Interlacing Results on Matrices Associated with Graphs

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Abstract

Given a graph G, the adjacency matrix, $A(G)$, the standard Laplacian, $L(G)$, and the normalized Laplacian, $\mathcal{L}(G)$, have been studied intensively. In this paper, interlacing inequalities are given for each of these three matrices under the two operations of removing an edge or a vertex from G. Examples are given to show that the inequalities are the best possible of their type. In addition, an interlacing result is proven for the adjacency matrix when two vertices of G are contracted. Among the results given are the following. Let G be a graph and let H be a graph obtained from G by removing an edge or a vertex of degree r. Let λ_i, $i = 1, 2, \ldots, n$ be the eigenvalues associated with $A(G)$, $L(G)$, or $\mathcal{L}(G)$ and let θ_i be the eigenvalues associated with $A(H)$, $L(H)$, or $\mathcal{L}(H)$ where both sets of eigenvalues are in nonincreasing order. In the case of removing a vertex so that $H = G - v$, for the normalized Laplacian we have $\lambda_{i-r+1} \geq \theta_i \geq \lambda_{i+r}$. For the standard Laplacian we have $\lambda_i \geq \theta_i \geq \lambda_{i+r}$. In the case of removing an edge so that $H = G - e$, where e is an edge incident on a vertex of degree 1, for the normalized Laplacian we have $\lambda_i \geq \theta_i \geq \lambda_{i+1}$.

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1 Introduction

All graphs in this paper are simple graphs, namely, finite graphs without loops or parallel edges. Let G be a graph, and let $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. Two vertices are adjacent if they are two end-vertices of an edge and two edges are adjacent if they share a common end-vertex. A vertex and an edge are incident if the vertex is one end-vertex of the edge. For any vertex $v \in V(G)$, let d_v denote the degree of v.

Suppose that $V(G) = \{v_1, v_2, \ldots, v_n\}$. An $n \times n$ (0,1)-matrix $A := A(G) = (a_{ij})$ is called the *adjacency matrix* of G if

$$a_{ij} = \begin{cases} 1 & \text{if } v_i v_j \in E(G), \\ 0 & \text{otherwise.} \end{cases}$$

The eigenvalues of $A(G)$ have been studied extensively. We refer to Biggs [2] and Schwenk and Wilson [9] for literature in this area.

The *standard Laplacian* $L := L(G) = (L_{ij})$ of a graph G of order n is the $n \times n$ matrix L defined as follows:

$$L_{ij} = \begin{cases} d_{v_i} & \text{if } v_i = v_j, \\ -1 & \text{if } v_i v_j \in E(G), \\ 0 & \text{otherwise.} \end{cases}$$

We note that for any graph G, its standard Laplacian $L(G) = L$ can be written as $L = SS^T$, where S is the matrix whose rows are indexed by the vertices and whose columns are indexed by the edges of G such that each column corresponding to an edge $e = v_i v_j$ (with $i < j$) has entry 1 in the row corresponding to v_i, and entry -1 in the row corresponding to v_j, and has zero entries elsewhere. Since $L = SS^T$, L is positive semidefinite and has nonnegative eigenvalues. Furthermore, 0 is always an eigenvalue of L since the vector $(1, 1, \ldots, 1)^T$ is a corresponding eigenvector. In fact, as noted by Mohar [8], the multiplicity of the eigenvalue 0 is equal to the number of connected components of the graph G.

The *normalized Laplacian* of G is the $n \times n$ matrix $\mathcal{L} := \mathcal{L}(G) = (\mathcal{L}_{ij})$ given by

$$\mathcal{L}_{ij} = \begin{cases} 1 & \text{if } v_i = v_j \text{ and } d(v_i) \neq 0, \\ -\frac{1}{\sqrt{d_{v_i} d_{v_j}}} & \text{if } v_i v_j \in E(G), \\ 0 & \text{otherwise.} \end{cases}$$

Let T denote the diagonal matrix with the (i, i)-th entry having value d_{v_i}. We can write $\mathcal{L} = T^{-1/2} L T^{-1/2} = T^{-1/2} S S^T T^{-1/2}$, with the convention that $T^{-1}(i, i) = 0$ if $d_{v_i} = 0$. It can be easily seen that all eigenvalues of $\mathcal{L}$ are real and non-negative. In fact, if λ is an eigenvalue of $\mathcal{L}$, then $0 \leq \lambda \leq 2$. As pointed out in [4, p. 2], the eigenvalues of the normalized Laplacians are in a "normalized" form, and the spectra of the normalized Laplacians relate well to other graph invariants for general graphs in a way that the other two definitions fail to do. The advantages of this definition are perhaps due to the fact that it is consistent with the eigenvalues in spectral geometry and in stochastic processes.

