A Generalization of the Anderson - Ellison Methodology for Z-cyclic DTWh(p) and OTWh(p)

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Abstract

I. Anderson and L. Ellison [7] demonstrated the existence of Z-cyclic Directed Triplewhist Tournament Designs and Z-cyclic Ordered Triplewhist Tournament Designs for all primes $p \equiv 9 \pmod{16}$. It is shown here that their methodology can be generalized completely to deal with primes of the form $p \equiv (2^k + 1) \pmod{2^{k+1}}$, $k \geq 4$.

keywords: Whist Tournaments, Triplewhist Tournaments, Directedwhist Tournaments, Orderedwhist Tournaments, Directed Triplewhist Tournaments, Ordered Triplewhist Tournaments, Z-Cyclic Designs,

1 Introduction

A whist tournament on v players, denoted Wh(v), is a $(v, 4, 3)$ (near) resolvable BIBD. Each block, (a, b, c, d), of the BIBD is called a whist game. For such a game, the partnership $\{a, c\}$ opposes the partnership $\{b, d\}$. The design is subject to the (whist) conditions that every player partners every other player exactly once and opposes every other player exactly twice. Each (near) resolution class of the design is called a round of the Wh(v). It has been known since the 1970s that Wh(v) exist for all $v \equiv 0, 1 \pmod{4}$ [5].

In a whist game (a, b, c, d) the opponent pairs $\{a, b\}$, $\{c, d\}$ are called opponents of the first kind and the opponent pairs $\{a, d\}$, $\{b, c\}$ are called opponents of the second kind. A Wh(v) with the property that every player opposes every other player exactly once as an opponent of the first kind and exactly once as an opponent of the second kind is called a triplewhist tournament on v players, and is denoted by TWh(v). E. H. Moore [16], in his classic paper "Tactical Memoranda I-III", introduced the triplewhist specialization. TWh(v) do not exist for $v = 4, 5, 9, 12, 13$ and exist for all other $v \equiv 0, 1 \pmod 4$ with the possible exception of $p = 17$ [2].

In a whist game one can also refer to left hand opponents and right hand opponents. These relationships are the obvious ones associated with the players seated at a table with a at the North position, b at the East position, c at the South position and d at the West position. A whist tournament is said to be a directed whist tournament on v players, DWh(v), if every player has every other player exactly once as a left hand opponent and exactly once as a right hand opponent. DWh(v) were introduced by R. D. Baker [11] and are known to exist for all $v = 4n + 1$ and for all $v = 4n, n \geq 12$ [10].

Another whist specialization, an ordered whist tournament design, was introduced by Y. Lu [15]. In this case, each opponent must be played once when playing North-South, and once when playing East-West. Necessarily, the number of games a player plays (i.e., $v - 1$) must be even. Abel, Costa and Finizio [1] have shown that OWh($4n + 1$) exist for all $n \geq 1$.

A (triple, directed, ordered)whist design is said to be Z-cyclic if the players are elements in $Z_m \cup \mathcal{A}$ where $m = v$, $\mathcal{A} = \emptyset$ when $v \equiv 1 \pmod 4$ and $m = v - 1$, $\mathcal{A} = \{\infty\}$ when $v \equiv 0 \pmod 4$ and it is further required that the rounds also be cyclic in the sense that the rounds can be labeled, say, $R_1, R_2, \ldots$ in such a way that R_{j+1} is obtained by adding $+1 \pmod m$ to every element in R_j.

Z-cyclic designs are particularly appealing for the simplicity of their presentation, one merely provides an initial round, R_1, and all remaining rounds are obtained by development of the initial round. The existence of Z-cyclic (directed, ordered, triple) whist tournament designs is an open problem of considerable interest. It is a fact that Z-cyclic DWh(v) cannot exist for $v \equiv 0 \pmod 4$ [12].

