On *b*-Eccentricity in Graphs

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Abstract

The distance $d(u, v)$ between a pair of vertices u and v in a connected graph G is the length of a shortest path joining them. A vertex v of a connected graph G is an eccentric vertex of a vertex u if v is a vertex at greatest distance from u; while v is an eccentric vertex of G if v is an eccentric vertex of some vertex of G. A vertex v of G is a boundary vertex of a vertex u if $d(u, w) \leq d(u, v)$ for each neighbour w of v. A vertex v is a boundary vertex of G if v is a boundary vertex of some vertex of G. It is easy to see that for a vertex u, its eccentric vertices are boundary vertices for u; but not conversely. In this paper, we introduce a new type of eccentricity called *b*-eccentricity and we study its properties.

Keywords. *b*-eccentricity, *b*-radius, *b*-diameter, *b*-center, *b*-periphery, *b*-self centered graph.

2000 Mathematics Subject Classification: 05C12

1 Introduction

Let G be a non-trivial connected graph. The distance $d(u, v)$ between two vertices u and v of G is the length of a shortest u-v path in G. For a vertex v of G, the eccentricity $e(v)$ is the distance between v and a vertex farthest from v. The minimum eccentricity among the vertices of G is the radius $r(G)$ of G and the maximum eccentricity is its diameter $d(G)$. A vertex v in G is called a *central vertex* if $e(v) = r(G)$ and the peripheral vertex if $e(v) = d(G)$; and the set of all center vertices is denoted by C

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and its induced subgraph $\langle C \rangle$ is called the center of the graph. The set of all peripheral vertices is denoted by P and its induced subgraph $\langle P \rangle$ is called the periphery of the graph. If $r(G) = d(G)$, then G is called a self centered graph. A vertex v in G is called an eccentric vertex of a vertex u if $d(u, v) = e(u)$, that is, every vertex at greatest distance from u is an eccentric vertex of u. A vertex v is an eccentric vertex of G if v is an eccentric vertex of some vertex of G. Consequently, if v is an eccentric vertex of u and w is a neighbor of v, then $d(u, w) \leq d(u, v)$. A vertex v may have this property, however, without being an eccentric vertex of u. In [2,3,4], a vertex v is defined to be a boundary vertex of a vertex u if $d(u, w) \leq d(u, v)$ for all $w \in N(v)$. In [2], Caceres et al. proved that boundary set of any graph G is geodetic, that is, every vertex in G lies on some shortest path joining two boundary vertices.

A vertex v is a boundary vertex of G if v is a boundary vertex of some vertex of G. A graph G in which each vertex is a boundary vertex is called a self-boundary graph. A vertex in a graph is called complete (or extreme or simplical) if the subgraph induced by its neighborhood is complete. For graph theoretic terminology, we follow [1,5].

2 Main Results

The b-eccentricity $e_b(u)$ of a vertex u is defined by $e_b(u) = min\{d(u, w) : w$ is a boundary of $u\}$. Minimum b-eccentricity among the vertices of G is the b-radius $r_b(G)$ of G and the maximum b-eccentricity is its b-diameter $d_b(G)$. A vertex v is called a b-central vertex if $e_b(v) = r_b(v)$ and a b-peripheral vertex if $e_b(v) = d_b(v)$. A graph G is called a b-self centered graph if $d_b(G) = r_b(G)$. The set of all b-central and b-peripheral vertices of a graph are denoted by C_b and P_b respectively. The induced subgraph $\langle C_b \rangle$ of all b-central vertices is called the b-center $C_b(G)$ of G and the induced subgraph $\langle P_b \rangle$ of all b-peripheral vertices is called b-periphery $P_b(G)$ of G.

A graph G and its b-eccentricities are given in Figure 1. In the graph G, $r_b(G) = 1$ and $d_b(G) = 4$. Also $C_b(G) = K_2$ and $P_b(G) = \overline{K_2}$.