One of our goals is to describe all eigenvalue interlacing results for $A(G)$, $L(G)$, and $\mathcal{L}(G)$, associated with the removal of an edge or vertex. Three of the six possible cases have been resolved: eigenvalue interlacing result on the adjacency matrix when a vertex is removed; eigenvalue interlacing result on the standard Laplacian when an edge is removed; and eigenvalue interlacing result on the normalized Laplacian when an edge is removed. We complete the picture by obtaining best possible interlacing results for the three remaining situations. Some other interesting related results are also given along the way. In particular, let G be a graph of order n and $H = G - e$, where e is an edge incident on a vertex of G of degree 1. If $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n = 0$ and $\theta_1 \geq \theta_2 \geq \ldots \geq \theta_n = 0$ are the eigenvalues of $\mathcal{L}(G)$ and $\mathcal{L}(H)$, respectively, then $\lambda_i \geq \theta_i$ for each $i = 1, 2, 3, 4, \ldots, n$.

2 Known Interlacing Results

Eigenvalue interlacing provides a useful tool for obtaining inequalities and regularity results concerning the structure of graphs in terms of eigenvalues of adjacency matrices and Laplacians. Much research has been done in this area. For a survey of literature, we refer to Haemers [5]. The following result is known as Cauchy's interlacing theorem.

Theorem 2.1 *Let A be a real $n \times n$ symmetric matrix and B be an $(n-1) \times (n-1)$ principal submatrix of A. If*

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \quad \text{and}$$
$$\theta_1 \geq \theta_2 \geq \ldots \geq \theta_{n-1}$$

are the eigenvalues of A and B, respectively, then

$$\lambda_i \geq \theta_i \geq \lambda_{i+1} \quad \text{for each } i = 1, 2, 3, 4, \ldots, n-1.$$

Now, instead of deleting one row and column, what if we delete q rows and the corresponding q columns? The following theorem is a generalized version of Cauchy's theorem. It can be proved by using Cauchy's interlacing theorem iteratively. For a direct proof, we may use the Courant-Fisher theorem, see page 190, [6].

Theorem 2.2 *Let A be an $n \times n$ real symmetric matrix and B be an $r \times r$, $1 \leq r \leq n$, principal submatrix of A, obtained by deleting $n-r$ rows and the corresponding columns from A. If*

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$$
$$\theta_1 \geq \theta_2 \geq \ldots \geq \theta_r$$

are the eigenvalues of A and B, respectively, then for each integer i such that
$1 \leq i \leq r$

$$\lambda_i \geq \theta_i \geq \lambda_{i+n-r}.$$

Let G be a graph of order n and let $H = G - v$, where v is a vertex of G. Theorem 2.1 gives an interlacing property of the eigenvalues of $A(G)$ and the eigenvalues of $A(H)$, which we refer to as the vertex version of the interlacing property.

Theorem 2.3 *Let G be a graph and $H = G - v$, where v is a vertex of G. If*

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \quad \textit{and}$$
$$\theta_1 \geq \theta_2 \geq \ldots \geq \theta_{n-1}$$

are the eigenvalues of $A(G)$ and $A(H)$, respectively, then

$$\lambda_i \geq \theta_i \geq \lambda_{i+1} \quad \textit{for each } i = 1, 2, 3, 4, \ldots, n-1.$$

Theorem 2.1 does not directly apply to the standard Laplacian (or the normalized Laplacian) of G and H since the principal submatrices of a standard Laplacian (or a normalized Laplacian) may no longer be the standard Laplacian (or the normalized Laplacian) of a subgraph. However, the following result given in van den Heuvel [7] reflects an edge version of the interlacing property.

Theorem 2.4 *Let G be a graph and $H = G - e$, where e is an edge of G. If*

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n = 0 \quad \textit{and}$$
$$\theta_1 \geq \theta_2 \geq \ldots \geq \theta_n = 0$$

are the eigenvalues of $L(G)$ and $L(H)$, respectively, then

$$\lambda_i \geq \theta_i \geq \lambda_{i+1} \quad \textit{for each } i = 1, 2, 3, 4, \ldots, n-1.$$

Since the trace of $\mathcal{L}$ is n when there are no isolated vertices, it is impossible to have an exactly parallel result to Theorem 2.4. However, through the use of Harmonic eigenfunctions, the following result was recently established, [3].