When a whist tournament design satisfies two or more of the specializations mentioned above then the design is refered to in terms of these specializations, e.g. a Z-cyclic directed triplewhist design, a directed ordered design, etc. The notations for directed triplewhist, ordered triplewhist and directed orderedwhist are DTWh(v), OTWh(v) and DOWh(v), respectively. The designation of Z-cyclic is usually written out. Since OWh(v) and Z-cyclic DWh(v) do not exist for $v \equiv 0 \pmod 4$ the investigation into multi-specializations of whist tournament designs has focused on $v \equiv 1 \pmod 4$. Abel et al. [1] have shown that DOWh($4n + 1$) exist for all $n \geq 1$. Anderson and Finizio [8] have shown that Z-cyclic DTWh(p) exist for all primes $p \equiv 5 \pmod 8$, $p \geq 29$. Additionally, they have shown the existence of Z-cyclic DTWh(p) for certain other primes $p < 10,000$ [9].

Anderson and Ellison [6] have shown that Z-cyclic OTWh(p) exist for all primes $p \equiv 5 \pmod 8$, $p \geq 29$. These same authors, in [7], have shown the existence of Z-cyclic DTWh(p) and Z-cyclic OTWh(p) for all primes $p \equiv 9 \pmod{16}$. In this latter paper it is established that the existence of a Z-cyclic DOTWh(v) is impossible. In this present study it is shown that the methodology employed by Anderson and Ellison in [7] is completely generalizable to primes of the form $p \equiv (2^k + 1) \pmod{2^{k+1}}$, $k \geq 4$, thereby providing a procedure for the establishment of the existence of Z-cyclic DTWh(p) and Z-cyclic OTWh(p), $p \equiv (2^k + 1) \pmod{2^{k+1}}$, $k \geq 4$.

2 Some Preliminary Materials

When $v \equiv 1 \pmod 4$ it is conventional that the initial round of a Z-cyclic Wh(v) is the round that omits 0. Using symmetric differences it follows that a collection of n games (a_i, b_i, c_i, d_i), $i = 1, \ldots, n$ form the initial round of a Z-cyclic triplewhist tournament on $v = 4n + 1$ players if

$$\bigcup_{i=1}^{n} \{a_i, b_i, c_i, d_i\} = Z_{4n+1} \setminus 0, \tag{2.1}$$

$$\bigcup_{i=1}^{n} \{\pm(a_i - c_i), \pm(b_i - d_i)\} = Z_{4n+1} \setminus 0, \tag{2.2}$$

$$\bigcup_{i=1}^{n} \{\pm(a_i - b_i), \pm(c_i - d_i)\} = Z_{4n+1} \setminus 0, \tag{2.3}$$

and

$$\bigcup_{i=1}^{n} \{\pm(a_i - d_i), \pm(c_i - b_i)\} = Z_{4n+1} \setminus 0. \tag{2.4}$$

If, in addition,

$$\bigcup_{i=1}^{n} \{b_i - a_i, c_i - b_i, d_i - c_i, a_i - d_i\} = Z_{4n+1} \setminus 0, \tag{2.5}$$

then these games form the initial round of a Z-cyclic DTWh(v). On the other hand, if, in addition to (2.1) - (2.4), we have that

$$\bigcup_{i=1}^{n} \{a_i - b_i, a_i - d_i, c_i - b_i, c_i - d_i\} = Z_{4n+1} \setminus 0, \tag{2.6}$$

then these games form the initial round of a Z-cyclic OTWh(v).

For convenience of reference the differences (2.2) are called the *partner differences*, (2.3) the *opponents first kind differences*, (2.4) the *opponent second kind differences*, (2.5) the *first forward differences* and (2.6) the *ordered differences.*

Example 2.1 The initial round of a Z-cyclic DTWh(29) is given by the following 7 tables [8].

(1, 13, 15, 14) (24, 22, 12, 17) (25, 6, 27, 2) (20, 28, 10, 19)
(16, 5, 8, 21) (7, 4, 18, 11) (23, 9, 26, 3).

Example 2.2 The 7 tables listed below constitute the initial round of a Z-cyclic OTWh(29) [6].