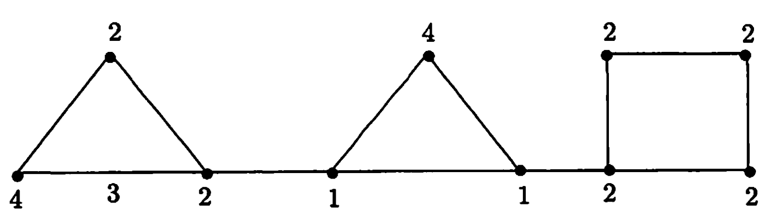

Figure 1. The graph G and its b-eccentricity

Remark 2.1. *Let u be a boundary vertex in a connected graph G. If u is complete, then $e_b(w) = 1$ for all $w \in N(u)$. But the converse need not be true. In the graph G of Figure 2, u is a boundary vertex of v and $e_b(w) = 1$ for all $w \in N(u)$ while $\langle N(u) \rangle$ is not complete.*

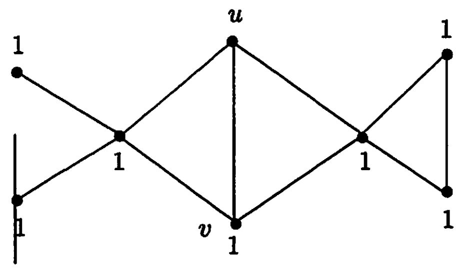

Figure 2. The graph G

Next we provide a result which establishes bounds for b-diameter of a graph.

Result 2.2. *For any connected graph G of order n, $r_b(G) \leq d_b(G) \leq n - 1$.*

We now verify the sharpness of the above inequality.

Proposition 2.3. *There exists a family of b-self centered graphs which are not self centered graphs.*

Proof. We construct a graph G_n by using K_2 and two copies of odd cycle C_n. Let the vertices of K_2 be v_1 and v_2. Let $u_{11}, u_{12}, u_{13}, \ldots, u_{1n}$ and $u_{21}, u_{22}, \ldots, u_{2n}$ be the vertices of the two copies of C_n. Merge the vertex u_{11} with the vertex v_1 and merge the vertex u_{21} with v_2 and let the resulting graph be G_n . The boundary vertices of u_{11} in G are $u_{1\left(\frac{n+1}{2}\right)}, u_{1\left(\frac{n+3}{2}\right)}, u_{2\left(\frac{n+1}{2}\right)}, u_{2\left(\frac{n+3}{2}\right)}$.

Also $d\left(u_{11}, u_{1\left(\frac{n+1}{2}\right)}\right) = d\left(u_{11}, u_{1\left(\frac{n+3}{2}\right)}\right) = \frac{n-1}{2}$.

But $d\left(u_{11}, u_{2\left(\frac{n+1}{2}\right)}\right) = d\left(u_{11}, u_{2\left(\frac{n+3}{2}\right)}\right) = \frac{n+1}{2}$. Thus $e_b(u_{11}) = \frac{n-1}{2}$. Similarly $e_b(u_{21}) = \frac{n-1}{2}$. The vertices $u_{1i}, i = 2, 3, \ldots, n-1$ has its boundary vertices at minimum distance within the set $\{u_{12}, u_{13}, u_{14}, \ldots, u_{1(n-1)}\}$. The vertices $u_{2i}, i = 2, 3, \ldots, n-1$ has its boundary vertices at minimum distance within the set $\{u_{22}, u_{23}, u_{24}, \ldots, u_{2(n-1)}\}$. From this it is easy to verify that $e_b(u) = \frac{n-1}{2}$ for all $u \in V(G)$. Thus G_n is a b-self centered graph but it is not a self centered graph. □

Proposition 2.4. *There exist graphs G_n such that $d_b(G_n) = n - 1$.*

Proof. The graph $G_n = P_n, n \geq 1$ has the required property. □

Next we give some graphs having prescribed b-radius and b-diameter.

Theorem 2.5. *For each pair a, b of positive integers with $a \leq b$, there exists a connected graph G with $r_b(G) = a$ and $d_b(G) = b$.*

Proof. **Case i.** $a = 1$ and $b \geq 1$.
The path P_{b+1} has the required property.

Case ii. $a \geq 2$ and $b > a$.
For given a, consider the two copies of cycle C_{2a+1} and for given b, consider the path $P_{2(b-a)+1}$. Let $u_{11}, u_{12}, \ldots, u_{1(2a+1)}$ and $u_{21}, u_{22}, \ldots, u_{2(2a+1)}$ be the vertices of the two copies of the cycle C_{2a+1}. Let $v_1, v_2, \ldots, v_{2(b-a)+1}$ be the vertices of the path $P_{2(b-a)+1}$. Merge the vertex u_{11} with v_1 and merge the vertex u_{21} with $v_{2(b-a)+1}$. Let the resulting graph be G. Then

$$e_b(u_{1i}) = e_b(u_{2i}) = a = r_b(G) \text{ for } i = 1, 2, \ldots, 2a + 1.$$

Also
$e_b\left(v_{2((b-a)+2)/2}\right) = e_b\left(v_{b-a+1}\right) = b = d_b(G)$.
$e_b\left(v_{b-a+1-i}\right) = b - i, i = 1, 2, \ldots, b - a - 1$.
$e_b\left(v_{b-a+1+i}\right) = b - i, i = 1, 2, 3, \ldots, b - a - 1$.