Theorem 2.5 *Let G be a graph and let $H = G - e$, where e is an edge of G. If*

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \quad \textit{and}$$
$$\theta_1 \geq \theta_2 \geq \ldots \geq \theta_n$$

are the eigenvalues of $\mathcal{L}(G)$ and $\mathcal{L}(H)$, respectively, then

$$\lambda_{i-1} \geq \theta_i \geq \lambda_{i+1} \quad \textit{for each } i = 1, 2, 3, 4, \ldots, n,$$

where $\lambda_0 = 2$ and $\lambda_{n+1} = 0$.

3 New Interlacing Results

We first consider how the eigenvalues of the standard Laplacian of the graphs G and $H = G - v$ interlace, where the degree of the vertex v is r. It turns out that a modification of the proof of Theorem 2.4 yields the following result.

Theorem 3.1 *Let G be a graph of order n and $H = G - v$, where v is a vertex of G of degree r. If*

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n = 0$$
$$\theta_1 \geq \theta_2 \geq \ldots \geq \theta_{n-1} = 0$$

are the eigenvalues of $L(G)$ and $L(H)$, respectively, then

$$\lambda_i \geq \theta_i \geq \lambda_{i+r} \quad \textit{for each } i = 1, 2, 3, 4, \ldots, n,$$

where $\lambda_i = 0$ for $i \geq n+1$.

Proof: We know that $L(G) = S_G S_G^T$ and that the eigenvalues of $S_G S_G^T$ are nonnegative. Let H be a subgraph of G obtained by deleting a vertex of degree r from G and let $L(H)$ be the standard Laplacian of the graph H. Now, for any $n \times m$ matrix A, the spectrum of the matrices AA^T and $A^T A$ coincide except for the multiplicity of the eigenvalue 0. In particular, the positive eigenvalues of $L(G)$ are the same as the positive eigenvalues of $S_G^T S_G$. Observe that $L(H) = S_H S_H^T$ can be obtained by deleting the r

columns and rows corresponding to the edges incident on the deleted vertex, so that $S_H^T S_H$ is an order $m-r$ principal submatrix of $S_G^T S_G$ (where m is the number of edges in G). Hence, the theorem follows from the generalized version of the Cauchy's Interlacing Theorem, Theorem 2.2. □

In the above theorem, we cannot improve the gap on the right by reducing it from r to $r-1$, as shown by considering the complete bipartite graph. It was noted by W. Anderson and T. Morley in [1] that the eigenvalues of the standard Laplacian of the complete bipartite graph $K_{m,n}$ on $m+n$ vertices are $m+n$, m, n, 0 with multiplicities 1, $n-1$, $m-1$, 1, respectively. Without loss of generality, assume $m > n$. Then the eigenvalues of the graph $K_{m,n-1}$ are $m+n-1$, m, $n-1$, 0 with multiplicities 1, $n-2$, $m-1$, 1, respectively. Since $\theta_n = n-1$, $\lambda_{n+1} = \lambda_{n+2} = \ldots = \lambda_{m+n-1} = n$, and $\lambda_{m+n} = 0$, we have $\theta_n = n-1 \geq 0 = \lambda_{m+n}$, where the gap is m, which is the degree of the vertex removed from $K_{m,n}$.

We now consider how the eigenvalues of the normalized Laplacian of the graphs G and $H = G - v$ interlace, where the degree of the vertex v is r. First of all, we look at a few results that will be of great help in proving a main result of this paper. Now we know how the eigenvalues of the normalized Laplacian of the graphs G and H interlace, where the graph H is obtained by deleting r edges from the graph G. We can apply Theorem 2.5 iteratively r times and we have the following proposition.

Proposition 3.2 *Let G be a graph and let H be a subgraph of G obtained by deleting r edges. If*

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \quad \text{and}$$
$$\theta_1 \geq \theta_2 \geq \ldots \geq \theta_n$$

are the eigenvalues of $\mathcal{L}(G)$ and $\mathcal{L}(H)$, respectively, then

$$\lambda_{i-r} \geq \theta_i \geq \lambda_{i+r}, \qquad \text{for each } i = 1, 2, \ldots, n$$

with the convention of

$$\lambda_i = 2 \quad \text{for each} \quad i \leq 0,$$
$$\lambda_i = 0 \quad \text{for each} \quad i \geq n+1.$$

The proof of Theorem 3.6 will heavily depend on the Courant-Fischer theorem.