(1, 3, 26, 13) (23, 11, 18, 9) (7, 21, 8, 4) (16, 19, 10, 5)
(20, 2, 27, 28) (25, 17, 12, 6) (24, 14, 15, 22).

Definition 2.1 *Let q denote a prime or a prime power. If θ is a primitive element for $GF(q)$ then for each non-zero element $y \in GF(q)$ there exists a unique integer i such that $y = \theta^i$. If $c \mid (q-1)$ and $i \equiv j \pmod{c}$ then it is customary to write $y \in C_j^c$. C_j^c, $j < c$, is often refered to as the j-th cyclotomic class of index c. It is also common to say that y is a j-th power modulo c.*

To say that y is a 0-th power modulo c is synonomous with y is a c-th power.

3 The Primary Constructions

For the remainder of this study we consider p to be a prime such that $p \equiv (2^k + 1) \pmod{2^{k+1}}$, $k \geq 4$. That is to say, $p = 2^k t + 1$, where t is odd and $k \geq 4$. For convenience we utilize the notation of Liaw [13] and set $d = 2^k$, $m = 2^{k-1}$ and $n = 2^{k-2}$.

Construction 1. Let r denote a primitive root of p, x denote a non-square in Z_p and consider the following collection of $(p-1)/4$ games.

$$(1, x, x^m, -x) \otimes r^{dj+2i}, \ 0 \leq i \leq n-1, \ 0 \leq j \leq t-1. \tag{3.7}$$

Since $-1 = r^{mt}$ it follows that the indices of the elements in the set $\{r^{2i}, xr^{2i}, x^m r^{2i}, -xr^{2i} : 0 \leq i \leq n-1,\}$ constitute a complete set of residues modulo d. Therefore the games (3.7) exhaust the set $Z_p \setminus 0$. Note that the partner differences in the *base table* are $\pm(x^m - 1)$ and $\pm 2x$. Again, the fact that $-1 = r^{mt}$ allows for the conclusion that the indices of the elements in the set $\{(x^m-1)r^{2i}, -(x^m-1)r^{2i}, 2xr^{2i}, -2xr^{2i} : 0 \leq i \leq n-1,\}$ constitute a complete set of residues modulo d if $2x(x^m - 1)$ is a nonsquare.

Since 2 is a square [3] and x is a non-square it follows that the partner differences exhaust $Z_p \setminus 0$ if $x^m - 1$ is a square. Similar logic applied to the (*base table*) opponents first kind and opponents second kind differences indicate that the games (3.7) constitute the initial round of a Z-cyclic TWh(p) if the following conditions are satisfied:

a. $x \neq \square$,

b. $x^m - 1 = \square$,

c. $(x-1)(x^{m-1}+1) = \square$,

d. $(x+1)(x^{m-1}-1) = \square$.

The first forward differences in the base table are $x - 1$, $x(x^{m-1} - 1)$, $-x(x^{m-1} + 1)$, $x + 1$. To assist in the analysis of these differences let us assume that exactly one of $x + 1$, $x - 1$ is a square. That is to say, we assume that $x^2 - 1 \neq \square$. Note, for example, that this assumption together with (d) above shows that the differences $x - 1$ and $x(x^{m-1} - 1)$ are such that their product is a square. The same conclusion is true for the pair $x + 1$, $-x(x^{m-1} + 1)$. Thus, the indices of the elements in the set $\{(x-1)r^{2i}, x(x^{m-1}-1)r^{2i}, -x(x^{m-1}+1)r^{2i}, (x+1)r^{2i} : 0 \leq i \leq n-1,\}$ constitute a complete set of residues modulo d if

$$\frac{x(x^{m-1}-1)}{x-1} \in C_m^d, \tag{3.8}$$

and

$$\frac{-x(x^{m-1}+1)}{x+1} \in C_m^d. \tag{3.9}$$

Now, (3.8) will be satisfied if $x^{m+1}(x^{m-2} + x^{m-3} + \ldots + x + 1) \in C_0^d$. Since $-1 \in C_m^d$, (3.9) is satisfied if $x(x^{m-2} - x^{m-3} + \ldots + x^2 - x + 1) \in C_0^d$. Thus the following theorem is established.

