mathcal{P}^*, \mathcal{L}^*)$ is a finite linear space, with $v = 45 = 7^2 - 4$ points, $b = 60 = 7^2 + 7 + 4$ lines, and with constant point degree 8, and $|\ell| \in \{3, 7\}$ for every $\ell \in \mathcal{L}^*$, that is it is an $\mathbb{L}(7, 4)$ with $b = 7^2 + 7 + 4$ lines.

For an $\mathbb{L}(n, d)$, with $z(n - d) = d(d - 1)$ and $b = n^2 + n + z$ lines, we have [7]:

- *Every point lies on a unique line of length $n - d$ and on n lines of length n.*
- *$\mathbb{L}(n, d)$ has $n^2 - d$ lines of length n and $n + d + z$ lines of length $n - d$.*

Hence, the structure consisting of the points and lines of length n of an $\mathbb{L}(n, d)$ is a symmetric configuration, and so the following non-existence criterion for $\mathbb{L}(n, d)$ easily follows.

Proposition 2.1. *If there is an $\mathbb{L}(n, d)$ with $z(n - d) = d(d - 1)$, $z \geq 1$, then it exists a symmetric configuration $(n^2 - d)_n$.*

Since there is no 32_6 [3] it follows that there exists no $\mathbb{L}(6, 4)$ $(z = 6)$. Similarly, since there is no 22_5 [3] it follows that there is no $\mathbb{L}(5, 3)$ $(z = 3)$. In [7] it is proved that there is no $\mathbb{L}(6, 3)$ $(z = 2)$, but this follows also from the fact that there is no 33_6 [5].

Remark 2.2. If $(\mathcal{P}, \mathcal{L})$ is a finite linear space, for any point–line pair (p, ℓ), with $p \notin \ell$, let $\pi(p, \ell) = [p] - |\ell|$ denote the number of lines passing through p and missing ℓ. Since the lines of an $\mathbb{L}(7, 4)$ have length $n = 7$ or $n - 4 = 3$, and all the points have degree $n + 1 = 8$, we have $\pi(p, \ell) \in \{1, 5\}$ for every point–line pair (p, ℓ) of $\mathbb{L}(7, 4)$, with $p \notin \ell$. So, the $\mathbb{L}(7, 4)$ found, is an example of finite $\{1, 5\}$–semiaffine linear space of order n with $b > n^2 + n + 1$ lines [cf [2], Section 4.8, *Research problems*].

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