Theorem 3.3 (Courant-Fischer) *For a real, symmetric $n \times n$ matrix A with eigenvalues*

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_n$$

we have

$$\lambda_k = \max_{g^{(k+1)}, g^{(k+2)}, \ldots, g^{(n)} \in \mathbb{R}^n} \quad \min_{\substack{g \perp g^{(k+1)}, g^{(k+2)}, \ldots, g^{(n)} \\ g \neq 0}} \frac{g^T A g}{g^T g}$$

and

$$\lambda_k = \min_{g^{(1)}, g^{(2)}, \ldots, g^{(k-1)} \in \mathbb{R}^n} \quad \max_{\substack{g \perp g^{(1)}, g^{(2)}, \ldots, g^{(k-1)} \\ g \neq 0}} \frac{g^T A g}{g^T g}.$$

Lemma 3.4 *Let G be a graph on n vertices, let $L = L(G)$ be the standard Laplacian of G, and let $f = (f_1, \ldots, f_n)^T$ be a column vector in $\mathbb{R}^n$. Then,*

$$f^T L f = \sum_{i \sim j} (f_i - f_j)^2$$

where $\sum_{i \sim j}$ runs over all unordered pairs $\{i, j\}$ for which v_i and v_j are adjacent.

Proof: Lemma 3.4 directly follows from the definition of L. □

Lemma 3.5 *Suppose that for real a, b and γ*

$$a^2 - 2\gamma^2 \geq 0, \qquad b^2 - \gamma^2 > 0, \qquad \text{and} \qquad \frac{a^2}{b^2} \leq 2.$$

Then

$$\frac{a^2 - 2\gamma^2}{b^2 - \gamma^2} \leq \frac{a^2}{b^2}.$$

Proof. The result follows from

$$\frac{a^2 - 2\gamma^2}{b^2 - \gamma^2} = \frac{a^2}{b^2} \frac{1 - 2\gamma^2/a^2}{1 - \gamma^2/b^2} \leq \frac{a^2}{b^2}$$

since the final inequality holds when $\frac{\gamma^2}{b^2} \leq \frac{2\gamma^2}{a^2}$ which is equivalent to $a^2/b^2 \leq 2$. □

In the special case where the edge removed is incident on a vertex of degree 1, we can improve on Theorem 2.5 by showing that the eigenvalues do not increase when an edge is removed. Note that the corresponding result for the standard Laplacian is a trivial special case of Theorem 2.4. The proof of the theorem for the normalized Laplacian is more involved and must deal with the fact that removing an edge decreases the degree of an adjacent vertex, thereby increasing the magnitude of some (possibly many) off-diagonal elements.

Theorem 3.6 *Let G be a graph of order n and $H = G - e$, where e is an edge incident on a vertex of G of degree 1. If*

$$\begin{aligned} \lambda_1 &\geq \lambda_2 \geq \ldots \geq \lambda_n = 0 \\ \theta_1 &\geq \theta_2 \geq \ldots \geq \theta_n = 0 \end{aligned}$$

are the eigenvalues of $\mathcal{L}(G)$ and $\mathcal{L}(H)$, respectively, then

$$\lambda_i \geq \theta_i \quad \textit{for each } i = 1, 2, 3, 4, \ldots, n.$$

Proof: We adapt the Courant-Fischer theorem to the Laplacian using harmonic eigenfunctions. Recall that

$$\mathcal{L} = T^{-1/2} L T^{-1/2}.$$

We assume that $T^{1/2}$ is invertible, that is, there are no vertices of degree zero.

For vectors g and $g^{(j)}$ define the vectors

$$f = T^{-1/2} g \qquad \text{and} \qquad f^{(j)} = T^{1/2} g^{(j)}.$$

Note that

$$g \perp g^{(1)}, g^{(2)}, \ldots, g^{(k-1)}$$

if and only if

$$f \perp f^{(1)}, f^{(2)}, \ldots, f^{(k-1)}.$$

The notation $f \perp f^{(1)}, \ldots, f^{(k-1)}$ means that f is orthogonal to $\operatorname{span}(f^{(1)}, \ldots, f^{(k-1)})$.

Applying the Courant-Fischer theorem to get the eigenvalues λ_k of $\mathcal{L}$ gives

$$\lambda_k = \min_{g^{(1)}, g^{(2)}, \ldots, g^{(k-1)} \in \mathbb{R}^n} \quad \max_{\substack{g \perp g^{(1)}, g^{(2)}, \ldots, g^{(k-1)} \\ g \neq 0}} \quad \frac{g^T T^{-1/2} L T^{-1/2} g}{g^T g}$$

$$= \min_{f^{(1)},f^{(2)},\ldots,f^{(k-1)}\in\mathbb{R}^n} \max_{\substack{f\perp f^{(1)},f^{(2)},\ldots,f^{(k-1)}\\ f\neq 0}} \frac{f^TLf}{f^TTf}$$