Theorem 3.1 *If there exists an element $x \in Z_p \setminus 0$ such that*

a. $x \neq \square$,

b. $x^2 - 1 \neq \square$,

c. $\prod_{i=2}^{k-1}(x^{2^{k-i}} + 1) \neq \square$,

d. $x^{m+1}(\sum_{i=0}^{m-2} x^i) \in C_0^d$, *e.* $x(\sum_{i=0}^{m-2}(-x)^i) \in C_0^d$,

then the games (3.7) form the initial round of a Z-cyclic DTWh(p).

The ordered differences for the base table are $-(x-1)$, $x+1$, $x(x^{m-1}-1)$, $x(x^{m-1}+1)$. Arguing as above, the following theorem can be established.

Theorem 3.2 *If there exists an element $x \in Z_p \setminus 0$ such that*

a. $x \neq \square$,

b. $x^2 - 1 \neq \square$,

c. $\prod_{i=2}^{k-1}(x^{2^{k-i}} + 1) \neq \square$,

d. $x(\sum_{i=0}^{m-2} x^i) \in C_0^d$,

e. $x^{m+1}(\sum_{i=0}^{m-2}(-x)^i) \in C_0^d$,

then the games (3.7) form the initial round of a Z-cyclic OTWh(p).

If r is a primitive root of p, the *power sequence* $\{r^i : i = 0, 1, \ldots, p-2\}$ and generates the multiplicative group in GF(p) $= Z_p$. If $x = r^i$ and $y = r^j$ we say that x and y are $|i-j|$ "powers apart" in this power sequence.

Construction 2. Let r denote a primitive root of p, x denote a non-square in Z_p and consider the following collection of $(p-1)/4$ games.

$$(1, x^{m-1}, x^m, -x^{m-1}) \otimes r^{dj+2i},\ 0 \leq i \leq n-1,\ 0 \leq j \leq t-1. \tag{3.10}$$

Since 1 and x^m are squares that are m powers apart in the power sequence of r and x^{m-1}, $-x^{m-1}$ are non-squares that are mt powers apart in this same power sequence, it easily follows that the games (3.10) exhaust the set $Z_p \setminus 0$. The partner differences in the base table are $\pm(x^m - 1)$ and $\pm 2x^{m-1}$. Again, it easily follows that Condition (2.2) is satisfied if $(x^m - 1)$ is a square. The opponents first kind differences are $\pm(x^{m-1} - 1)$ and $\pm x^{m-1}(x+1)$. Since x^{m-1} is a non-square, Condition (2.3) is satisfied if $(x+1)(x^{m-1}-1)$ is a square. Similarly, Condition (2.4) is satisfied if $(x-1)(x^{m-1}+1)$ is a square. Thus the conditions sufficient for Construction 2 to be the initial round of a Z-cyclic TWh(p) are precisely the same as those for Construction 1.

The first forward differences in the base table are $x^{m-1} - 1$, $x^{m-1}(x-1)$, $-x^{m-1}(x+1)$, $x^{m-1} + 1$. Once again we assume that $x^2 - 1 \neq \square$. It then follows that the pair $x^{m-1} - 1$, $x^{m-1}(x-1)$ is such that their product is a square. Likewise for the pair $-x^{m-1}(x+1)$, $x^{m-1}+1$. We conclude that Condition (2.5) is satisfied if

$$\frac{(x^{m-1}-1)}{x^{m-1}(x-1)} \in C^d_m, \tag{3.11}$$

and

$$\frac{(x^{m-1}+1)}{-x^{m-1}(x+1)} \in C^d_m. \tag{3.12}$$

Now, (3.11) will be satisfied if $x(x^{m-2}+x^{m-3}+\ldots+x+1) \in C^d_0$. (3.12) will be satisfied if $x^{m+1}(x^{m-2}-x^{m-3}+\ldots+x^2-x+1) \in C^d_0$. Thus the following theorem is established.