Case iii. $a = b \geq 2$.
Consider the graph G constructed in Proposition 2.3. When we take $n = 2a + 1$ in Proposition 2.3, we get the required property. □

3 b-Center and b-Periphery of Graphs

In this section, we prove that every connected graph is a b-center of some connected graph and b-periphery of some connected graph.

The b-center of a connected graph G need not lie within a block of G. The graph G given in Figure 3 satisfies this property.

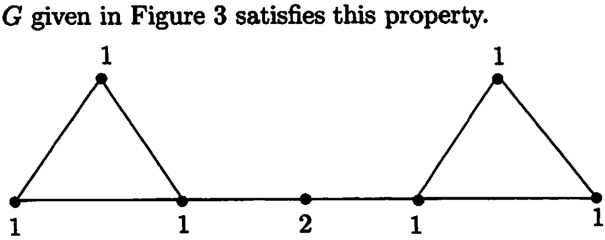

Figure 3. The graph G

If G is a complete graph, then G is a b-center and b-periphery of itself. Now we assume that G is not a complete graph.

Theorem 3.1. *Every graph G is a b-center of some connected graph H.*

Proof. Let G be a graph. Replace every edge uv by K_3 if uv lie on a cycle and by $K_4 - x$ if uv does not lie on a cycle in such a way that the 3-degree vertices x and w in $K_4 - x$ are coincided with u and v respectively. Let $v_{11}, v_{12}, v_{21}, v_{22}, \ldots, v_{l1}, v_{l2}$ be the vertices of degree 2 in the copies of $K_4 - x$ where l is the number of edges which are not lie on cycle, in the resulting graph be H. Then

$$e_b(u) = 1 \text{ for all } u \in V(G) \text{ and } e_b(y) = 2$$

for all newly added vertices y.

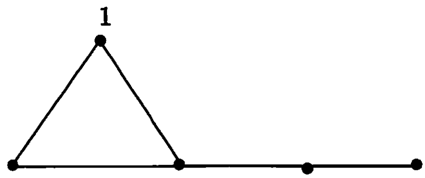

Figure 4. The graph G

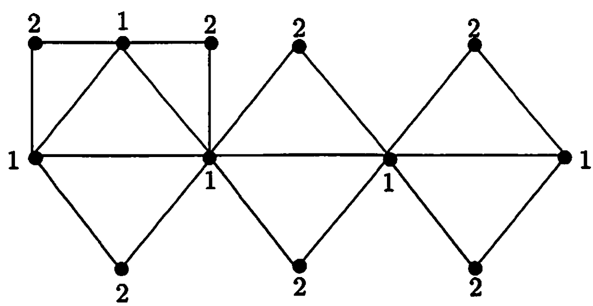

Figure 5. The graph H with b-eccentricity

□

Remark 3.2. *The construction of the graph H given in Theorem 3.1 is not unique.*

Theorem 3.3. *Every connected graph G is a b-periphery of some connected graph H.*

Proof. Consider a complete graph K_m. Replace exactly one edge of K_m by K_3 and denote the resulting graph by F and the newly added vertex by v.

With each vertex of G, merge the 2-degree vertex of F and let the resulting graph be H. Then $e_b(u) = 2$ for all $u \in V(G)$ in H and $e_b(w) = 1$ for all vertices of K_m in H. Thus G is a b-Periphery of H.

Figure 6. The graph G

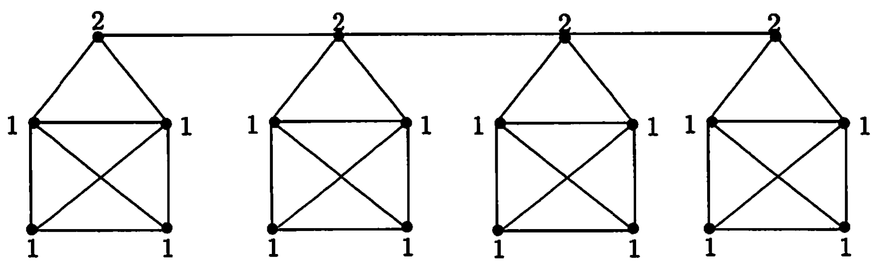

Figure 7. The graph H with b-eccentricity

□

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