$$= \min_{f^{(1)},f^{(2)},\ldots,f^{(k-1)}\in\mathbb{R}^n} \max_{\substack{f\perp f^{(1)},f^{(2)},\ldots,f^{(k-1)}\\ f\neq 0}} \frac{\sum_{i\sim j}(f_i - f_j)^2}{\sum_j f_j^2 d_j}$$

where f_j is the j-th component of f, $d_j = d(v_j)$ is the degree of v_j, and $\sum_{i\sim j}$ runs over all unordered pairs $\{i,j\}$ for which v_i and v_j are adjacent. The second line depends on the invertibility of T so that maximizing over vectors $f^{(k)}$ is equivalent to maximizing over vectors $g^{(k)}$. The final line depends on Lemma 3.4. The vector f can be viewed as a function $f(v)$ on the set of vertices that maps v_j to f_j. The function $f(v)$ is a harmonic eigenfunction.

Without loss of generality we assume that an edge between the particular vertices v_1 and v_2 is removed, where the degree of the vertex v_1 is 1 and consider the eigenvalues θ_k of the Laplacian of the modified graph. Two changes occur in the Courant-Fischer theorem when an edge is removed. The degrees of v_1 and v_2 are decreased from 1 and $d(v_2)$ to 0 and $d(v_2)-1$, respectively, so that

$$\sum_j f_j^2 d_j \to \sum_j f_j^2 d_j - f_1^2 - f_2^2.$$

Also, since v_1 and v_2 are no longer adjacent, the sum no longer includes the pair $\{1,2\}$ so that

$$\sum_{i\sim j}(f_i - f_j)^2 \to \sum_{i\sim j}(f_i - f_j)^2 - (f_1 - f_2)^2$$

Note that the sum $\sum_{i\sim j}$ still runs over vertices that are adjacent in the *original graph*; in applying the theorem to the modified graph we explicitly subtract out $(f_1 - f_2)^2$ instead of modifying the index set of the sum.

Thus

$$\theta_k = \min_{f^{(1)},f^{(2)},\ldots,f^{(k-1)}\in\mathbb{R}^n} \max_{\substack{f\perp f^{(1)},f^{(2)},\ldots,f^{(k-1)}\\ f\neq 0}} \frac{\sum_{i\sim j}(f_i - f_j)^2 - (f_1 - f_2)^2}{\sum_j f_j^2 d_j - f_1^2 - f_2^2}$$

$$= \min_{f^{(1)},f^{(2)},\ldots,f^{(k-1)}\in\mathbb{R}^n} \max_{\substack{f\perp f^{(1)},f^{(2)},\ldots,f^{(k-1)}\\ f\neq 0\\ f_1=-f_2}} \frac{\sum_{i\sim j}(f_i - f_j)^2 - 4f_2^2}{\sum_j f_j^2 d_j - 2f_2^2}$$

$$\leq \min_{f^{(1)},f^{(2)},\ldots,f^{(k-1)}\in\mathbb{R}^n} \max_{\substack{f\perp f^{(1)},f^{(2)},\ldots,f^{(k-1)} \\ f\neq 0 \\ f_1=-f_2}} \frac{\sum_{i\sim j}(f_i - f_j)^2}{\sum_j f_j^2 d_j}$$

$$\leq \min_{f^{(1)},f^{(2)},\ldots,f^{(k-1)}\in\mathbb{R}^n} \max_{\substack{f \perp f^{(1)},f^{(2)},\ldots,f^{(k-1)} \\ f\neq 0}} \frac{\sum_{i\sim j}(f_i - f_j)^2}{\sum_j f_j^2 d_j} = \lambda_k$$

In line 2, we use the fact that the expression does not depend on f_1. In line 3 we use Lemma 3.5. The lemma is applicable since the inequality $\lambda_k \leq 2$, which holds for every eigenvalue of any normalized Laplacian, implies

$$\frac{\sum_{i\sim j}(f_i - f_j)^2}{\sum_j f_j^2 d_j} \leq 2.$$

Hence,

$$\lambda_k \geq \theta_k \quad \text{for each } k = 1,2,3,4,\ldots,n$$

when the edge incident to a vertex of degree 1 is removed.