Theorem 3.3 *If there exists an element $x \in Z_p \setminus 0$ such that*

a. $x \neq \Box$,

b. $x^2-1 \neq \Box$,

c. $\prod_{i=2}^{k-1}(x^{2^{k-i}}+1) \neq \Box$,

d. $x(\sum_{i=0}^{m-2} x^i) \in C^d_0$,

e. $x^{m+1}(\sum_{i=0}^{m-2}(-x)^i) \in C^d_0$,

then the games (3.10) form the initial round of a Z-cyclic DTWh(p).

The ordered differences for the base tables are $-(x^{m-1}-1)$, $x^{m-1}+1$, $x^{m-1}(x-1)$, $x^{m-1}(x+1)$. Arguing as above, the following theorem can be established.

Theorem 3.4 *If there exists an element $x \in Z_p \setminus 0$ such that*

a. $x \neq \Box$,

b. $x^2-1 \neq \Box$,

c. $\prod_{i=2}^{k-1}(x^{2^{k-i}}+1) \neq \Box$,

d. $x^{m+1}(\sum_{i=0}^{m-2} x^i) \in C^d_0$,

e. $x(\sum_{i=0}^{m-2}(-x)^i) \in C^d_0$, *then the games (3.10) form the initial round of a Z-cyclic OTWh(p).*

One can discern a certain reciprocity in the sufficient conditions associated with the production of DTWh(p) and OTWh(p) via Constructions 1 and 2. If it is desired to obtain such designs for a given p then one could investigate the satisfaction of the sufficient conditions indicated in Theorems 3.1, 3.2, 3.3, 3.4 and then employ the appropriate construction. On the other hand, suppose it is desired to investigate the existence of DTWh(p) and/or OTWh(p) for all p via these two constructions. One approach to this latter concern is to use Weil's Theorem (see Theorem 4.1 below). In the application of Weil's Theorem it is beneficial to have an efficient set of sufficient conditions in order to keep the asymptotic bound as small as possible via the methods discussed in Section 4. Theorem 3.5 serves this purpose for Constructions 1 and 2.

Theorem 3.5 *If there exists an element $x \in Z_p \setminus 0$ such that*

a. $x \neq \square$,

b. $x^2 - 1 \neq \square$,

c. $\prod_{i=2}^{k-1}(x^{2^{k-i}} + 1) \neq \square$,

d. $x(\sum_{i=0}^{m-2} x^i) \in C_0^m$,

e. $x^2(\sum_{i=0}^{m-2} x^i)(\sum_{i=0}^{m-2}(-x)^i) \in C_m^d$,

then there exists both a Z-cyclic DTWh(p) and a Z-cyclic OTWh(p).

Proof: Clearly it suffices to show that Conditions (d) and (e) guarantee the corresponding conditions in each of Theorems 3.1, 3.2, 3.3, 3.4. For convenience, set $f(x) = \sum_{i=0}^{m-2} x^i$ and $g(x) = \sum_{i=0}^{m-2}(-x)^i$. Condition ($d$) leads to two possibilities: (i) $xf(x) \in C_m^d$ but $xf(x) \notin C_0^d$ and (ii) $xf(x) \in C_0^d$. Now, Case(i) implies that $(x^m)(xf(x) \in C_0^d$. Furthermore, Case (i) together with Condition (e) imply that $xg(x) \in C_0^d$. Consequently, Construction 1 yields a Z-cyclic DTWh(p) via Theorem 3.1 and Construction 2 yields a Z-cyclic OTWh(p) via Theorem 3.4. Case (ii) together with Condition (e) imply that $xg(x) \in C_m^d$ and, hence, $(x^m)(xg(x)) \in C_0^d$. Thus, Construction 1 yields a Z-cyclic OTWh(p) via Theorem 3.2 and Construction 2 yields a Z-cyclic DTWh(p) via Theorem 3.3. ∎

4 Asymptotic Existence

For the set of sufficient conditions listed in Theorem 3.5 it is possible to find, for each k, a number N, the "asymptotic bound", such that for all $p > N$ of the form $p \equiv (2^k + 1) \pmod{2^{k+1}}$ the conditions are guaranteed to be satisfied. One approach for determining such an N is outlined below and uses the following theorem of Weil.

Theorem 4.1 *[14] Let q be a prime or a prime power and let χ be a multiplicative character of $GF(q)$ of order $s > 1$. Let $f \in GF(q)[x]$ be a monic polynomial of positive degree that is not an s-th power of a polynomial. Let b be the number of distinct roots of f in its splitting field over $GF(q)$. Then for every $a \in GF(q)$ we have $|\sum_{x\in GF(q)} \chi(af(x))| \leq (b-1)\sqrt{q}$.*

Let $s|(q-1)$. A multiplicative character of order s will be denoted by χ_s and can be defined by $\chi_s(y) = e^{(2\pi i)j/s}$ for $y \in C_j^s$ and $\chi_s(0) = 0$.
For what follows, set $\alpha(x) = x^2 - 1$, $\beta(x) = \prod_{i=2}^{k-1}(x^{2^{k-i}} + 1)$, $\gamma(x) = x(\sum_{i=0}^{m-2} x^i)$, and $\delta(x) = (x^m)(x^2)(\sum_{i=0}^{m-2} x^i)(\sum_{i=0}^{m-2}(-x)^i)$. Additionally, set $H_s(z) = 1 + \chi_s(z) + \chi_s(z^2) + \ldots + \chi_s(z^{s-1})$. Consider the sum

$$S = \sum_{x\in Z_p} (1-\chi_2(x))(1-\chi_2(\alpha(x)))(1-\chi_2(\beta(x)))(H_m(\gamma(x)))(H_d(\delta(x))). \tag{4.13}$$

Note that $|S| = 4d^2|A|$ where A is the set of elements in Z_p that satisfy the conditions of Theorem 3.5. Typically one uses Weil's theorem to obtain a lower bound on $|S|$ but in order to do so in (4.13) each factor must be expressed in terms of the multiplicative character of order d. This is accomplished via the following well known Lemma.

Lemma 4.2 *Let q be a prime or a prime power. Suppose that s_1, s_2 are positive integers such that $s_1|s_2|(q-1)$. Then $\chi_{s_1}(y) = \chi_{s_2}(y^{s_2/s_1})$.*

Proof: The result is trivial if $y = 0$. Let θ denote a primitive element of GF(q) and set $s_2 = \ell s_1$. If $y \neq 0$, there exist unique integers a, b, c, u, w such that $y = \theta^c$, $y \in C_a^{s_1}$, $y \in C_b^{s_2}$ and $us_2 + b = c = ws_1 + a$. Thus $\chi_{s_2}(y^{s_2/s_1}) = e^{(2\pi i\ell a)/s_2} = \chi_{s_1}(y)$. ∎

Thus,

$$S = \sum_{x\in Z_p} (1-\chi_d(x^m))(1-\chi_d((\alpha(x))^m))(1-\chi_d((\beta(x))^m))(H_d((\gamma(x))^2))(H_d(\delta(x))). \tag{4.14}$$

Since $(-1)^m \in C_0^d$, S has the value 0 when $x = 0$. Expanding (4.14) and applying Theorem 4.1 to all but the constant term (= 1) yields $|S| > p - \mu\sqrt{p}$, where μ is a function of k. We conclude that $|A| > 0$ if $p > \mu^2$. $\mu^2 = N$ is the asymptotic bound beyond which the conditions of Theorem 3.5 are guaranteed to hold. For example, in [7] $\mu = 1453$. Preliminary investigation indicates that for $k = 4$, $\mu = 15949$ and for $k = 5$, $\mu = 145549$. Thus it seems that a complete analysis is possible for $k = 4$ but doubtful for $k = 5$.

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