We have assumed throughout that T is invertible (i.e., there is no vertex of degree zero). However, this is not restrictive; the inequality holds in general. If $d(v) = 0$ for q vertices then the normalized Laplacian can be permuted so that

$$P\mathcal{L}P^T = \begin{pmatrix} \mathcal{L}_1 & 0 \\ 0 & 0_{q\times q} \end{pmatrix}$$

for some permutation matrix P. Thus for $\mathcal{L}$,

$$\lambda_{n-q} = \lambda_{n-q+1} = \cdots = \lambda_n = 0.$$

The removal of an edge then only affects $\mathcal{L}_1$ so that the theorem can be applied to the submatrix. The additional zero eigenvalues, $\theta_{n-q+1} = \theta_{n-q+2} = \cdots = \theta_n = 0$ satisfy

$$\lambda_k \geq \theta_k$$

for $k = n-q+1,\cdots,n$. Interlacing bounds for all other θ_k follow from the interlacing theorem applied to $\mathcal{L}_1$. □

Now consider removing j edges from v_1 to get eigenvalues $\theta_k^{(j)}$ (where $\theta_k^{(j)}$ represents the k-th eigenvalue of the normalized Laplacian of the subgraph obtained by deleting j edges of the original graph from v_1). By applying the edge interlacing result, Theorem 2.5, we have

$$\theta_k^{(j)} \leq \theta_{k-1}^{(j-1)} \leq \theta_{k-2}^{(j-2)} \leq \ldots \leq \theta_{k-j}^{(0)} = \lambda_{k-j}.$$

Thus, from Theorem 3.6, we have the following

$$\theta_k^{(d_1)} \leq \theta_k^{(d_1-1)} \leq \lambda_{k-(d_1-1)},$$

where $d_1 = d_{v_1}$. Using this fact and Proposition 3.2, we have the following new result.

Theorem 3.7 *Let G be a graph of order n and H be a subgraph of G obtained by deleting the r edges from a vertex v of degree r. If*

$$\begin{aligned} \lambda_1 &\geq \lambda_2 \geq \ldots \geq \lambda_n = 0 \\ \theta_1 &\geq \theta_2 \geq \ldots \geq \theta_n = 0 \end{aligned}$$

are the eigenvalues of $\mathcal{L}(G)$ and $\mathcal{L}(H)$, respectively, then

$$\lambda_{i-r+1} \geq \theta_i \geq \lambda_{i+r} \quad \textit{for each } i = 1, 2, 3, 4, \ldots, n,$$

where $\lambda_i = 2$ for $i \leq 0$ and $\lambda_i = 0$ for $i \geq n+1$.

Hence, if we remove a vertex of degree r from the graph G to obtain the subgraph H of G, we will have $\theta_{n-1} = \theta_n = 0$, which will give us the following result.

Theorem 3.8 *Let G be a graph of order n and $H = G - v$, where v is a vertex of G of degree r. If*

$$\begin{aligned} \lambda_1 &\geq \lambda_2 \geq \ldots \geq \lambda_n = 0 \\ \theta_1 &\geq \theta_2 \geq \ldots \geq \theta_{n-1} = 0 \end{aligned}$$

are the eigenvalues of $\mathcal{L}(G)$ and $\mathcal{L}(H)$, respectively, then

$$\lambda_{i-r+1} \geq \theta_i \geq \lambda_{i+r} \quad \textit{for each } i = 1, 2, 3, 4, \ldots, n-1,$$

where $\lambda_i = 2$ for $i \leq 0$ and $\lambda_i = 0$ for $i \geq n+1$.

In the above theorem, we cannot improve the gap on left by reducing it from $r-1$ to $r-2$, as shown by considering the complete graph. It was noted by Chung in [4] that the eigenvalues of the normalized Laplacian of the complete graph K_n on n vertices are 0 and $\frac{n}{n-1}$ with multiplicities 1 and $n-1$ respectively. Hence, the eigenvalue of the normalized Laplacian of the subgraph $K_{n-1} = K_n - v$ on $n-1$ vertices are 0 and $\frac{n-1}{n-2}$ with multiplicities 1 and $n-2$ respectively. Since $\theta_{n-2} = \frac{n-1}{n-2} \not\leq \frac{n}{n-1} = \lambda_1$, we have $\theta_{n-2} = \frac{n-1}{n-2} \leq 2 = \lambda_0 = \lambda_{n-2-(n-1)+1}$, where the gap is $n-2$, which is 1 less than the degree of the vertex removed from K_n. Similarly, we cannot improve the gap on the right by reducing it from r to $r-1$, as shown by considering the star graph. It was noted by Chung in [4] that the eigenvalues of the star graph S_n are 0, 1, 2 with multiplicities 1, $n-2$, 1, respectively. Hence, if we remove the center vertex from the star graph S_n on n vertices, the eigenvalues of the new subgraph are 0 with multiplicity $n-1$. Since $\theta_1 = 0 \not\geq 1 = \lambda_{n-1}$, we have $\theta_1 = 0 \geq 0 = \lambda_n = \lambda_{1+(n-1)}$, where the gap is $n-1$, which is the degree of the vertex removed from S_n.

Finally, using Cauchy's interlacing result on two pairs of matrices, we can prove the following edge version of the interlacing result for the adjacency matrix. This completes the picture regarding the six cases of eigenvalue interlacing results for $A(G)$, $L(G)$, and $\mathcal{L}(G)$, associated with the removal of an edge or vertex.

Theorem 3.9 *Let G be a graph and let $H = G - e$, where e is an edge of G. If*

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \quad \text{and}$$
$$\theta_1 \geq \theta_2 \geq \ldots \geq \theta_n$$

are the eigenvalues of $A(G)$ and $A(H)$, respectively, then

$$\lambda_{i-1} \geq \theta_i \geq \lambda_{i+1} \quad \text{for each } i = 2, 3, 4, \ldots, n-1,$$

$\theta_1 \geq \lambda_2$, and $\theta_n \leq \lambda_{n-1}$.

Proof: Let $P = G - v$, where v is a vertex of G that is incident to edge e and $A(P)$ be the adjacency matrix of the graph P. Let

$$\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_{n-1}$$

be the eigenvalues of $A(P)$. Then $A(P)$ is a principal submatrix of both $A(G)$ and $A(H)$. Hence, from Cauchy's interlacing result we have the following

$$\lambda_i \geq \gamma_i \geq \lambda_{i+1} \quad \text{for each } i = 1, 2, 3, 4, \ldots, n-1 \quad \text{and}$$
$$\theta_i \geq \gamma_i \geq \theta_{i+1} \quad \text{for each } i = 1, 2, 3, 4, \ldots, n-1$$

Then, we have the following,

$$\lambda_{i-1} \geq \gamma_{i-1} \geq \theta_i \geq \gamma_i \geq \lambda_{i+1}$$

Hence, the theorem follows. □

In the above theorem, we cannot improve the gap, as shown by considering the Petersen graph G. The eigenvalues of $A(G)$ are 3, 1 (with multiplicity 5), and -2 (with multiplicity 4). The eigenvalues of $A(H)$, where the graph H is obtained by deleting any edge of G, are $2.8558, 1.4142, 1, 1, 1, 0.3216, -1.4142, -2, -2$, and -2.1774. Hence, $\lambda_2 = 1 \not\geq 1.4142 = \theta_2$ and $\theta_1 = 2.8558 \not\geq 3 = \lambda_1$.

Let G be a graph and $x \in V(G)$. The neighborhood of x is

$$N(x) = \{y \ : \ xy \in E(G)\}.$$

For any two vertices u and v of G, we use $G/\{u, v\}$ to denote the graph obtained from G by contracting u and v to one vertex, i.e., $G/\{u, v\}$ is the graph obtained from G by deleting the vertices u and v and adding a new vertex (uv) such that the neighborhood of (uv) is the union of the neighborhoods of u and v When u and v are adjacent, $G/\{u, v\}$ is the graph obtained from G by contracting the edge uv. Contraction of edges and vertices has many applications in graph theory. By contracting two nonadjacent vertices with nonintersecting neighborhoods we obtain the following interlacing result.

Theorem 3.10 *Let G be a graph and let u and v be two distinct vertices of G. Define $H = G/\{u, v\}$ and let*

$$\begin{aligned} \lambda_1 &\geq \lambda_2 \geq \dots \geq \lambda_n \quad \textit{and} \\ \theta_1 &\geq \theta_2 \geq \dots \geq \theta_{n-1} \end{aligned}$$

be the eigenvalues of $A(G)$ and $A(H)$, respectively. Then

$$\lambda_{i-1} \geq \theta_i \geq \lambda_{i+2}, \quad \textit{for each } i = 2, 3, 4, \dots, n-2,$$

where $\theta_1 \geq \lambda_3$ and $\lambda_{n-2} \geq \theta_{n-1}$.

If we assume that $N(u) \cap (N(v) \cup \{v\}) = \emptyset$ then, depending on the sign of θ_i, the above inequalities can be strengthened in one of two ways. Let k be such that $\theta_k \geq 0$ and $\theta_{k+1} < 0$. Then

$$\theta_i \geq \lambda_{i+1} \quad \textit{for each } i = 1, 2, \dots, k$$

and

$$\lambda_i \geq \theta_i \quad \textit{for each } i = k+1, k+2, \dots, n-1.$$

Proof: The matrix $A(H)$ can be obtained from $A(G)$ by removing the rows and columns associated with u and v and adding a new row and column for the contracted vertex. The inequality $\lambda_{i-1} \geq \theta_i \geq \lambda_{i+2}$ follows from Theorem 2.2.

Let us now suppose that $N(u) \cap (N(v) \cup \{v\}) = \emptyset$, and let k be such that $\theta_k \geq 0$ and $\theta_{k+1} < 0$. We assume without loss of generality that $u = v_1$ and $v = v_2$ and consider the graph $\hat{H}$ obtained by deleting every edge from v_1 and adding a corresponding edge to v_2. The eigenvalues of the graph $\hat{H}$ differ from those of H only in that $\hat{H}$ has an additional vertex, v_1, of degree zero giving an additional zero eigenvalue. For $x \in \mathbb{R}^n$ we have

$$\frac{x^T A(G) x}{x^T x} = 2\frac{\sum_{j\sim l} x_j x_l}{\sum_j x_j^2}$$

If we let J be the set of indices of vertices adjacent to v_1 then

$$\frac{x^T A(\hat{H}) x}{x^T x} = 2\frac{\sum_{j\sim l} x_j x_l + \sum_{j\in J}(x_2 x_j - x_1 x_j)}{\sum_j x_j^2}.$$

The min-max part of the Courant-Fischer theorem gives

$$\begin{aligned}
\hat{\theta}_i &= \min_{y^{(1)},\ldots,y^{(i-1)}\in\mathbb{R}^n} \max_{\substack{x\perp y^{(1)},\ldots,y^{(i-1)}\\ x\neq 0}} 2\frac{\sum_{j\sim l} x_j x_l + \sum_{j\in J}(x_2 x_j - x_1 x_j)}{\sum_j x_j^2} \\
&\geq \min_{y^{(1)},\ldots,y^{(i-1)}\in\mathbb{R}^n} \max_{\substack{x\perp y^{(1)},\ldots,y^{(i-1)}\\ x_1 = x_2\\ x\neq 0}} 2\frac{\sum_{j\sim l} x_j x_l + \sum_{j\in J}(x_2 x_j - x_1 x_j)}{\sum_j x_j^2} \\
&= \min_{y^{(1)},\ldots,y^{(i-1)}\in\mathbb{R}^n} \max_{\substack{x\perp y^{(1)},\ldots,y^{(i-1)},e_1-e_2\\ x\neq 0}} 2\frac{\sum_{j\sim l} x_j x_l}{\sum_j x_j^2} \\
&\geq \min_{y^{(1)},\ldots,y^{(i)}\in\mathbb{R}^n} \max_{\substack{x\perp y^{(1)},\ldots,y^{(i)}\\ x\neq 0}} 2\frac{\sum_{j\sim l} x_j x_l}{\sum_j x_j^2} = \lambda_{i+1}
\end{aligned}$$

where in the above $i = 1, 2, \ldots, n-1$. The max-min part of the Courant-Fischer theorem can be used to show that $\hat{\theta}_i \leq \lambda_{i-1}$ for $i = 2, 3, \ldots, n$. Thus for the eigenvalues $\hat{\theta}_i$ of $A(\hat{H})$ we have $\lambda_{i-1} \geq \hat{\theta}_i \geq \lambda_{i+1}$ for $i = 2, 3, \ldots, n-1$ as well as $\hat{\theta}_n \leq \lambda_{n-1}$ and $\hat{\theta}_1 \geq \lambda_2$. Since the eigenvalues of $A(H)$ and $A(\hat{H})$ differ only in that the latter set includes an additional zero eigenvalue we have $\theta_i = \hat{\theta}_i$ for $i = 1, 2, \ldots, k$ from which we get $\theta_i = \hat{\theta}_i \geq \lambda_{i+1}$ for

$i = 1, 2, \ldots, k$. Similarly we have $\theta_i = \hat{\theta}_{i+1}$ for $i = k+1, k+2, \ldots, n-1$ so that $\theta_i = \hat{\theta}_{i+1} \leq \lambda_i$ for $i = k+1, k+2, \ldots, n-1$. □

We conclude by observing that the inequality $\lambda_{i-1} \geq \theta_i \geq \lambda_{i+1}$ holds in a surprising number of cases, including for $A(G)$ and $\mathcal{L}(G)$ in the case of removing an edge. In the case of contracting two vertices, the inequality holds for $\mathcal{L}(G)$ (proven in [3]). Except for the adjacency matrix in the case of edge removal, none of these is an obvious consequence of Cauchy's interlacing theorem. This raises a natural question of what other operations on graphs might lead to similar interlacing results.

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